S. Ponnusamy

Foundations of Complex Analysis

With 114 Illustrations

Dedicated to

Professor O.P. Juneja

and

Professor M.S. Rangachari

Preface

The main aim of this book is to present the concepts and techniques of complex function theory in a way that will give the reader maximum assistance in mastering the fundamentals of the theory. The book is designed to serve as a text for a first course on the classical theory complex functions or as a supplement to other standard texts. The selection and the sequencing of the contents are the result of the experiences I had during the course of my studies and teaching. As a prerequisite the reader is expected to have adequate knowledge of the elements real analysis. However, in an effort to make the book accessible to a wider audience, I have tried my best to minimize the prerequisites and to keep the exposition at an elementary level. Thus I have included a number of examples motivating the ideas involved in most of the theorems and definitions. Most of the exercises have been provided with hints for their solutions. Some theorems have been given more than one proof to help the reader acquire a deeper understanding of the theory. The present edition has a large number of illuminating new examples, observations, exercises, and some additional materials covering some advanced topics.

Within each chapter, all the numbered items (except figures) eg. Corollary, equation, Example, Lemma, Proposition, Remark, Theorem are numbered consecutively as they appear. For the sake of convenience, the sign ■ signals the end of the proofs of Theorem, Corollary, Lemma and Proposition whereas the sign ● indicates the end of Remark, Observation and Example.

It is a great pleasure for me to express my gratitude to Prof. M.S. Rangachari who has not only encouraged me to write the first edition of the book but also has read several drafts with care and patience. During my doctoral work at the Indian Institute of Technology Kanpur, I had opportunities to be a tutor to graduate and undergraduate students on this subject. In this context, I would like to thank my teacher Prof. O.P. Juneja who introduced me to the theory and has been a source of inspiration since then.

Many people have given me help, encouragement, and inspiration. I should take this opportunity to mention just a few. I express my deep gratitude to Professors R. Balasubramanian, O. Martio, St. Ruscheweyh, C.S. Seshadri, V. Singh, M. Vuorinen, and G.P. Youvaraj whose continued encouragement has been invaluable for widening my research interest. Particularly, I am deeply indebted to Prof. M. Vuorinen for his continued support in bringing out this edition. In addition, a number of my research visits to work with him has helped me to learn a lot.

It is my pleasant duty to thank Prof. G.P. Youvaraj who has been my best friend and to this work. He has proofread the final manuscript and his constructive criticisms has improved the presentation lot. I take this opportunity to thank Prof. Roger W. Barnard for his support and friendship during my visit to Texas Tech University. I greatly appreciate his careful reading of the final version of the manuscript and making suggestions. I welcome and appreciate any suggestions for improvements and comments regarding errors and misprints.

I must also record my appreciation due to my daughter Abirami and son Ashwin for their understanding and love during the long period that I have taken to complete this second edition of the book. Above all, my deepest gratitude goes to my wife Booma (alias Geetha) for her infinite patience, continued support, and loving encouragements in all walks of my life.

Finally, I wish to express my thanks to the Center for Continuing Education at the Indian Institute of Technology, Madras, India, for its support in the preparation of the manuscript. Also, I thank Mr. N.K. Mehra, publisher and Managing Director of Narosa Publishing House Pvt. Ltd. who has been superb.

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Contents

Chapter 1

Complex Numbers

In this chapter we review basic results such as fundamental algebraic and topological properties of complex numbers. We assume that the reader is acquainted with the familiar properties of the real number system. In Section 1.1, we introduce complex numbers. Section 1.3 provides a way of obtaining solutions of a quadratic equation in complex variable. Section 1.2 discusses polar representation of complex numbers whereas Section 1.4 gives an easy method of finding solutions of a rational power of complex numbers. In Section 1.5, we introduce topological properties of the complex plane. In Section 1.6, we present several fundamental theorems concerning convergence of sequences and series of complex numbers.

1.1 Definition of Complex Numbers

Consider ordered pairs of real numbers (x, y). The word '*ordered*' means that (x, y), (y, x) are distinct unless $x = y$. We denote the set of all ordered pairs of real numbers by $\mathbb{C}$. We shall call $\mathbb{C}$ as the set of all complex numbers. In $\mathbb{C}$, we define addition (+) and multiplication ($\times$ or juxtaposition) between two such ordered pairs (x_1, y_1), (x_2, y_2) by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

and

$$(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2). \tag{1.2}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then we say that

$$z_1 = z_2 \Longleftrightarrow x_1 = x_2 \text{ and } y_1 = y_2.$$

In particular, $z = (x, y) = (0, 0) \Longleftrightarrow x = 0$ and $y = 0$.

We can easily check the following simple properties for equality of ordered pairs making it an equivalence relation: For any z_1, z_2 and z_3 in $\mathbb{C}$,

(i) $z_1 = z_1$

(ii) $z_1 = z_2 \Longrightarrow z_2 = z_1$

(iii) $z_1 = z_2$ and $z_2 = z_3 \Longrightarrow z_1 = z_3$.

The associative and commutative laws for addition and the multiplication and distributive laws etc., follow easily from the properties of the field of real numbers $\mathbb{R}$. Further, it is clear from (1.1) and (1.2) that $(0,0)$ is the additive identity, $(1,0)$ is the multiplicative identity, $(-x,-y)$ is the additive inverse of $z=(x,y)$ and

$$\frac{1}{z} := \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$$

is the multiplicative inverse of $z=(x,y) \neq (0,0)$.[1] Given two complex numbers z_1 and z_2,

- there is a unique complex number, say z_3, such that $z_1 + z_3 = z_2$. If $z_j = (x_j, y_j)$ $(j = 1,2,3)$, then $z_3 = (x_2 - x_1, y_2 - y_1)$ and is denoted by $z_2 - z_1$. `[Subtraction]`
- for $z_2 \neq (0,0)$, there is a unique z_3 such that $z_1 = z_2 z_3$. In fact, $z_3 = z_1.(1/z_2)$ since $z_2 z_3 = z_2.z_1.(1/z_2) = (z_2.(1/z_2)).z_1 = 1.z_1 = z_1$. The complex number z_3 is otherwise written as $z_3 = z_1/z_2$. `[Division]`

The symbol commonly used for a complex number is not (x,y) but $x+iy$, x,y real. Following Euler, we define $i := (0,1)$ in the complex number system $\mathbb{C}$ of ordered pairs. We write a real number x as $(x,0)$. Then according to (1.2)

$$i^2 = (0,1)(0,1) = (-1,0), \quad i^3 = i^2 \cdot i = (-1,0)(0,1) = (0,-1),$$

and $i^4 = i^2 \cdot i^2 = (1,0)$. Also, $x + iy = (x,0) + (0,1)(y,0) = (x,y)$. The above discussion shows that $\mathbb{C}$ is also a field. Further, writing a real number x as $(x,0)$ and noting that

$$(x_1,0) + (x_2,0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1,0)(x_2,0) = (x_1 x_2, 0),$$

$\mathbb{R}$ turns out to be a subfield of $\mathbb{C}$. The association $x \mapsto (x,0)$ shows that we can always treat $\mathbb{R}$ as a subset of $\mathbb{C}$. Complex numbers of the form $(x,0)$ are said to be *purely real* or just real. Those of the form $(0,y)$ are said to be *purely imaginary* whenever $y \neq 0$. In particular, we have, with the above identification of $\mathbb{R}$, $i^2 = -1$. Every (complex number) $z = (x,y) \in \mathbb{C}$, denoted now by $x+iy$, admits a unique representation[2]

$$(x,y) = (x,0) + (0,1)(y,0) = x + iy, \quad \text{with} \quad x,y \in \mathbb{R}.$$

[1] We use ':=' to abbreviate "defined by" or "written as".

[2] From now on if we write $z = x + iy \in \mathbb{C}$, x,y are real numbers unless otherwise stated.

'Zero' viz. $(0,0) = 0 + i0$ is the only complex number both real and purely imaginary. The *conjugate* of a complex number $z = x + iy$ is the complex number $\overline{z} := x - iy$. Note that $z = \overline{z}$ if and only if $x + iy = x - iy$, i.e. $y = 0$, i.e. z is purely real. The inverse or reciprocal z^{-1} of a complex number $z = x + iy \neq 0$ is

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{x^2 + y^2} = \left(\frac{x}{x^2 + y^2}\right) - i\left(\frac{y}{x^2 + y^2}\right),$$

which was defined earlier as the multiplicative inverse of z. We call x and y the *real part* and *imaginary part* of $z = x + iy$, respectively. We write

$$\operatorname{Re} z := x \text{ and } \operatorname{Im} z := y; \ \operatorname{Re} z = \frac{z + \overline{z}}{2} \text{ and } \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

We know that ordered pairs of real numbers represent points in the geometric plane referred to a pair of rectangular axes. We then call the collection of ordered pairs as $\mathbb{R}^2$ and the two axes as the x-axis and y-axis. Because $(x, 0) \in \mathbb{R}^2$ corresponds to real numbers, the x-axis is called the *real axis* and since $iy = (0, y) \in \mathbb{R}^2$ is purely imaginary, the y-axis is called the *imaginary axis*.

Now, we can visualize $\mathbb{C}$ as a plane with $x + iy$ as points in $\mathbb{R}^2$ and we simply refer to it as the *finite complex plane* or simply *complex plane*. Depending on the problems on hand, we use $x + iy$ or (x, y), to represent a complex number.

1.3. Theorem. *The field $\mathbb{C}$ cannot be totally ordered in consistence with the usual order on $\mathbb{R}$. (Total ordering means that if $a \neq b$, then either $a < b$ or $a > b$).*

Proof. Suppose that such a total ordering exists on $\mathbb{C}$. Then for $i \in \mathbb{C}$ we would have either $i > 0$ or $i < 0$ since $i \neq 0$. This means that in either case

$$-1 = i \cdot i = (-i)(-i) > 0$$

which is not true in $\mathbb{R}$. This observation shows that such an ordering is impossible in $\mathbb{C}$. ∎

Theorem 1.3 means that the expressions $z_1 > z_2$ or $z_1 < z_2$ have no meaning unless z_1 and z_2 are real.

1.4. Concepts of modulus/absolute value. The *modulus* or *absolute value* of $x \in \mathbb{R}$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

As it stands, there is no natural generalization of $|\cdot|$ to $\mathbb{C}$, because, as we have seen in Theorem 1.3, there is no total ordering on $\mathbb{C}$. However, we interpret $|x|$ geometrically as the distance from x to the origin (zero) of the real line. It is this fact which leads us to define the *modulus* of a complex number $z = x + iy \in \mathbb{C}$ by $|z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.It is easy to derive the following simple facts:

(i) $|z| = 0$ iff[3] $z = 0$.

(ii) $|\mathrm{Re}\, z| \leq |z|$, $|\mathrm{Im}\, z| \leq |z|$, $|z| = |\overline{z}|$, $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$, $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$. Equality holds in the first two inequalities iff $\mathrm{Im}\, z = 0$ and $\mathrm{Re}\, z = 0$, respectively.

(iii) $|z_1 z_2| = |z_1|\ |z_2|$, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$.

(iv) $\mathrm{Re}\,(z_1 + z_2) = \mathrm{Re}\, z_1 + \mathrm{Re}\, z_2$ and $\mathrm{Im}\,(z_1 + z_2) = \mathrm{Im}\, z_1 + \mathrm{Im}\, z_2$.

1.5. Example. It is easy to see that if $z_1 - \overline{z}_2$ and $z_1 z_2$ are both real, then either z_1 and z_2 are both real or one is the conjugate of the other.

Indeed, there is nothing to prove if one of z_1 and z_2 is zero. Therefore, we may assume that both z_1 and z_2 are non-zero. By hypothesis

$$z_1 z_2(z_1 - \overline{z}_2) = z_1(z_1 z_2 - |z_2|^2)$$

which is real. Therefore, we must have either z_1 is real or $z_1 z_2 - |z_2|^2 = 0$ which is equivalent to saying that either z_1 is real or $z_1 = \overline{z}_2$. This proves the required conclusion. ●

1.6. Example. We wish to show that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$,

(ii) $|\,|z_1| - |z_2|\,| \leq |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

It is also easy to see that (i) and (ii) are equivalent.

To prove these, we first observe that if $z_1 + z_2 = 0$, then there is nothing to prove. If $z_1 + z_2 \neq 0$, then $|z_1 + z_2| \neq 0$. Since $\mathrm{Re}\, z \leq |z|$, we have

$$\left|\frac{z_1}{z_1 + z_2}\right| + \left|\frac{z_2}{z_1 + z_2}\right| \geq \mathrm{Re}\left(\frac{z_1}{z_1 + z_2}\right) + \mathrm{Re}\left(\frac{z_2}{z_1 + z_2}\right) = 1 \tag{1.7}$$

from which (i) follows. To prove the second inequality, we write $z_1 = z_2 + (z_1 - z_2)$ so that, by (i),

$$|z_1| \leq |z_2| + |z_1 - z_2|, \quad \text{i.e.} \quad |z_1| - |z_2| \leq |z_1 - z_2|. \tag{1.8}$$

Similarly, we obtain

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|. \tag{1.9}$$

[3]We use Paul Halmos's notation 'iff' to abbreviate the words 'if and only if'.

Combining (1.8) and (1.9) we obtain (ii). Actually, (ii) implies (i) can be seen similarly. Thus, (i) and (ii) are equivalent.

Finally, we discuss the equalities in (i) and (ii). If one of z_1, z_2 is zero, then equality in (i) holds obviously. If $z_1 + z_2 = 0$, then equality in (i) is possible only when $z_1 = z_2 = 0$. So we assume that $z_1 \neq 0$, $z_2 \neq 0$, $z_1 + z_2 \neq 0$ and note that $|z| = |\text{Re}\, z| \iff \text{Im}\, z = 0$. From (1.7), we see that the equality in (i) holds iff for each $k = 1, 2$,

$$\text{Im}\left(\frac{z_k}{z_1+z_2}\right) = 0 \quad \text{and} \quad \text{Re}\left(\frac{z_k}{z_1+z_2}\right) \geq 0.$$

That is, iff

$$\frac{|z_1\overline{z}_2|}{|z_1+z_2|^2} = \frac{\text{Re}\,(z_1\overline{z}_2)}{|z_1+z_2|^2}, \quad \text{i.e.} \quad |z_1\overline{z}_2| = \text{Re}\,(z_1\overline{z}_2).$$

Equivalently, this holds when $z_1\overline{z}_2$ is a non-negative real number. This is, in fact, equivalent to the relation $z_1 = tz_2$ with $t \geq 0$, since $z_1\overline{z}_2 = z_1|z_2|^2/z_2$. One can easily check that the same condition holds for the equality in (ii). This completes the problem. ●

Here is another proof of the two inequalities of Example 1.6:

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(\overline{z}_1+\overline{z}_2) \\ &= |z_1|^2 + 2\text{Re}\,(z_1\overline{z}_2) + |z_2|^2 \\ &\leq |z_1|^2 + 2|z_1\overline{z}_2| + |z_2|^2, \text{ since } \text{Re}\, z \leq |\text{Re}\, z| \leq |z|, \\ &= |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 \\ &\leq (|z_1| + |z_2|)^2; \end{aligned}$$

that is, $|z_1 + z_2| \leq |z_1| + |z_2|$ which is known as the *triangle inequality* and it is clear that the equality holds iff $|z_1\overline{z}_2| = \text{Re}\,(z_1\overline{z}_2)$; i.e. $z_1 = tz_2$, where $t \geq 0$. Moreover, by finite induction, it is easy to see that

$$\left|\sum_{j=1}^n z_j\right| \leq \sum_{j=1}^n |z_j|.$$

The equality holds when z_j $(j = 1, \dots, n)$ lie on the same ray emanating from the origin.

1.10. Remark. If $z_1 = tz_2$ where $t \geq 0$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, then it follows that $x_1 = tx_2$ and $y_1 = ty_2$ so that $(0,0)$, (x_1, y_1) and (x_2, y_2) lie on a ray from $(0,0)$. In other words, the points in $\mathbb{R}^2$ corresponding to z_1 and z_2 are on the ray from the origin. Note that this means that z_1, z_2 are linearly dependent. ●

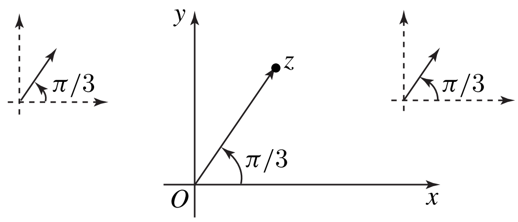

Figure 1.1: Geometric representation of $\sqrt{3}/2 + i/\sqrt{2}$.

1.11. Remark. For $z = x + iy$, we have $|z|^2 \geq 2|x|\,|y|$ so that

$$2|z|^2 = |z|^2 + |z|^2 \geq x^2 + y^2 + 2|x|\,|y| = (|x| + |y|)^2.$$

Hence, for any $z = x + iy \in \mathbb{C}$, we have $|z| \leq |x| + |y| \leq \sqrt{2}|z|$ and it is clear that the constant $\sqrt{2}$ cannot be replaced by a smaller number unless $x = 0$ or $y = 0$. The latter inequality becomes equality if $x = \pm y$. ●

1.2 Geometric Interpretation

As we have pointed out in Section 1.1, for every point of the z-plane there is one and only one complex number z and conversely. A complex number $z = x + iy$ may be thought of as a vector or as a directed line segment, in the complex plane and pictured as an arrow from the origin to the point (x, y) in $\mathbb{C}$; or as any vector obtained by translating this vector parallel to itself, i.e. one whose starting point is irrelevant. For instance the point

$$z = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

determines the vector from the origin to this point and so the directed line segment from $(a - \sqrt{3}/2, b - 1/2)$ to (a, b) also represents the same complex number $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ for any $(a, b) \in \mathbb{R}^2$. Both vectors have the same length and point in the same direction. Thus we have an infinite number of line segments that we can draw in order to represent a complex number, because all such segments have the same length and point in the same direction (see Figures 1.1 and 1.2). Here $|z|$ is the length of the vector z. Some of the simple geometric relations between z, $\overline{z}$ and $-z$ are outlined in Figure 1.3. Geometrically, addition of two complex numbers corresponds to the vector addition of vectors representing them. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers such that $0, z_1$ and z_2 are not collinear. Then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + x_2, y_1 + y_2).$$

We plot these vectors in Figure 1.4. As indicated in Figure 1.4, the vector $z_1 + z_2$ is the diagonal of the parallelogram with z_1 and z_2 as adjacent sides.

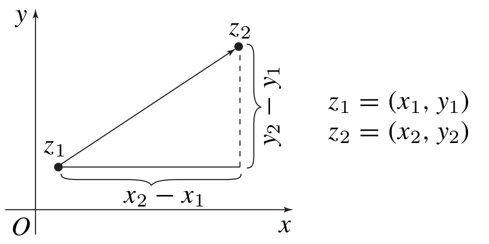

Figure 1.2: $z = z_2 - z_1 = (x_2 - x_1) + i(y_2 - y_1)$.

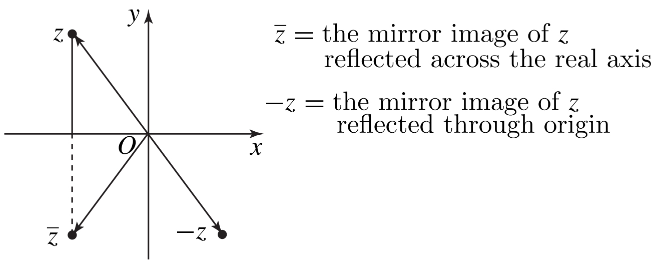

Figure 1.3: Relations between z, $\overline{z}$ and $-z$.

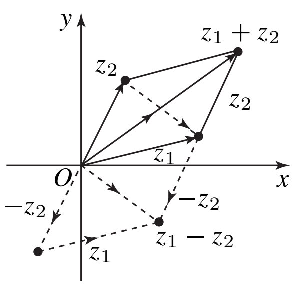

Figure 1.4: Law of parallelogram.

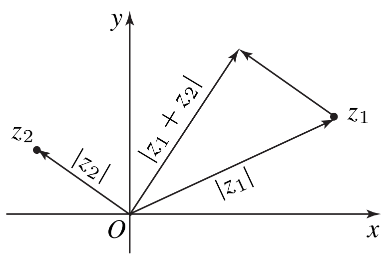

Figure 1.5: Triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.

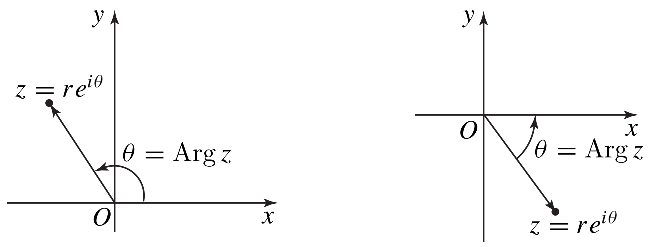

Figure 1.6: Argument of $z \neq 0$.

Now the triangle inequalities (see Example 1.6 and Figure 1.5) are obvious.

If r is the length of the vector represented by z, i.e. the distance from the origin to z, and θ is the angle measured from the positive x-axis to the radius vector joining z in the anti-clockwise sense and $z = x + iy$, then the coordinate systems, viz. the cartesian and polar systems, are related by (see Figure 1.6)

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{1.12}$$

and hence

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = y/x. \tag{1.13}$$

Thus, we define $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Any value of θ for which (1.12) (or (1.13)) holds is called *an argument* of $z \neq 0$ written as $\theta = \arg z$. Clearly z has an infinite number of distinct arguments. Any two distinct arguments of z differ by an integral multiple of 2π. (Since $z = 0 \iff |z| = 0$, $\arg 0$ is indeterminate). In order to specify a unique value of $\arg z$, we may restrict its value to some interval of length 2π. To make this statement precise, we introduce the concept of *"principal value"* of $\arg z$ as follows:

For an arbitrary $z \neq 0$, the principal value of $\arg z$ is defined to be the unique value that satisfies $-\pi < \arg z \leq \pi$ and it will be denoted by $\operatorname{Arg} z$. Thus, the relation between $\arg z$ and $\operatorname{Arg} z$ is given by

$$\arg z = \operatorname{Arg} z + 2k\pi, \quad k \in \mathbb{Z}.$$

For convenience the set of all the values of $\arg z$ is denoted by $* \arg z$. For example,

$$\operatorname{Arg} i = \frac{\pi}{2}, \ \operatorname{Arg}(1 - i) = -\frac{\pi}{4}, \ \operatorname{Arg}(-1) = \pi, \ \operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}.$$

While inverting the second equation in (1.13) we should note the following

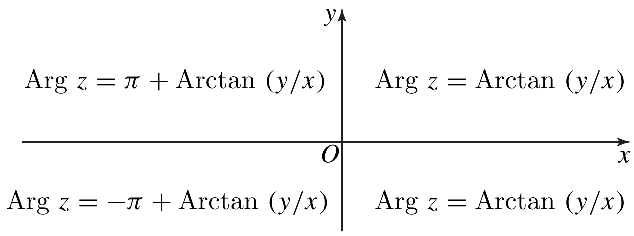

Figure 1.7: Description of argument of a complex number $z = x + iy$.

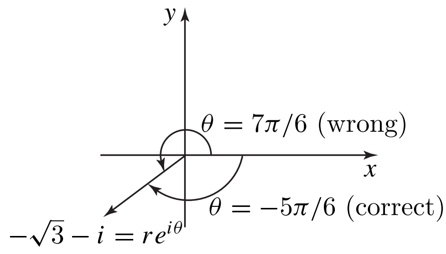

Figure 1.8: To specify the correct value when $\theta = \operatorname{Arg} z = -5\pi/6$.

(see Figure 1.7):

$$\operatorname{Arg} z = \begin{cases} \operatorname{Arctan}(y/x) & \text{if } x > 0 \\ \pi + \operatorname{Arctan}(y/x) & \text{if } x < 0,\, y \geq 0 \\ -\pi + \operatorname{Arctan}(y/x) & \text{if } x < 0,\, y < 0 \\ \pi/2 & \text{if } x = 0,\, y > 0 \\ -\pi/2 & \text{if } x = 0,\, y < 0 \end{cases}$$

where $\operatorname{Arctan} t$ is the principal value of the arctangent of a real number t satisfying the inequality $-\pi/2 < \arctan t \leq \pi/2$. The following are then true:

$$\sin(\arg z) = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos(\arg z) = \frac{x}{\sqrt{x^2 + y^2}}, \quad \text{and} \quad \tan(\arg z) = \frac{y}{x}.$$

For instance, we have the following:

(a) $\arg z = -\arg \overline{z}$ unless z is a negative real number, because otherwise $\arg z = \arg \overline{z} = \pi$. In fact, we have $* \arg z = -\arg \overline{z} + 2k\pi, \quad k \in \mathbb{Z}$.

(b) If $z = -\sqrt{3} - i$, then we have $\operatorname{Arg} z = -5\pi/6$ and so (see Figure 1.8)

$$* \arg z = -5\pi/6 + 2k\pi, \quad k \in \mathbb{Z}.$$

(c) The complex number $z = -1 - i$ may be written as $z = \sqrt{2}e^{i\theta}$, where

$$\theta = \operatorname{Arg} z + 2k\pi = -3\pi/4 + 2k\pi, \quad k \in \mathbb{Z}.$$

We get non-principal values if and only if $k \in \{\pm 1, \pm 2, \dots\}$.

(d) Similarly we easily derive $-\sqrt{10} + i\sqrt{30} = 2\sqrt{10}e^{i\theta}$ with $\theta = 2\pi/3 + 2k\pi$, $k \in \mathbb{Z}$.

Next we see that the polar representation of a complex number simplifies the task of describing the product of two complex numbers geometrically. To do this we consider two complex numbers z_1 and z_2 with polar representations: $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Now,

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}, \end{aligned}$$

which gives

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{1.14}$$

Note that $|e^{i\theta}| = 1$ and $e^{i\theta} = e^{i(\theta+2k\pi)}$, $k \in \mathbb{Z}$. Thus, we have

$$|z_1 z_2| = |z_1|\,|z_2| \tag{1.15}$$

and

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi} \tag{1.16}$$

in the sense that they are the same but for an integral multiple of 2π. For $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Thus, for $z_2 \neq 0$, we have

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \pmod{2\pi}. \tag{1.17}$$

If we combine (1.17) with the operation of subtraction, clearly the angle ϕ defined by

$$\phi = \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)$$

represents the angle at the vertex z_1 (see Figure 1.9). For instance, let us choose $z_1 = -1$ and $z_2 = -i$. Then $z_1 z_2 = i$ and hence $\mathrm{Arg}\,(z_1 z_2) = \pi/2$. Further $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \pi - \pi/2 = \pi/2$. Therefore, in this case, we obtain $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \mathrm{Arg}\,(z_1 z_2)$.

As another example, suppose $z_1 = -1$ and $z_2 = i$. Then we see that

$$\mathrm{Arg}\,(z_1 z_2) = -\pi/2 \quad\text{and}\quad \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = 3\pi/2.$$

Therefore, in this case, we obtain the correct answer by adding -2π to bring within the interval $(-\pi, \pi]$. These two examples show that when the

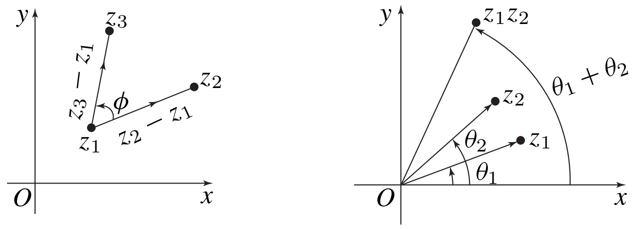

Figure 1.9: Angle at the vertex z_1.

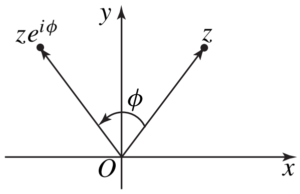

Figure 1.10: Rotation of z through an angle ϕ.

principal arguments are added together in a multiplication problem, the resulting argument need not be a principal value. Conversely, we also see that when the non-principal arguments are combined, a principal argument may result. From this observation, we also see that if we are given two complex numbers z_1 and z_2, then z_1 and z_2 are equal iff $|z_1| = |z_2|$ and $\arg z_1 = \arg z_2 \pmod{2\pi}$.

(i) Suppose that $z_1 = e^{i\phi}$, where ϕ is real and $z_2 = z$, any complex number. Then $z_1 z_2 = e^{i\phi} z$ is obtained by rotating z through an angle ϕ (see Figure 1.10).

(ii) The set of all points given by the equation $\dfrac{\pi}{6} < \arg(z - 2 - 3i) \leq \pi$ is geometrically represented in Figure 1.11.

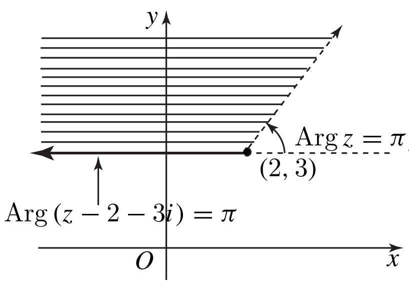

Figure 1.11: The set of all points such that $\frac{\pi}{6} < \operatorname{Arg}(z - 2 - 3i) \leq \pi$.

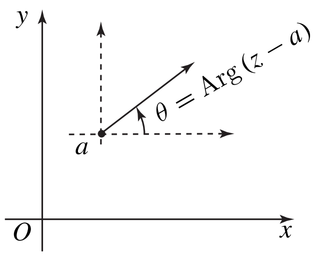

Figure 1.12: The straight line $\mathrm{Arg}\,(z-a)=\theta$, where $a=m+in,\ m,\, n>0$.

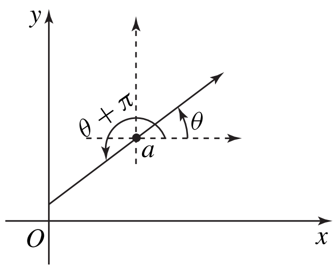

Figure 1.13: The straight line $\arg(z-a)=\theta+\pi$.

(iii) The set of points described by $S=\{z:\ |\arg z-\pi/2|<\pi/2\}$ represents the upper half-plane, viz $\{z:\ \mathrm{Im}\, z>0\}$. Two more examples of this kind are given in Figures 1.12 and 1.13.

(iv) Suppose z_1 and z_2 are non-zero complex numbers. Then

$$\begin{aligned} z_1\overline{z}_2+\overline{z}_1 z_2=0 &\iff \frac{z_1}{\overline{z}_1}=-\left(\frac{z_2}{\overline{z}_2}\right), \text{ i.e. } \frac{z_1^2}{|z_1|^2}=-\frac{z_2^2}{|z_2|^2} \\ &\iff 2\arg z_1=\pi+2\arg z_2+2k\pi \\ &\iff \arg z_1=\arg z_2+(2k+1)\pi/2, \quad k\in\mathbb{Z}. \end{aligned}$$

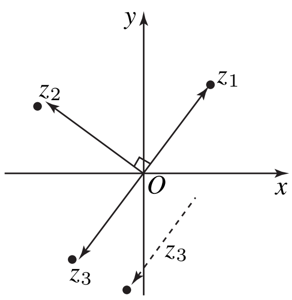

Figure 1.14: Description for perpendicular and parallel vectors.

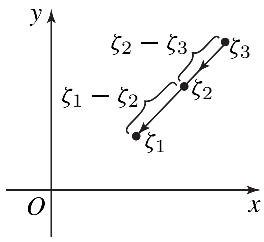

Figure 1.15: Description for parallel vectors on the same straight line.

This means that the vectors z_1 and z_2 are perpendicular (see Figure 1.14) iff $z_1\overline{z}_2 + \overline{z}_1 z_2 = 0$, i.e. $\mathrm{Re}\,(z_1\overline{z}_2) = 0$ which is same as writing $z_1 = isz_2$, for some real number s. Note that perpendicularity is equivalent to the Pythagorean equation $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$.

(v) The line through points ζ_1 and ζ_2 is perpendicular to the line through points ζ_3 and ζ_4 iff (the vectors) $\zeta_1 - \zeta_2$ and $\zeta_3 - \zeta_4$ are perpendicular. Thus, we conclude that the two lines are perpendicular if and only if there exists a real number s such that

$$\frac{\zeta_1 - \zeta_2}{\zeta_3 - \zeta_4} = is, \quad \text{i.e.} \quad \arg\left(\frac{\zeta_1 - \zeta_2}{\zeta_3 - \zeta_4}\right) = \frac{\pi}{2} \pm 2k\pi \quad \text{or} \quad -\frac{\pi}{2} \pm 2k\pi$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(vi) Similarly, for nonzero $z_1, z_2 \in \mathbb{C}$ it follows that

$$\begin{aligned} z_1\overline{z}_2 - \overline{z}_1 z_2 = 0 &\iff 2\mathrm{Im}\,(z_1\overline{z}_2) = 0 \\ &\iff \mathrm{Im}\left(\frac{z_1}{z_2}\right) = 0, \quad \text{i.e.} \quad \frac{z_1}{z_2} \text{ is real} \\ &\iff \arg z_1 = \arg z_2 + k\pi, \quad k \in \mathbb{Z}. \end{aligned}$$

This means vector z_1 is parallel to vector z_2 iff $z_1 = tz_2$, where t is real.

(vii) Thus, three complex numbers ζ_j $(j = 1, 2, 3)$ lie on the same straight line iff $\zeta_2 - \zeta_3 = \lambda_1(\zeta_3 - \zeta_1)$ for some real λ_1 (see Figure 1.15). Similarly we also have

$$\zeta_3 - \zeta_1 = \lambda_2(\zeta_1 - \zeta_2) \quad \text{and} \quad \zeta_1 - \zeta_2 = \lambda_3(\zeta_2 - \zeta_3)$$

for the real numbers λ_2 and λ_3.

The points $\zeta_1, \zeta_2, \zeta_3$ are collinear iff

$$\lambda_1(\zeta_3 - \zeta_1) + \lambda_2(\zeta_1 - \zeta_2) + \lambda_3(\zeta_2 - \zeta_3) = 0$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Equivalently,

$$(\lambda_2 - \lambda_1)\zeta_1 + (\lambda_3 - \lambda_2)\zeta_2 + (\lambda_1 - \lambda_3)\zeta_3 = 0.$$

Writing $\lambda_2 - \lambda_1 = \mu_1$, $\lambda_3 - \lambda_2 = \mu_2$ and $\lambda_1 - \lambda_3 = \mu_3$, we can state the condition as $\sum_{j=1}^{3} \mu_j \zeta_j = 0$ and $\sum_{j=1}^{3} \mu_j = 0$ for some real μ_j $(j = 1, 2, 3)$ not all zero.

1.3 Square roots

Since the square of a real number is nonnegative, $x^2 = a$ has a real solution only if $a \geq 0$.

Now we consider the question of square roots in $\mathbb{C}$, for instance $w = i, -i$ are the solutions of $w^2 = -1$. Moreover, our next theorem shows that if $z \in \mathbb{C}$, then there is a solution $w \in \mathbb{C}$ for $w^2 = z$. Theorem 1.18 below gives a purely algebraic proof of this fact. Later we obtain this result, using polar coordinates, as a special case of a more general result.

1.18. Theorem. *For a given $z = x + iy \in \mathbb{C}$, the solutions of $w^2 = z$ are given by*

$$w = \pm \left[\sqrt{\frac{|z| + x}{2}} + i \,\mathrm{sgn}\,(y) \sqrt{\frac{|z| - x}{2}} \right], \quad \mathrm{sgn}\,(y) = \begin{cases} +1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0. \end{cases}$$

Proof. We have to solve the equation $(u + iv)^2 = x + iy$ for u and v. This equation gives

(1.19) $$u^2 - v^2 = x \quad \text{and} \quad 2uv = y$$

and therefore, $u^2 + v^2 = |z|$. Using this and the first equation in (1.19), we have

$$u^2 = \frac{|z| + x}{2} \quad \text{and} \quad v^2 = \frac{|z| - x}{2}.$$

(Note that $|z| \pm x \geq 0$). From the second equation in (1.19) we observe that

$$y > 0 \Longleftrightarrow uv > 0; \quad y < 0 \Longleftrightarrow uv < 0.$$

Therefore, selecting u and v so that their product has the same sign as that of y, we obtain the required conclusion. ∎

1.20. Corollary. *For every complex number z with $|z| = 1$ and $\mathrm{Re}\, z \geq 0$, there exists a complex number w with $|w| = 1$ such that $w^2 = z$ and $|\mathrm{Im}\, w| \leq (1/\sqrt{2})|\mathrm{Im}\, z|$.*

Proof. Let $z = x + iy$ with $x \geq 0$ and $|z| = 1$. Then $x^2 + y^2 = 1$ and $x \geq x^2$. Using Theorem 1.18, we have (since $|z|^2 = 1 = |z|$)

$$|\mathrm{Im}\, w|^2 = \left| \sqrt{\frac{|z|^2 - x}{2}} \right|^2 = \frac{x^2 + y^2 - x}{2} \leq \frac{x + y^2 - x}{2} = \frac{|\mathrm{Im}\, z|^2}{2}$$

which completes the proof. ∎

1.21. Remark. For a given complex number z, the following facts are easy to obtain:

(i) There are two values of w such that $w^2 = z$ and these two values are called the *square roots* of z.

(ii) Each of the two values of w is real iff z is real and positive.

(iii) Each of the two values of w is purely imaginary iff z is real and negative.

(iv) The two values of w coincide with zero iff $z = 0$.

(v) For $\alpha, \beta \in \mathbb{C}$, the equation

$$z^2 + \alpha z + \beta = 0 \tag{1.22}$$

has solutions

$$z = \frac{-\alpha + w}{2}, \tag{1.23}$$

where w is such that $w^2 = \alpha^2 - 4\beta$ as obtained in Theorem 1.18. Here the procedure adopted to obtain the solution is the same as for equations with real coefficients, viz. that of completing the square.

(vi) Suppose that the equation (1.22) has real roots, say $z = x$. Then we have $x^2 + \alpha x + \beta = 0$, and $x^2 + \overline{\alpha} x + \overline{\beta} = 0$. Thus, eliminating x, the equation (1.22) has real roots if

$$(\overline{\beta} - \beta)^2 = (\overline{\alpha} - \alpha)(\alpha\overline{\beta} - \overline{\alpha}\beta); \quad \text{i.e.} \quad (\mathrm{Im}\,\beta)^2 + (\mathrm{Im}\,\alpha)(\mathrm{Im}\,(\alpha\overline{\beta})) = 0.$$

Similarly we see that the condition under which (1.22) has purely imaginary roots is that $(\mathrm{Re}\,\alpha)(\mathrm{Re}\,(\alpha\overline{\beta})) = (\mathrm{Im}\,\beta)^2$.

(vii) On the other hand suppose that the equation (1.22) has a complex root, say $z = z_1$ such that $|z_1| = 1$. Then we have

$$z_1^2 + \alpha z_1 + \beta = 0. \tag{1.24}$$

Since $|z_1| = 1$, $z_1\overline{z}_1 = 1$ so that $\overline{z}_1 = 1/z_1$. So (1.24) yields

$$z_1 + \alpha + \beta\overline{z}_1 = 0 \quad \text{and} \quad \overline{z}_1 + \overline{\alpha} + \overline{\beta} z_1 = 0.$$

Now, eliminating $\overline{z}_1$ from the above two equations we get

$$(1 - |\beta|^2)z_1 = -(\alpha - \beta\overline{\alpha}).$$

Thus, (1.22) has roots on $|z| = 1$ only if $|\alpha - \beta\overline{\alpha}| = |1 - |\beta|^2|$. ●

1.25. Remark. Let $*\sqrt{a + ib}$ denote the two square roots of $a + ib$ given by Theorem 1.18. More generally we use the notation $*\sqrt[n]{a + ib}$ to denote the n n-th roots of $a + ib$. It is easy to see that

$$*\sqrt{3 + 4i} = \pm(2 + i), \; *\sqrt{-3 + 4i} = \pm(1 + 2i), \; *\sqrt{2i} = \pm(1 + i). \quad ●$$

In Section 1.4, De Moivre's Theorem will be used to solve the equation $w^n = z$ for w when z is given. In fact, there are exactly n n-th roots of any non-zero complex number z.

1.4 Rational Powers of a Complex Number

Repeated application of (1.14) yields

$$(re^{i\theta})^m = r^m e^{im\theta} \tag{1.26}$$

or equivalently, $|z^m| = |z|^m$ and $\arg(z^m) = m \arg z \pmod{2\pi}$, where m is a natural number. If $z = re^{i\theta} \neq 0$, repeated application of (1.17) shows that for a natural number m, $z^{-m} = (re^{i\theta})^{-m} = r^{-m}e^{-im\theta}$. Of course, we define $z^0 = 1$ for $z \neq 0$. In particular, if we let $|z| = 1$, i.e. $z = e^{i\theta}$, then we obtain the following:

1.27. Theorem. (De Moivre's Theorem) *If m is an integer, then $(e^{i\theta})^m = e^{im\theta}$, i.e. $(\cos\theta + i\sin\theta)^m = \cos m\theta + i \sin m\theta$.*

Further, the identity (1.26) is especially useful for finding n-th roots of a complex number $z_0 \neq 0$, when n is a natural number. For, if we have $z^n = z_0$ with $z = re^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$, then

$$r^n e^{in\theta} = r_0 e^{i\theta_0} \implies r = \sqrt[n]{r_0} \text{ and } n\theta = \theta_0 + 2k\pi,$$

where $r = \sqrt[n]{r_0}$ is the unique positive n-th root of r_0 (> 0). Hence, all the roots of $z = z_0^{1/n}$ are given by

$$\sqrt[n]{|z_0|}e^{i(\theta_0+2k\pi)/n} \tag{1.28}$$

where k is any integer. Each value of $k = 0, 1, \dots, n-1$ gives a different value of z. Any other value of k merely repeats one of the values of z corresponding to $k = 0, 1, \dots, n-1$, since $e^{2\pi ik} = 1$. Thus, there are exactly n n-th roots of $z_0 \neq 0$. Also (1.28) shows that the n n-th roots of z_0 actually lie on a circle centered at the origin and having radius equal to $\sqrt[n]{|z_0|}$. Each of the roots obtained from (1.28) has the same modulus and the arguments are equally spaced. Geometrically the n n-th roots of $z_0 \neq 0$ are located at the vertices of a regular n-sided polygon inscribed in the circle of radius $\sqrt[n]{|z_0|}$. Thus, we have proved

1.29. Theorem. *Given a nonzero complex number $z = re^{i\theta}$, the equation $w^n = z$ has precisely n distinct solutions given by*

$$w_k = \sqrt[n]{r}e^{i(\theta+2k\pi)/n},$$

where $k = 0, 1, \dots, n-1$, and $\sqrt[n]{r}$ denotes the positive n-th root of $r = |z|$ and $\theta = \operatorname{Arg} z$.

For instance, the n-th roots of unity are given by

$$\omega_k = e^{i2k\pi/n}, \ k = 0, 1, \dots, n-1. \tag{1.30}$$

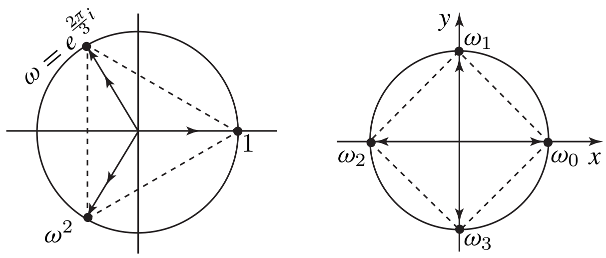

Figure 1.16: The n-th roots of 1 when $n = 3, 4$.

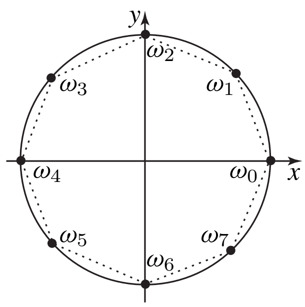

Figure 1.17: The n-th roots of 1 when $n = 8$.

The n-th roots of 1 when n equals 4 and 8 are pictured in Figures 1.16 and 1.17.

Similarly, the 6-th roots of i are $e^{i(4k+1)\pi/12}$, $k = 0, 1, \dots, 5$. Further, we easily derive that the values of $[(1-i)/(\sqrt{3}+i)]^{1/6}$ are

$$2^{-1/12}e^{i\theta},\ \theta = (4k\pi - 5\pi)/12,\ k = 0, 1, \dots, 5.$$

1.31. Remark. It is easily verified that (1.28) is valid when n is a negative integer, since $z^{1/n} = (1/z)^{-1/n}$. Further, we can easily show that if m and n are integers having no common factors then all the values of $z_0^{m/n}$ are $\sqrt[n]{|z_0|^m}e^{i[(m/n)\theta_0+(2mk\pi/n)]}$, $k = 0, 1, \dots, n-1$. •

If we set $\omega = e^{2\pi i/n}$, then all the n-th roots of unity are expressed by $1, \omega, \omega^2, \omega^3, \dots, \omega^{n-1}$. From (1.28) it is clear that if $\omega \neq 1$ is an n-th root of unity, then the others are $\omega^2, \omega^3, \dots, \omega^{n-1}, 1$. Hence if c is any one of n-th roots of z_0, then all the n-th roots of z_0 are given by

$$c, c\omega, \omega^2, \dots, c\omega^{n-1}.$$

Now, we see some of the immediate consequences of the above observations. Since $\omega^n - 1 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) = 0$. If $\omega \neq 1$, then for $n > 1$, we have

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0 \quad (\omega = e^{i2\pi/n}).$$

Thus, in particular, by equating real and imaginary parts on both sides

$$\sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1 \quad\text{and}\quad \sum_{k=1}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = 0.$$

Further, it is also evident that the sum of the products of all n-th roots of unity, taken $2, 3, \ldots, n-1$ respectively at a time, is zero, since they are the roots of the equation $z^n - 1 = 0$. Now, for $\omega \neq 1$, the condition $\omega^n = 1$ gives that

$$\begin{aligned} 1 + 2\omega + \cdots + n\omega^{n-1} &= \frac{(1-\omega)(1 + 2\omega + \cdots + n\omega^{n-1})}{1-\omega} \\ &= \frac{1 + \omega + \omega^2 + \cdots + \omega^{n-1} - n\omega^n}{1-\omega} \\ &= \frac{-n}{1-\omega}. \end{aligned}$$

Let h be any integer which is not a multiple of n, i.e. $h = np + q$ $(0 < q < n)$. Then $\omega^h = \omega^q$, $q < n$ which implies that the sets

$$\{1, \omega^h, \omega^{2h}, \ldots, \omega^{(n-1)h}\} \quad\text{and}\quad \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$$

are the same in some order. Hence if h is not a multiple of n or h is an integer such that $0 < |h| < n$, then

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0.$$

If h is a multiple of n, then $\omega^h = \omega^{np} = (\omega^n)^p = 1$ and so we have

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = n.$$

Similarly, it is evident that

$$1 - \omega^h + \omega^{2h} - \cdots + (-1)^{n-1}\omega^{(n-1)h} = \frac{1 - (-1)^n \omega^{nh}}{1 + \omega^h} = \frac{1 - (-1)^n}{1 + \omega^h}.$$

That is, if ω is an n-th root of unity,

$$\sum_{k=1}^{n} (-1)^{k-1} \omega^{(k-1)h} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{1 + \omega^h} & \text{if } n \text{ is odd.} \end{cases}$$

1.32. Example. We wish to prove that all circles that passes through α and $(\overline{\alpha})^{-1}$ intersect the circle $|z| = 1$ at right angles if $|\alpha| \neq 1$, $\alpha \neq 0$.

To do this, we consider the equation of the circle C with center at z_0 and radius r: $C = \{z \in \mathbb{C} : |z - z_0| = r\}$. If C passes through the points $\alpha\,(\neq 0)$ and $1/\overline{\alpha}$, then we have

$$|\alpha - z_0|^2 = r^2, \quad \text{i.e.} \quad |\alpha|^2 + |z_0|^2 - 2\ \mathrm{Re}\,(\overline{\alpha} z_0) = r^2$$

and

$$|1 - \overline{\alpha} z_0|^2 = |\alpha|^2 r^2, \quad \text{i.e.} \quad 1 + |\alpha|^2 |z_0|^2 - 2\ \mathrm{Re}\,(\overline{\alpha} z_0) = |\alpha|^2 r^2.$$

Subtract the latter from the former to get $(1 - |\alpha|^2)(1 + r^2 - |z_0|^2) = 0$. This implies either $|\alpha| = 1$ or $|z_0|^2 = 1 + r^2$ holds. The second condition yields the required conclusion if $|\alpha| \neq 1$. ●

1.33. Example. Discuss the nature of the set

$$S = \{z : |z - a| + |z + a| = 2c\}. \tag{1.34}$$

For two complex numbers z_1 and z_2 we know that

$$|z_1 - z_2|,\ |z_1 + z_2| \leq |z_1| + |z_2|$$

so that $|2a| = |z + a - (z - a)| \leq |z + a| + |z - a| = 2c$. Thus, there are complex numbers satisfying (1.34) only if $|a| \leq c$.

Suppose that $z \in S$. Then

$$2|z| = |z + a + z - a| \leq |z + a| + |z - a| = 2c, \quad \text{i.e.} \quad \max_{z \in S} |z| = c.$$

Here the maximum is attained at $z = ce^{-i\theta}$ where $\theta = \arg(c/a)$. If $z \in S$, then we see that the sum of the distances from the point z to the given points a and $-a$, is equal to the constant $2c$. This means that S represents an equation of an ellipse with foci at $\pm a$. For instance, if a is real then the equation of the ellipse using rectangular coordinates is given by

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - a^2} = 1.$$

Its center is at the origin and its semi-major and semi-minor axes are equal to c and $\sqrt{c^2 - a^2}$, respectively.

If a is real, then we can discuss the set described in (1.34) in the following way: Let $z = x + iy$. Then, $|z - a| + |z + a| = 2c$ is equivalent to

$$|x + iy - a| + |x + iy + a| = 2c$$

$$\begin{aligned}
&\iff |x + iy - a| = 2c - |x + iy + a| \\
&\iff |x + iy - a|^2 = (2c - |x + iy + a|)^2 \\
&\iff (x - a)^2 + y^2 = 4c^2 + (x + a)^2 + y^2 - 4c|x + a + iy|
\end{aligned}$$

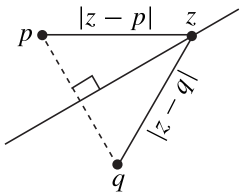

Figure 1.18: Equidistant points from p and q: $|z-p| = |z-q|$.

$$
\begin{aligned}
&\iff (c|x+a+iy|)^2 = (c^2+ax)^2 \\
&\iff c^2[(x+a)^2+y^2] = (c^2+ax)^2 \\
&\iff (c^2-a^2)x^2 + c^2y^2 = c^2(c^2-a^2) \\
&\iff \frac{x^2}{c^2} + \frac{y^2}{c^2-a^2} = 1.
\end{aligned}
$$

Using the technique of this example one can show that the set

$$S = \{z : |z-a| - |z+a| = 2c\}$$

describes a hyperbola with foci at $z = a$ and $z = -a$. ●

1.35. Example. If p and q are two distinct points in $\mathbb{C}$, then it is easy to see that the set described by $\{z : |z-p| = |z-q|\}$ gives the straight line that is the perpendicular bisector of the line segment joining p and q (see Figure 1.18). Next we see that the set of points described by

$$|z-p| = \alpha|z-q| \quad (p \neq q,\ 0 < \alpha < 1) \tag{1.36}$$

is a circle. Upon squaring (1.36) we get

$$
\begin{aligned}
|z-p|^2 = \alpha^2|z-q|^2 \iff & |z|^2 - 2\mathrm{Re}\left[z\left(\frac{\overline{p}-\alpha^2\overline{q}}{1-\alpha^2}\right)\right] = \frac{\alpha^2|q^2| - |p|^2}{1-\alpha^2} \\
\iff & \left|z - \frac{p-\alpha^2 q}{1-\alpha^2}\right|^2 = \frac{|p-\alpha^2 q|^2}{(1-\alpha^2)^2} + \frac{\alpha^2|q|^2 - |p|^2}{1-\alpha^2} \\
& = \frac{\alpha^2|p-q|^2}{(1-\alpha^2)^2}, \\
\iff & \left|z - \frac{p-\alpha^2 q}{1-\alpha^2}\right| = \frac{\alpha|p-q|}{1-\alpha^2}.
\end{aligned}
$$

Thus, the set of the points described by (1.36) is a circle. ●

1.5 Topology of the Complex Plane

In the earlier sections we discussed some of the algebraic and geometric properties of the complex field or plane as the case may be. In this section we study some of the topological properties of the complex plane. This is required for our study of Complex Analysis. If X is any set, then the function $d : X \times X \to \mathbb{R}$ is called a *metric* or *a distance function* if it satisfies the following conditions for all a, b and c in X:

(i) $d(a,b) \geq 0$

(ii) $d(a,b) = 0 \Longleftrightarrow a = b$

(iii) $d(a,b) = d(b,a)$

(iv) $d(a,c) \leq d(a,b) + d(b,c)$.

The set X together with a metric, i.e. (X,d) or, in short X, is called *a metric space*. As we have seen earlier, the function

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \ (z,z') \mapsto |z - z'|,$$

has the following properties:

(a) $|z - z'| \geq 0$

(b) $|z - z'| = 0 \Longleftrightarrow z = z'$

(c) $|z - z'| = |z' - z|$

(d) $|z - w| \leq |z - z'| + |z' - w|$, where $z, z', w \in \mathbb{C}$,

where $d(z,z') = |z - z'|$ is called the Euclidean metric. Thus, $\mathbb{C}$ is a metric space with the Euclidean metric d. For instance, we have

(a) If $X = \mathbb{R}$ and $d(x,x') = |x - x'|$, then $(\mathbb{R}, d)$ is a metric space.

(b) If $Y \subset X$ and (X,d) is a metric space, then so is the restriction (Y,d).

(c) Besides the Euclidean metric we have another natural metric known as the *maximum metric* on $\mathbb{C}$. This is defined as

$$d(x + iy, x' + iy') = \max\{|x - x'|, |y - y'|\}.$$

For a detailed discussion on metric spaces, we refer to the book by Ponnusamy [8]. In the Euclidean metric space $(\mathbb{C}, d)$, an open ball

$$\Delta(z_0;\epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

is called *an open disk of radius* $\epsilon > 0$ *centered* at $z_0 \in \mathbb{C}$ or an ϵ *neighborhood* or simply a *neighborhood* of z_0. Geometrically, $\Delta(z_0;\epsilon)$ is just the disk centered at z_0 consisting of all points at a distance less than ϵ from z_0. Evidently, $\Delta(z_0;\infty) = \mathbb{C}$ for any $z_0 \in \mathbb{C}$.

We use the term *deleted neighbourhood of* z_0 to denote a set of the form

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}, \ \text{i.e.} \ \ \Delta(z_0;\epsilon) \setminus \{z_0\}.$$

Figure 1.19: Half planes.

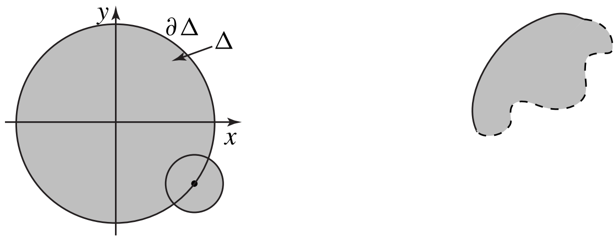

Figure 1.20: Description for a non-open set.

We define $\partial\Delta(z_0;\epsilon) = \{z \in \mathbb{C} : |z - z_0| = \epsilon\}$, the circle of radius $\epsilon > 0$ centered at z_0. Throughout this book, we use the notation

$$\Delta_R := \Delta(0;R) = \{z \in \mathbb{C} : |z| < R\} \quad \text{and} \quad \Delta := \Delta_1.$$

The unit disk Δ, as we shall see in later chapters, plays a crucial role in the theory of functions of a complex variable.

A subset $S \subseteq \mathbb{C}$ is called *open* (in $\mathbb{C}$) if for every $z_0 \in S$ there is a $\delta > 0$ such that $\Delta(z_0;\delta) \subset S$. This means that some disk around z_0 lies entirely in S. For instance, the interior of a circle, the entire plane $\mathbb{C}$, and the half-planes given by $\operatorname{Re} z < \alpha$, $\operatorname{Im} z > \alpha$ and $\operatorname{Im} z < \alpha$, are all examples of open sets (see Figure 1.19). Here α is an arbitrary fixed real number. On the other hand "the interior of a circle union circumference" does not form an open set, since no neighborhood of a point on the circumference lies entirely within the set; for instance, $S = \{z : |z| \leq 1\}$ is not open (see Figure 1.20). Observe that $\mathbb{R}$ when considered as a subset of $\mathbb{C}$ is not open.

To show that the disk $\Delta(z_0;R)$ *is open,* let $\zeta \in \Delta(z_0;R)$. If we choose δ such that $0 < \delta < R - |\zeta - z_0|$, then $\Delta(\zeta;\delta) \subset \Delta(z_0;R)$. As $\zeta \in \Delta(z_0;R)$ being arbitrary, this proves that $\Delta(z_0;R)$ is open. On the other hand, this fact is geometrically clear.

The complement of a set $S \subseteq \mathbb{C}$ is $\mathbb{C}\backslash S := \{z \in \mathbb{C} : z \notin S\}$ and is usually denoted by S^c. A set $S \subseteq \mathbb{C}$ is said to be *closed* if its complement $\mathbb{C}\backslash S$ is open. For each $\epsilon > 0$, the set $\overline{\Delta}(z_0;\epsilon) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$

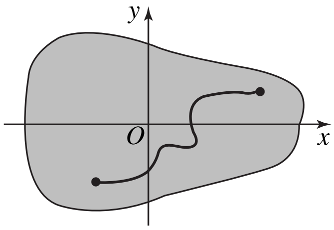

Figure 1.21: Description for connected set.

is closed and consequently we call it a *closed disk.* We write $\overline{\Delta}(0;\epsilon)$ as $\overline{\Delta}_\epsilon$ and $\overline{\Delta}_1$ simply as $\overline{\Delta}$. Here are some examples of closed sets.

(a) $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$; $\Delta^c = \{z \in \mathbb{C} : |z| \geq 1\}$; $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$;

(b) $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq 0\}$; $\{z \in \mathbb{C} : \mathrm{Im}\, z = 1\}$; $\{z \in \mathbb{C} : |z-4| \geq |z|\}$.

(c) The entire plane $\mathbb{C}$ and the empty set $\emptyset$.

There is another way of characterizing a closed set S using the notion of *limit point* of S. A point z_0 is a *limit point* of a set S if every $\Delta(z_0;\epsilon)$ contains a point of S other than z_0. The point z_0 itself may or may not belong to the set S. For example, $z_0 = 0$ is a limit point of

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$$

but $0 = z_0 \notin S$. Similarly if $\Delta = \{z : |z| < 1\}$, then each point on $|z| = 1$ is a limit point of Δ but again does not belong to the disk Δ. Each $z \in \Delta$ is also a limit point of S.

1.37. Example. We easily have the following:

(a) $S = \{z : z = x+iy$ with x and y rational$\}$ is neither open nor closed.

(b) $S = \{z : z = 2\} \cup \{z : |z| < 2\}$ is neither open nor closed.

(c) $\mathbb{C}$, $\emptyset$ are both open and closed.

(d) $S = \{z : 0 < |z| \leq 1\}$ is neither open nor closed.. ●

As an alternative characterization of closed sets in $\mathbb{C}$, we have "*a set S is closed $\Longleftrightarrow$ S contains all its limit points.*" We also note that, not every point of a closed set S need be a limit point of S; for instance, if

$$S = \{z : z = 0 \text{ or } z = 1/n, \text{ for positive integers } n\},$$

then $z = 0$ is the only limit point of S (which is in S) and therefore, S is closed. Note that no other point of S is a limit point of S, since, for

$$z_0 = \frac{1}{n} \text{ and } \epsilon = \frac{1}{n(n+1)},$$

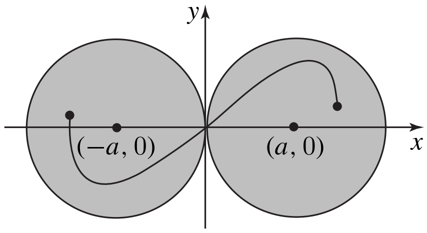

Figure 1.22: The set described by S_1.

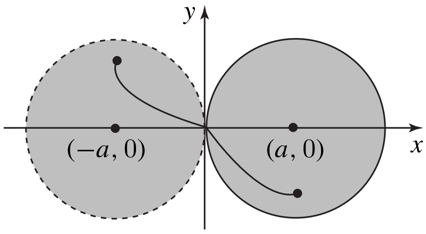

Figure 1.23: Set described by S_2 and S_3.

the disk $\Delta(z_0;\epsilon)$ contains no point of S other than z_0 itself. Points in a set S which are not limit points are called *isolated points* of the set S. Further, it is clear from the definition that each element of an open set is a limit point of the set.

A boundary point of a set S is a point for which every neighborhood contains at least one point of S and at least one point not in S. The *boundary* of S, denoted by ∂S, is defined as the set of all the boundary points. For instance, consider

$$S = \{z \in \mathbb{C} : |z-1| \leq 1\}.$$

The point $z_0 = 1+i$ is a boundary point of S since every δ-neighborhood of it has a non-empty intersection with both S and S^c. Although in this case

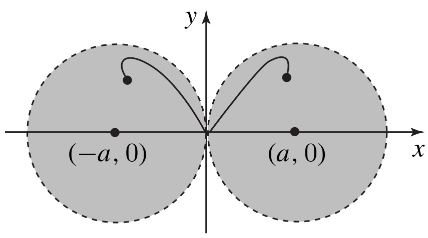

Figure 1.24: The set described by S_3.

$z_0 \in S$, this need not always be so. For example, $z_0 = 1 + i$ is a boundary point but not in $S = \{z \in \mathbb{C} : |z - 1| < 1\}$.

The boundary ∂S is always closed in $\mathbb{C}$, and $\overline{S}$, the *closure* of S, is defined by $\overline{S} = S \cup \partial S$.

A point z_0 is called *an interior point* of S if there exists a $\delta > 0$ such that $\Delta(z_0; \delta) \subset S$. The *interior of* S, denoted by $\operatorname{Int} S$, is the set of all interior points of S. Thus, it is clear from the definition that "*A set is open* $\Longleftrightarrow$ *each of its points is an interior point.*"

A set $S \subseteq \mathbb{C}$ is said to be *separated* (or *disconnected*) if there exist two nonempty disjoint open sets A and B such that $S \subseteq A \cup B$, $S \cap A \neq \emptyset$, and $S \cap B \neq \emptyset$. If S is not disconnected, it is called *connected.*

In any particular situation we generally can tell at a glance if a given set S in the complex plane is connected, as illustrated in Figure 1.21. If $a > 0$,

$$\begin{aligned} S_1 &= \{z : |z - a| \leq a \ \text{ or } \ |z + a| \leq a\} \\ S_2 &= \{z : |z - a| \leq a \ \text{ or } \ |z + a| < a\}, \\ S_3 &= \{z : |z - a| < a \ \text{ or } \ |z + a| < a\}, \\ S_4 &= \{z : |z^2 - 1| < 1\}, \end{aligned}$$

then S_1 and S_2 are connected (see Figure 1.23) whereas S_3 and S_4 are not connected (see Figure 1.24).

The function $\gamma : [0, 1] \to \mathbb{C}$, defined by $\gamma(t) = (1 - t)z_0 + tz_1$ is called the *line segment* with end points z_0 and z_1 and is designated by $[z_0, z_1]$. If $\gamma(t) \in S$ for each $t \in [0, 1]$, then the line segment $[z_0, z_1]$ is said to be contained in S. A *polygonal line* from z_0 to z_n is a finite union of segments of the form

$$[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n].$$

The points z_0 and z_n are then said to be polygonally connected. If the segment $[z_k, z_{k+1}]$ is contained in S, $k = 0, 1, \ldots, n-1$, then the polygonal line from z_0 to z_n is said to be contained in S. A set S is said to be *polygonally connected* if any two points of S can be connected by a polygonal line contained in S. In other words, S is connected iff each pair of points z, ζ of S can be connected by an arc lying in S. For instance, any open disk $\Delta(z_0; \delta)$ is polygonally connected. For, if $z_1, z_2 \in \Delta(z_0; \delta)$ and $\gamma(t) = (1 - t)z_1 + tz_2$,

$$|\gamma(t) - z_0| = |(1 - t)(z_1 - z_0) + t(z_2 - z_0)| \leq (1 - t)\delta + t\delta = \delta$$

and so for each $t \in [0, 1]$, $\gamma(t) \subset \Delta(z_0; \delta)$.

A *domain* is a nonempty open connected set in $\mathbb{C}$. A domain together with some, none, or all of its boundary points is referred to as a *region.* For instance, if

$$S_5 = \{z \in \mathbb{C} : \operatorname{Re} z < a, \ \ a\text{-real}\} \ \text{ and } \ S_6 = \{z \in \mathbb{C} : \operatorname{Re} z \leq a, \ \ a\text{-real}\},$$

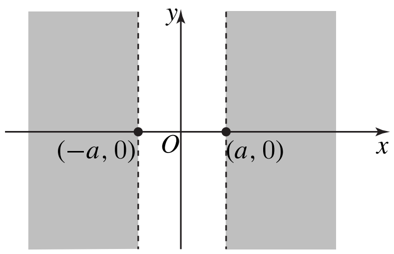

Figure 1.25: The set described by S_7.

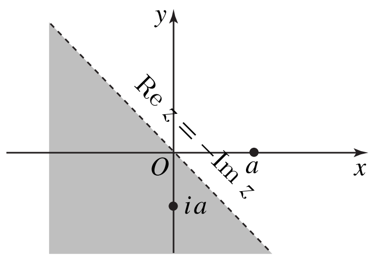

Figure 1.26: The set $\{z = x + iy : y < -x\}$.

then S_5 describes a domain whereas S_6 is a region but is not a domain, since the set defined by S_6 is not open but connected. On the other hand, the set

$$S_7 = \{z \in \mathbb{C} : |\mathrm{Re}\, z| > a \;\text{ for some } a > 0\}$$

does not constitute a domain. Note that S_7 is open but not connected (see Figure 1.25). Further, the set

$$S_8 = \{z \in \mathbb{C} : a < \mathrm{Re}\, z \le b \;\text{ for some } a < b\},$$

called an infinite strip, is not a domain but is a region. Also, $S_9 = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < |\mathrm{Re}\, z|\}$ is disconnected while $S_{10} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \le |\mathrm{Re}\, z|\}$ is connected. Similarly, the set

$$S_{11} = \{z \in \mathbb{C} : |z + ia| < |z - a| \text{ for some } a > 0\}$$

is connected and open. Note that (see Figure 1.26)

$$\begin{aligned} |z + ia| < |z - a| &\iff |z + ia|^2 < |z - a|^2 \\ &\iff |z|^2 + |a|^2 + 2\mathrm{Re}\,(-iza) < |z|^2 + |a|^2 - 2\mathrm{Re}\,(za) \\ &\iff -2\mathrm{Re}\,(iza) < -2\mathrm{Re}\,(za) \\ &\iff y < -x, \quad \text{since} \quad a > 0. \end{aligned}$$

A set S is *bounded* if there is an $R > 0$ such that $S \subset \overline{\Delta}_R$. Geometrically S is contained in a closed disk centered at 0 and radius R. A set that cannot be enclosed by any $\overline{\Delta}_R$ for $R > 0$ is called *unbounded.* A simple example is an infinite strip, S_8 above. Sets which are closed as well as bounded in $\mathbb{C}$ are called *compact* sets in $\mathbb{C}$. Of course, the complex plane $\mathbb{C}$ is not compact as it is not bounded in $\mathbb{C}$. Note that $\mathbb{C}$ is closed because $\mathbb{C}\backslash\mathbb{C} = \emptyset$ is open. Thus, $\mathbb{C}$ (with usual metric) is not a compact metric space. On the other hand, $\overline{\Delta}_R$ is compact.

A set S is *countable* if the elements of S can be placed in a one-to-one correspondence with the set of positive integers. For instance, each of the sets defined by,

$$\begin{aligned} A &= \{2n : n = 1, 2, \dots\}, \\ B &= \{2n + 1 : n = 1, 2, \dots\}, \\ C &= \{r : \ r \text{ rational}\}, \end{aligned}$$

is countable whereas the set of irrationals and the set of reals are both *not countable*, i.e. *uncountable.*

1.6 Sequences and Series

An infinite sequence of complex numbers is a list of points $z_1, z_2, \dots, z_n, \dots$ of $\mathbb{C}$ listed in some order. More explicitly a mapping $\mathbb{N} \to \mathbb{C}$, $n \mapsto z_n$, is called *a sequence.* This is briefly denoted by $\{z_n\}_{n\geq 1}$ (or simply by $\{z_n\}$ when there is no confusion) with the understanding that z_n is the n-th term of the sequence. A sequence is thus merely an assignment of a specific point z_n to each $n \in \mathbb{N}$.

Suppose $\{z_n\}$ is a sequence of points (complex numbers) in $\mathbb{C}$ and that $\{n_k\}$ is a strictly increasing sequence of natural numbers. Then the sequence $\{z_{n_k}\}$ (think of z_{n_k} as a_k) is called a *subsequence* of the sequence $\{z_n\}$. For instance,

$$\{z_{k+1}\}, \ \{z_{2k}\}, \ \{z_{2k+5}\}, \ \text{ and } \{z_{2^k}\}$$

are some subsequences of $\{z_n\}$. Roughly speaking subsequences are obtained by deleting some of the terms from the sequence under consideration. Of course, $\{z_n\}$ is trivially a subsequence of itself. For instance, $\{z_{2k+1}\}$ may be obtained by removing the terms $z_2, z_4, z_6, \dots$. We remark that the notion of sequences is not confined to sequences of complex numbers. In later chapters, we shall also consider sequences of sets and sequences of functions.

A sequence $\{z_n\}$ is said to converge to a point z_0, and write $z_n \to z_0$, if for every $\epsilon > 0$ there exists an $N(\epsilon) \in \mathbb{N}$ such that

$$|z_n - z_0| < \epsilon, \quad \text{for all } \ n > N(\epsilon).$$

That is, $z_n \to z_0$ if $|z_n - z_0| \to 0$, as a real sequence. If a sequence fails to converge, it is said to *diverge.*

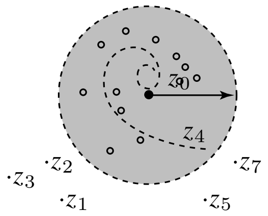

Figure 1.27: Description for a limit of a sequence.

Geometrically, $z_n \to z_0$ if every neighborhood of z_0 contains all but finitely many terms of the sequence $\{z_n\}$; such a point z_0 is called a *limit* of the sequence (see Figure 1.27). Sometimes we write

$$\lim_{n\to\infty} z_n = z_0$$

instead of $z_n \to z_0$. A point z_0 is a *limit* of the sequence $\{z_n\}$ if there exists a subsequence that converges to z_0.

To have a better understanding of series that will be introduced later, we must define the notion of limit superior (resp. limit inferior) of a sequence $\{r_n\}$ of real numbers. If $\{r_n\}$ is bounded above (resp. bounded below) and has at least one convergent subsequence, then limit superior (resp. limit inferior), written

$$L = \limsup_{n\to\infty} r_n \quad \text{or} \quad \overline{\lim}_{n\to\infty} r_n \quad \left(\text{resp. } l = \liminf_{n\to\infty} r_n \quad \text{or} \quad \underline{\lim}_{n\to\infty} r_n\right),$$

is the least upper bound (resp. the greatest lower bound) of the limits of all convergent subsequences of $\{r_n\}$. If no such L (resp. l) exists, we set

$$L = \limsup_{n\to\infty} r_n = +\infty \quad \left(\text{resp. } \ l = \liminf_{n\to\infty} r_n = -\infty\right).$$

Note that this is possible only if $\{r_n\}$ is an unbounded sequence. Of course, if $\{r_n\}$ happens to be a convergent sequence then we note that

$$\limsup_{n\to\infty} r_n = \lim_{n\to\infty} r_n = \liminf_{n\to\infty} r_n.$$

For instance, let

$$a_n = (-1)^{3n},\ b_n = 1 + (-1)^n,\ c_n = n(-1)^n,\ d_n = 1 - 4^{-n}.$$

Then we have

(i) $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$

(ii) $\limsup_{n\to\infty} b_n = 2$ and $\liminf_{n\to\infty} b_n = 0$

(iii) $\limsup_{n\to\infty} c_n = +\infty$ and $\liminf_{n\to\infty} c_n = -\infty$

(iv) $\limsup_{n\to\infty} d_n = 1 = \liminf_{n\to\infty} d_n$.

How about the sequence $\{e_n\}$, where $e_n = \sin(n\pi/2) + n\cos(n\pi/2)$?

For any complex number z, we have

$$|\mathrm{Re}\, z|,\ |\mathrm{Im}\, z| \le |z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|.$$

Now, suppose $z_n = x_n + iy_n \to z_0 = x_0 + iy_0$. Then for a given $\epsilon > 0$, there exists an $N(\epsilon)$ such that

$$|z_n - z_0| = |(x_n - x_0) + i(y_n - y_0)| < \epsilon \ \text{ for all } \ n > N(\epsilon).$$

This implies

$$|x_n - x_0| < \epsilon \ \text{ and } \ |y_n - y_0| < \epsilon \ \text{ for all } \ n > N(\epsilon);$$

i.e. $x_n \to x_0$, $y_n \to y_0$. Conversely, let $x_n \to x_0$ and $y_n \to y_0$. Then for a given $\epsilon > 0$, there exist $N_1(\epsilon)$ and $N_2(\epsilon)$ such that

$$|x_n - x_0| < \epsilon/2 \ \text{ for all } n > N_1(\epsilon)$$

and

$$|y_n - y_0| < \epsilon/2 \ \text{ for all } n > N_2(\epsilon).$$

Hence, for all $n > N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, we have

$$|z_n - z_0| \le |x_n - x_0| + |y_n - y_0| < \epsilon.$$

The above discussion shows that

$$z_n \to z_0 \iff \mathrm{Re}\, z_n \to \mathrm{Re}\, z_0 \ \text{ and } \ \mathrm{Im}\, z_n \to \mathrm{Im}\, z_0. \tag{1.38}$$

A sequence can have at most one limit if it exists; for, let $z_n \to z_0$ and $z_n \to z_0^*$. Then for a given $\epsilon > 0$, we have

$$\begin{aligned} |z_0 - z_0^*| &= |(z_n - z_0^*) - (z_n - z_0)| \\ &\le |z_n - z_0^*| + |z_n - z_0| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \quad \text{for sufficiently large } n. \end{aligned}$$

Recall that if $x,\ y \in \mathbb{R}$ and $x < y + \epsilon$ for any $\epsilon > 0$, then $x \le y$. For, suppose that $x - y > 0$. Then for $\epsilon = x - y$, the hypothesis yields $\epsilon < \epsilon$. This contradiction shows that $x \le y$.

Note also that $z_0,\ z_0^*$ are independent of n and $|z_0 - z_0^*| < \epsilon = 0 + \epsilon$ for every ϵ. So $|z_0 - z_0^*| \le 0$, and hence we must have $|z_0 - z_0^*| = 0$, i.e. $z_0 - z_0^* = 0$ or $z_0 = z_0^*$. This shows that a limit of a sequence is unique.

1.39. Remark. If $z_n \to \ell$, then the limit ℓ is a limit point of the sequence $\{z_n\}$. However, the converse is not true. For example, let

$$z_n = (-1)^n + i\frac{n}{n+1}, \; n \geq 1.$$

Then $1+i$ and $-1+i$ are the limit points of the set $\{z_1, z_2, \ldots\}$, but $z_n \not\to 1+i$, or $-1+i$. Similarly, for $z_n = 2^{1-n} + n + (-1)^n n$ $(n \geq 1)$, we have $z_n \not\to 0$, but 0 is the limit point of the set $S = \{z_1, z_2, \ldots\}$. ●

1.40. Example. To illustrate the concept of the limit of a sequence, we shall present some more examples.

(a) For $z_n = \frac{1}{n} + \frac{(n-1)i}{n}$, we have $z_n \to i$ since $|z_n - i| = \frac{1}{n}|1-i| = \frac{\sqrt{2}}{n} \to 0$. If we write $z_n = x_n + iy_n$ with $x_n = \frac{1}{n}$ and $y_n = \frac{n-1}{n}$ as real sequences, we have $x_n \to 0 = x_0$ and $y_n \to 1 = y_0$ $(z_0 = x_0 + iy_0 = i)$. Compare with the observation in (1.38).

(b) If $z_n = i^n/n$, then, for a given $\epsilon > 0$, there is an $N(\epsilon)$ such that[4]

$$|z_n - 0| = \frac{1}{n} < \epsilon \;\text{ for all } n > N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1, \qquad \text{i.e. } z_n \to 0.$$

(c) For $z_n = -1 + i/n$ and $z_n' = -1 - i/n$, we have $z_n \to -1$ and $z_n' \to -1$ whereas $\operatorname{Arg} z_n \to \pi$ and $\operatorname{Arg} z_n' \to -\pi$.

(d) The sequence $\{\operatorname{Arg}[(-1)^n/n]\}$ is divergent, because the sequence has the form $\pi, 0, \pi, 0, \ldots$ and hence has no limit. ●

A sequence $\{z_n\}$ is called *a Cauchy sequence* if for each $\epsilon > 0$ there exists an $N(\epsilon)$ such that $|z_n - z_m| < \epsilon$ for all $n, m > N(\epsilon)$. Using this definition, we can easily conclude the *Cauchy criterion for convergence in* $\mathbb{C}$*:*

$$\{z_n\} \text{ converges } \iff \{z_n\} \text{ is a Cauchy sequence.}$$

For, let $z_n = x_n + iy_n \to z_0 = x_0 + iy_0$. Then, $\operatorname{Re} z_n \to \operatorname{Re} z_0$, $\operatorname{Im} z_n \to \operatorname{Im} z_0$. As $|z_n - z_m| \leq |\operatorname{Re}(z_n - z_m)| + |\operatorname{Im}(z_n - z_m)|$, $\{z_n\}$ is clearly a Cauchy sequence.

Suppose $\{z_n\}$ is a Cauchy sequence. As

$$|\operatorname{Re}(z_n - z_m)|, \; |\operatorname{Im}(z_n - z_m)|\} \leq |z_n - z_m|,$$

we see that $\{\operatorname{Re} z_n\}$ and $\{\operatorname{Im} z_n\}$ are Cauchy sequences in $\mathbb{R}$. Because of the completeness property (see [11]) of $\mathbb{R}$, $\{\operatorname{Re} z_n\}$ and $\{\operatorname{Im} z_n\}$ converge and hence, $\{z_n\}$ converges.

A convergent sequence with limit zero is called a *null sequence.* If the sequence $\{z_n\}$ converges to z_0, then the sequence $\{z_n - z_0\}$ is a null sequence.

[4]Here $[x]$ denotes the greatest integer less than or equal to x

Since $\{z_n\} = z_0 + \{z_n - z_0\}$, any convergent sequence may be written as a sum of a fixed number and a null sequence. Conversely, if $\{z_n\} = z_0 + \{z'_n\}$, where $\{z'_n\}$ is a null sequence, then

$$|z_n - z_0| = |z'_n| < \epsilon \quad \text{for} \quad n > N(\epsilon)$$

and so the sequence $\{z_n\}$ is convergent.

1.41. Example. It is easy to see that $\{z^n\}$ is a null sequence for $|z| < 1$. To prove this, let $\epsilon > 0$ be given. We must find a value of N such that, for any $n > N$, $|z^n| = |z|^n < \epsilon$. This is certainly true if $z = 0$. So, for a fixed $z \neq 0$, we simply need to find an N such that, for any $n > N$,

$$n \ln |z| < \ln \epsilon, \quad \text{i.e.} \quad n > \ln \epsilon / \ln |z|$$

(since $\ln |z|$ is negative, for $|z| < 1$). This proves that if $|z| < 1$ then

$$|z|^n < \epsilon \quad \text{for} \quad n > N = [\ln \epsilon / \ln |z|] + 1.$$

Thus, $z^n \to 0$ as $n \to \infty$ if $|z| < 1$. For $|z| > 1$, the sequence $\{z^n\}$ does not converge to any point of $\mathbb{C}$. How about for points z on the unit circle $|z| = 1$? •

On many occasions we will need to work with an infinite series. Given a sequence $\{z_n\}$ of complex numbers the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} z_k$$

is called the sequence of *partial sums* of the (infinite) series $\sum_{k=1}^{\infty} z_k$. For convenience we shall use the equivalent forms:

$$\sum_{k=1}^{\infty} z_k \quad \text{or} \quad \sum_{k \geq 1} z_k$$

(which will be sometimes abbreviated as $\sum z_k$). Sometimes it is convenient to start the series with $k = 0$ (or perhaps even with some other integer p). We then write $\sum_{k=1}^{\infty} z_k$ or $\sum_{k=p}^{\infty} z_k$ whichever the case may be.

The series $\sum z_k$ is said to be *convergent* or *summable* or said to *converge to* s if $s_n \to s$. The number s is then called the sum of the series and we write $s = \sum z_k$. Otherwise, the series is said to be *divergent* or to *diverge.* If the series is convergent, the sequence of its partial sums is bounded.

1.42. Remark. As in the case of sequences, if we let

$$z_k = x_k + iy_k \quad \text{and} \quad s = u + iv, \ x_k, y_k, u, v \ \text{real}$$

then the series $\sum z_k$ converges to s iff $\sum x_k$ converges to u and $\sum y_k$ converges to v. ●

From the Cauchy criterion, i.e. "a sequence is convergent iff it is a Cauchy sequence", we see that

$$\sum z_k \text{ converges } \iff \{s_n\} \text{ is a Cauchy sequence.}$$

That is, "$\sum z_k$ *converges iff for every* $\epsilon > 0$ *there exists an* N *such that*

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} z_k\right| < \epsilon \textit{ for } n > m > N \text{ .''}$$

Writing $z_n = s_n - s_{n-1}$, we have, by choosing $m = n - 1$ in the above inequality

$$\sum z_k \text{ converges} \implies \lim_{n\to\infty} s_n = s \implies \lim_{n\to\infty} z_n = 0. \tag{1.43}$$

That is, "*a necessary condition for the series* $\sum z_n$ *to converge is that* $z_n \to 0$ *as* $n \to \infty$." However, this condition is not sufficient for the convergence of $\sum z_n$ as the harmonic series $\sum_{n\geq 1} 1/n$ shows.

1.44. Example.

(i) As the n-th term $(-1)^n$ of the series $\sum_{n\geq 1}(-1)^n$ does not converge to zero, the series diverges.

(ii) Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = 1/[(n+1)(n+2)]$. Then

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2} - \frac{1}{n+2}$$

which converges to $1/2$. Thus, $\sum_{n=1}^{\infty} a_n = 1/2$.

(iii) The series $\sum_{n=1}^{\infty} k^{-1/2}$ diverges to ∞, because

$$\begin{aligned} s_n = \sum_{k=1}^{n} k^{-1/2} &> \text{ number of terms times the last term} \\ &= n(n^{-1/2}) = n^{1/2}; \end{aligned}$$

so $\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} n^{1/2} = +\infty$. A similar argument may be used to show that $\sum_{n=1}^{\infty} k^{-\alpha}$ diverges to ∞ when $\alpha \in [0, 1)$. How about when $\alpha = 1$? If $\alpha = 1$, then we easily have

$$\begin{aligned} s_{2^n} &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &\quad + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2}. \end{aligned}$$

This shows that the partial sum of the series is unbounded and hence $\{s_n\}$ is divergent. Note also that, since $k^{-p} \geq k^{-1}$ for all $p \leq 1$ and $\sum_{k\geq 1} k^{-1}$ diverges, we deduce that the series $\sum_{k\geq 1} k^{-\alpha}$ diverges for $\alpha \leq 1$. •

The following result is easily established from the definition of convergence of a series. So we omit its proof.

1.45. Theorem. *If* $\sum z_k = s$ *and* $\sum w_k = t$ *then* $\sum (z_k \pm w_k) = s \pm t$ *and* $\sum cz_k = cs$, *where* c *is any complex constant.*

As with real series, a series $\sum z_k$ is said to *converge absolutely* or be *absolutely convergent* if the series $\sum |z_k|$ is convergent. Further, using the fact that

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \sum_{k=m+1}^{n} |z_k|,$$

we conclude that *"every absolutely convergent series is convergent."* Note that the converse of this result is generally false. For instance, the series $\sum_{k\geq 1} z_k$, where $z_k = (-1)^{k-1}/k$, is convergent, but not absolutely. This is because

$$\begin{aligned} |s_n - s_m| &= \left|\sum_{k=m+1}^{n} z_k\right| \\ &= \frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} - \cdots \frac{1}{n} \\ &\leq \frac{1}{m+1} < \epsilon \text{ for } n > m > N = \left[\frac{1}{\epsilon}\right] \geq \frac{1}{\epsilon} - 1 \end{aligned}$$

and so $\{s_n\}$ is a Cauchy sequence. Further one can show that the sum of this series is, in fact, $\ln 2$. But $\sum |z_k| = \sum_{k\geq 1} \frac{1}{k}$ diverges as shown earlier.

On the other hand, the series $\sum z_k$, where $z_k = (-1)^{k-1}/k^2$, is absolutely convergent (and hence convergent).

Analogous to Remark 1.42 we have "$\sum z_k$ *is absolutely convergent iff each of* $\sum x_k$ *and* $\sum y_k$ *is absolutely convergent.*" The following results are often useful.

1.46. Theorem. *If* $\sum_{k\geq 1} |w_k|$ *is convergent and if* $|z_k| \leq |w_k|$ *except for finitely many* k*'s then* $\sum_{k\geq 1} z_k$ *is absolutely convergent.*

The above theorem, popularly known as the *comparison test* for convergence, can be easily deduced from the Cauchy criterion for convergence. For, let $\epsilon > 0$ be given. Then there is an $N \geq 1$ such that for all $n > m \geq N$,

$$|s_n - s_m| \leq \sum_{k\geq m+1}^{n} |z_k| \leq \sum_{k\geq m+1}^{n} |w_k| < \epsilon.$$

This observation proves the theorem. The series $\sum |w_k|$ is called a majorant of $\sum |z_k|$.

1.47. Example. If $z_n = e^{in}\cos(n^2)/n^{3/2}$, then $|z_n| \leq n^{-3/2}$ and hence $\sum z_n$ is convergent. ●

1.48. Theorem. *Let $\sum z_k$ be a series with nonzero terms such that*

$$\limsup_{n\to\infty} L_n = L \quad \textit{and} \quad \liminf_{n\to\infty} L_n = l \quad \textit{where} \quad L_n = |z_{n+1}/z_n|.$$

Then we have the following:

(a) *If $L < 1$, the series converges absolutely.*
(b) *If $l > 1$, the series diverges.*
(c) *If $l \leq 1 \leq L$ no conclusion can be made concerning the convergence of the series.*

Proof. Clearly $L, l \geq 0$. Suppose that $L < 1$. Then, for λ with $L < \lambda < 1$, there exists an integer N such that

$$\left|\frac{z_{n+1}}{z_n}\right| < \lambda \text{ for all } n \geq N$$

so that

$$|z_n| = |z_N| \cdot \left|\frac{z_{N+1}}{z_N}\right| \cdot \left|\frac{z_{N+2}}{z_{N+1}}\right| \cdots \left|\frac{z_n}{z_{n-1}}\right| < |z_N|\lambda^{n-N}.$$

Therefore, $|z_{N+p}| < |z_N|\lambda^p$ for $p \geq 1$. Since $\sum_{p\geq 1} |z_N|\lambda^p$ is a convergent (geometric) series, $\sum_{n\geq 1} |z_n|$ is convergent (see also Example 2.54), by the comparison test. This proves (a).

Next, we let $l > 1$. Then, by hypothesis, there exists an integer N such that $l > k > 1$ and

$$\left|\frac{z_{n+1}}{z_n}\right| > k \quad \text{for all} \quad n \geq N.$$

Therefore, for all $n > N$,

$$|z_n| = |z_N| \cdot \left|\frac{z_{N+1}}{z_N}\right| \cdot \left|\frac{z_{N+2}}{z_{N+1}}\right| \cdots \left|\frac{z_n}{z_{n-1}}\right| > |z_N|k^{n-N} \to \infty.$$

Hence, $z_n \not\to 0$ as $n \to \infty$ and so $\sum z_n$ diverges. This proves (b).

The last case follows by considering the series with $z_n = 1/n$ and $z_n = 1/n^2$. ■

1.49. Corollary. *Let $\sum z_k$ be a series of non-zero complex terms such that*

$$\lambda = \lim_{n\to\infty} L_n, \quad \textit{where} \quad L_n = |z_{n+1}/z_n|.$$

(a) *If $\lambda < 1$, the series converges absolutely.*

(b) *If $\lambda > 1$, the series diverges.*

(c) *If $\lambda = 1$, the series may converge or diverge.*

1.50. Example. Consider a series $\sum z_k$ with $L_n = |z_{n+1}/z_n|$.

(i) If $z_n = (1+i)^n/n$, then $L_n = \frac{n\sqrt{2}}{n+1} = \sqrt{2} - \frac{\sqrt{2}}{n+1} \to \sqrt{2}$ as $n \to \infty$ and hence the series diverges.

(ii) If $z_n = (1+i)^n/n!$, then $L_n = \frac{\sqrt{2}}{n+1} \to 0$ as $n \to \infty$ and so the series converges absolutely.

(iii) If $z_n = (n+1)(1+i)^n/n!$, then $L_n = \sqrt{2}\frac{(n+2)}{(n+1)^2} \to 0$ as $n \to \infty$ and therefore the series converges absolutely.

(iv) If a, b, c and d are real such that $|a^2+b^2| < |c^2+d^2|$, then we see that the series, with $z_n = (a+ib)^n/(c+id)^n$, is absolutely convergent. ●

1.51. Theorem. *Let $\sum z_k$ be a series of complex terms such that*

$$\limsup_{n\to\infty} |z_n|^{1/n} = L.$$

(a) If $L < 1$, the series converges absolutely.

(b) If $L > 1$, the series diverges.

(c) If $L = 1$, the series may converge or diverge.

Proof. (a) If $L < 1$, choose $\lambda > 0$ so that $L < \lambda < 1$ Then for all sufficiently large values of n, we have

$$|z_n|^{1/n} < \lambda, \quad \text{i.e} \quad |z_n| < \lambda^n$$

and so the convergence of $\sum |z_n|$ follows from the comparison test with the geometric series $\sum \lambda^n$.

(b) If $L > 1$, $|z_n| > 1$ for infinitely many n so that $z_n \not\to 0$ as $n \to \infty$ and thus, $\sum z_n$ diverges if $L > 1$.

(c) Examples relating to the proof of Theorem 1.48 can be used to justify the last assertion. ■

The Cauchy product of two convergent infinite series of complex terms $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ is the series $\sum_{n\geq 0} c_n$, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, 2, \ldots .$$

Next we state the following theorem. Proof of (a) follows directly from the definition and the hypothesis.

1.52. Theorem.

(a) *If $\{z_n\}$ has a limit point at z_0, then there exist a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\{z_{n_k}\} \to z_0$ as $k \to \infty$.*

(b) *If the sequence $\{\sum_{k=1}^n a_k\}$ is bounded and $\{b_n\}$ is a decreasing null sequence of positive numbers, then $\sum_{n\geq 1} a_n b_n$ converges.*

Proof. To prove (b), by hypothesis, we note that there exists an M such that

$$\left|\sum_{k=1}^{n} a_n\right| \leq M \quad \text{for all } n.$$

Since $\{b_n\}$ is a null sequence, given any $\epsilon > 0$ there exists an N such that $b_n < \epsilon/(2M)$ for all $n \geq N$. Now, for all $n \geq N$

$$\left|\sum_{k=n+1}^{m} a_k b_k\right| \leq M\left[\sum_{k=n+1}^{m}(b_k - b_{k+1}) + (b_{n+1} + b_{m+1})\right] \leq 2Mb_{n+1} < \epsilon.$$

Thus, $\{\sum_{k=1}^m a_k b_k\}$ is a Cauchy sequence and the result follows by the Cauchy criterion. ■

1.7 Exercises

1.53. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The set of points z such that $|z + bi| < |z + b|$ is the half-plane $\{z = x + iy : y < x\}$.

(b) The inequality $|z_1 - z_2| < |\overline{z}_1 + z_2|$ holds provided either $\mathrm{Re}\, z_1 > 0$ and $\mathrm{Re}\, z_2 > 0$, or $\mathrm{Re}\, z_1 < 0$ and $\mathrm{Re}\, z_2 < 0$.

(c) Two complex numbers z_1 and z_2 whose sum and difference are real and purely imaginary, respectively, must satisfy $z_2 = \overline{z}_1$.

(d) The equation $az + b\overline{z} + c = 0$ has exactly one solution, iff $|a| \neq |b|$.

(e) If the sum and product of two complex numbers are both real then either both the complex numbers are real or one is the conjugate of the other.

(f) The inequalities $\mathrm{Re}\, z > 0$ and $|z - 1| < |z + 1|$ are equivalent.

(g) The inequalities $|z| < 1$ and $\mathrm{Re}\left(\dfrac{1+z}{1-z}\right) > 0$ are equivalent.

(h) The inequalities $\mathrm{Re}\, z \geq \delta > 0$ and $\left|\dfrac{1}{z} - \dfrac{1}{2\delta}\right| \leq \dfrac{1}{2\delta}$ are equivalent.

(i) The roots of the cubic equation $(z + \alpha\beta)^3 = \alpha^3$ $(\alpha \neq 0)$ form the vertices of a triangle each side of which is of length equal to $\sqrt{3}|\alpha|$.

(j) For $\mathrm{Re}\, z_j > 0$ $(j = 1, 2)$, $\mathrm{Arg}\,(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$.

(k) For any positive integer n, $|\mathrm{Im}\, z^n| \le n|\mathrm{Im}\, z|\, |z|^{n-1}$.

(l) If $|z| < 1$, then both $\mathrm{Re}\,(1+z) > 0$ and $\mathrm{Re}\,(1+z)^2 > 0$ do not hold simultaneously.

(m) If $z^2 = (\overline{z})^2$, then z is either real or purely imaginary.

(n) The equation $\dfrac{1+\cos\phi + i\sin\phi}{1-\cos\phi + i\sin\phi} = e^{i(\phi-\pi/2)}\cot(\phi/2)$ holds for each $\phi \neq 2n\pi$, $n \in \mathbb{Z}$.

(o) The non-real roots of $(1+z)^4 = 16z^4$ are $(-1 \pm 2i)/5$.

(p) The product of the distinct n-th roots of unity is $(-1)^{n-1}$.

(q) All the solutions of $z^4 + 81 = 0$ are $3[\pm 1 \pm i]/\sqrt{2}$.

(r) The complex roots of a quadratic equation have the property that one is the square of the other.

(s) If $z = a + ib$, where a and b both are integers, then

 (i) $|1 + z + z^2 + \cdots + z^n| \ge |z|^n$ if $a > 0$

 (ii) $|1 + z + z^2 + \cdots + z^n| \le |z|^n$ if $a < 0$.

(t) The set of complex numbers z such that $\arg(z - i) = \pi/3$ represents the equation of the straight line $y = \sqrt{3}x + 1$.

(u) If $\mathrm{Arg}\,(z+3) = \pi/3$, then the least value of $|z|$ is $3\sqrt{3}/2$.

(v) If $|z - (4 - 3i)| = 2$, then the greatest and least value of $|z|$ are 7 and 3, respectively.

(w) Convergence of $\{z_n\}$ implies the convergence of $\{Z_n\}$, where $Z_n = \frac{1}{n}\sum_{k=1}^{n} z_k$.

(x) For $z \neq a$ and $n \in \mathbb{N}$, we have $\left|\dfrac{z^n - a^n}{z - a}\right| \le \dfrac{|z|^n - |a|^n}{|z| - |a|}$.

(y) If ω is an n-th root of unity, then $\displaystyle\sum_{j=0}^{n-1} |z_1 + \omega^j z_2|^2 = n[|z_1|^2 + |z_2|^2]$.

Note: For $n = 2$, this reduces to the Parallelogram identity.

(z) If $z_n \neq 0$ and $z_n \to \ell \neq 0$, then $\mathrm{Arg}\, z_n \to \mathrm{Arg}\,\ell$.

1.54. For any two non-zero complex numbers z_1 and z_2, prove the following

(a) $\mathrm{Arg}\,(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 + 2\pi k_1$

(b) $\mathrm{Arg}\,(z_1/z_2) = \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2 + 2\pi k_2$,

where k_n $(n = 1, 2)$, which depends on z_1 and z_2, is given by

$$k_n = \begin{cases} 1 & \text{if } -2\pi < \mathrm{Arg}\, z_1 + (-1)^{n-1}\mathrm{Arg}\, z_2 \le -\pi \\ 0 & \text{if } -\pi < \mathrm{Arg}\, z_1 + (-1)^{n-1}\mathrm{Arg}\, z_2 \le \pi \\ -1 & \text{if } \pi < \mathrm{Arg}\, z_1 + (-1)^{n-1}\mathrm{Arg}\, z_2 \le 2\pi. \end{cases}$$

1.55. Prove Lagrange's identity (using the method of induction):

$$\left|\sum_{k=1}^{m} z_k w_k\right|^2 = \left(\sum_{k=1}^{m} |z_k|^2\right)\left(\sum_{k=1}^{m} |w_k|^2\right) - \sum_{k\leq j} |z_k \overline{w}_j - z_j \overline{w}_k|^2.$$

1.56. If z_1, z_2, z_3 are the vertices of an equilateral triangle, then show that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$. If $z_1 = 1 + i$ and $z_2 = 1 - i$, determine the two possible values of z_3 so that z_1, z_2, z_3 form an equilateral triangle.

1.57. Compute the limit of the sequence $\{z_n\}$ when z_n equals

$n^{1/n}$, $3^{-n+i\sqrt{n}}$, $2^n/n$, $n\sin(1/n)$, n^2e^{-n}, $i^{n!}$.

1.58. Test the convergence of $\sum_{n=1}^{\infty} z_n$ when z_n equals

$n(3+i)^n$, $n^3 3^n (2+i)^{-n}$, $n^{-1}i^n$, 5^{-n}, $(n!)^2/(3n!)$, $n^{-1/3}i^{3n}$.

1.59. Classify the following sets according to the properties open, closed, bounded, unbounded, compact:

(a) $S_1 = \{z : z = e^{2\pi ki/5},\ k = 0, 1, 2, 3, \dots\}$
(b) $S_2 = \{z : |z| > 2|z-1|\}$
(c) $S_4 = \{z : \mathrm{Re}\,(iz) < 1\}$
(d) $S_5 = \{z : |z+2| \geq |z|\}$
(e) $S_6 = \{z : z^2 + \overline{z}^2 = 1\}$.

1.60. Supply the geometric description of the following subsets of $\mathbb{C}$:

(a) $S_1 = \{z : 0 < \mathrm{Re}\,(iz) \leq 1\}$
(b) $S_2 = \left\{z : \mathrm{Im}\left(\frac{z-z_1}{z-z_2}\right) = 0,\ \text{for } z_1, z_2 \in \mathbb{C}\right\}$
(c) $S_3 = \left\{z : \mathrm{Re}\left(\frac{z-z_1}{z-z_2}\right) = 0,\ \text{for } z_1, z_2 \in \mathbb{C}\right\}$
(d) $S_4 = \{z : |z-1| = 3|z+1|\}$
(e) $S_5 = \{z : |z-1| = \mathrm{Re}\,(z)|\}$
(f) $S_6 = \{z : |z+1| > 2,\ 0 < \mathrm{Arg}\,(z-2) < \pi/4\}$
(g) $S_7 = \{z : -\pi/4 < \arg(z-k) < 3\pi/2,\ k = 1, 2, \dots, 5\}$
(h) $S_8 = \{z : -\pi/6 < \arg(z-1/k) < 13\pi/6,\ k = 1, 2, \dots, 5\}$.

1.61. Find all circles that are orthogonal to both $|z-1| = 4$ and $|z| = 1$.

Chapter 2

Functions, Limit and Continuity

In this chapter we introduce fundamental results concerning limits, continuity and uniform convergence of sequences and series of functions. We also introduce sets in the extended complex plane. This chapter lays the ground work for a careful treatment of analytic functions of a complex variable. In Section 2.1, we define functions in $\mathbb{C}$ and their elementary properties such as one-to-one and onto. In Section 2.2, we briefly present basic facts about limits and continuous functions of a complex variable. In Section 2.3, we formalize the notion of 'the point at infinity'. This helps to extend the notion of limit and continuity for functions defined on unbounded sets. This section also provides a convenient way for discussing the behavior of functions as $|z|$ gets large and for certain generalization of domains in the complex plane.

2.1 One-to-one and Onto Functions

Let A and B be two non-empty subsets of $\mathbb{C}$. A *function* from A to B is a rule, f, which assigns each $z_0 = x_0 + iy_0 \in A$ a unique element $w_0 = u_0 + iv_0 \in B$. The number w_0 is called the value of f at z_0 and we write $w_0 = f(z_0)$. If z varies in A then $w = f(z)$ varies in B. We say that f is a *complex function* of a complex variable in A. We say that f is a function defined on A. We also write

$$f : A \to B, \quad z \mapsto w = f(z). \tag{2.1}$$

Here z is called the independent variable; w the dependent variable and A the domain of f. If $S \subseteq A$, we can have $f : S \to B$ and we call this new function the restriction of f in (2.1) to S and denote it by $f\big|_S$.

Let us examine the graphical representation of (2.1), i.e. $w = f(z)$. By definition, for each $z = x + iy$,

$$w = f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z),$$

w being a complex number. Identifying $z = x + iy$ with $(x, y) \in \mathbb{R}^2$, we have the functions

$$(x, y) \mapsto \text{Re}\, f(z),\ (x, y) \mapsto \text{Im}\, f(z),$$

or equivalently

$$\text{Re}\, f : z \mapsto \text{Re}\, f(z), \quad \text{Im}\, f : z \mapsto \text{Im}\, f(z),$$

defined on the domain A, now considered a subset of $\mathbb{R}^2$. We observe that these are real-valued functions defined on $A \subset \mathbb{R}^2$ and denote them by $\text{Re}\, f$, $\text{Im}\, f$, respectively. Conversely, if $A \subset \mathbb{R}^2$ and we have two real-valued functions

$$u : A \to \mathbb{R},\ v : A \to \mathbb{R},$$

then, by defining

$$f(z) = u(x, y) + iv(x, y), \quad (x, y) \in A,$$

we obtain $f : A \to \mathbb{C}$, where A is now considered as a subset of $\mathbb{C}$. Thus, a complex function of a complex variable is completely determined by the functions $\text{Re}\, f$, $\text{Im}\, f$, known as the real and imaginary parts.

If we use the polar form for z, then f can be written as

$$f(z) = u(r, \theta) + iv(r, \theta).$$

To illustrate these ideas let $f(z) = 2z^2 - z + 1$, $z \in \mathbb{C}$. Then

$$f(z) = [2(x^2 - y^2) - x + 1] + i[4xy - y].$$

Thus, $(\text{Re}\, f)(z) = 2(x^2 - y^2) - x + 1$ and $(\text{Im}\, f)(z) = 4xy - y$. On the other hand if we write $z = re^{i\theta}$, then

$$f(z) = [2r^2 \cos 2\theta - r\cos\theta + 1] + i[2r^2 \sin 2\theta - r\sin\theta].$$

Since each of the variables z and w requires two dimensions for representation, a graphical representation cannot be given for a complex function. Because of this, we need to employ two different copies of the complex plane to describe the nature of a complex function of a complex variable. Thus, to every $z = x + iy$ of its domain in the z-plane, we determine the resulting values of $u = \text{Re}\, f$ and $v = \text{Im}\, f$ $(w = u + iv)$ and plot them in the w-plane (see Figure 2.1).

If f is defined on A and $S \subseteq A$, then $f(S) = \{f(z) : z \in S\}$ is called the *image* of the set S under f. The set $R = f(A)$ of all image points of A is called the *range* of f.

If $R \subseteq B$, the function f on A to B is called a mapping of A *into* B and if $R = B$, we say that the function f maps A *onto* the range R since every element $w \in R$ is an image of at least one point in A. Clearly, a mapping

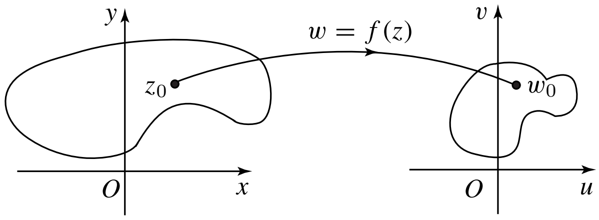

Figure 2.1: Description for mapping.

which is "onto" is "into" but the converse is not necessarily true. Point $w \in R$ is called an image of a point $z \in A$ and the point z is called the pre-image of w, under the mapping[5] $w = f(z)$. A mapping is said to be an *open mapping* if it maps open sets onto open sets.

We know that every complex number $z \in \mathbb{C}\setminus\{0\}$ can be uniquely written in polar form as

$$z(r,\theta) = re^{i\theta} \quad (r = |z|,\ \theta \in (-\pi, \pi]).$$

If we increase θ by 2π,

$$z(r,\theta + 2\pi) = re^{i(\theta+2\pi)} = re^{i\theta} = z(r,\theta)$$

returning to its original value. With this observation, the definition of single-valuedness of a function takes the following simple form in polar coordinates. A function f is said to be *single-valued* if f satisfies

$$f(z) = f(z(r,\theta)) = f(z(r,\theta + 2\pi)). \tag{2.2}$$

Otherwise, f is said to be *multiple-valued.*

Let us look at the function $f(z) = z^n$. Then, when n is an integer, we know that for every $w \neq 0$ there are n values of z satisfying $w = f(z) = z^n$. We have

$$f(z(r,\theta)) = r^n e^{in\theta}, \quad f(z(r,\theta + 2\pi)) = r^n e^{in\theta} e^{2\pi ni}.$$

So, the condition (2.2) is satisfied iff n is an integer. This shows that the function $f(z) = z^n$ is single-valued iff n is an integer. If n is not an integer, then $e^{2\pi ni} \neq 1$ and so f in this case is multiple-valued.

If the elements of A are complex numbers and those of B are real numbers, then we say that f is a real-valued function of a complex variable. Similarly, if the elements of A are real and those of B are complex numbers, then f is a complex-valued function of a real variable. Whenever we speak of a function we shall, unless otherwise stated explicitly, consider a single-valued function.

[5]'Mapping' is another word for function, transformation.

Suppose we have a mapping of a set A onto a set R. It may happen that some points of R have more than one pre-image. If each $w \in R$ is the image of precisely one point in A; that is

$$f(z_1) = f(z_2) \implies z_1 = z_2; \quad \text{or} \quad z_1 \neq z_2 \implies f(z_1) \neq f(z_2),$$

then the mapping $w = f(z)$ is said to be *one-to-one*. If the mapping $w = f(z)$ is one-to-one then the function f is said to be *univalent*, a fancy term for one-to-one functions. (Note that the function f is said to be univalent at z_0 if it is univalent in a neighborhood of z_0. This will be discussed later in this book.) In this case, we have a mapping from R into the z-plane with A as the range and R as the domain of definition. Denoting the latter mapping, called the *inverse* of f, by f^{-1} we write

$$z = f^{-1}(w) \text{ if } w = f(z).$$

Thus, if f maps A in a one-to-one fashion onto R, then there is an inverse mapping $z = f^{-1}(w)$ on R onto A. Note that $f^{-1}(w) = f^{-1}(f(z)) = z$. Observe that if $f : A \to R$ is univalent on A, then f^{-1} is defined on R and is univalent therein.

A function f with domain A and range B is called a *constant function* if B contains only one element, say c. In this case, we write $f(z) \equiv c$, $z \in A$.

A function $f(z)$ is said to be *bounded* on a subset $S \subseteq \mathbb{C}$ if there exists an $M > 0$ such that $|f(z)| \leq M$ for all $z \in S$.

Suppose we have a function f with domain D_1 and another function g with domain D_2. Suppose further that, for each $z \in D_1$, $f(z)$ is in D_2. Then for every $z \in D_1$ the association $g \circ f$ defined by $(g \circ f)(z) = g(f(z))$, is a function called the *composition* of f and g. We indicate this by

$$D_1 \xrightarrow{f} D_2 \xrightarrow{g} \mathbb{C}.$$

$$g \circ f$$

For instance, let $f(z) = 2z + 1$ and $g(z) = z^2 + 2$ on $\mathbb{C}$. Then,

$$g(f(z)) = (2z+1)^2 + 2 \text{ and } f(g(z)) = 2(z^2+2) + 1, \; z \in \mathbb{C}.$$

If $f(z) = u(z) + iv(z) \equiv u(x,y) + iv(x,y)$, where u, v are real-valued functions, then

$$\overline{f(z)} = u(x,y) - iv(x,y) \text{ and } f(\overline{z}) = u(x,-y) + iv(x,-y).$$

Thus, we note that $\overline{f(z)}$ and $f(\overline{z})$ are in general different functions. For instance, if $f(z) = (1-i)z$, f may be rewritten as $f(z) = (x+y) + i(y-x)$ so that $\overline{f(z)} = (1+i)\overline{z}$ and $f(\overline{z}) = (1-i)\overline{z}$.

2.3. Examples. Consider $w = f(z) = z^2$, $|z| \leq 1$. Here the domain of definition of f is $\overline{\Delta}$ and the range is also $\overline{\Delta}$. An angular region with

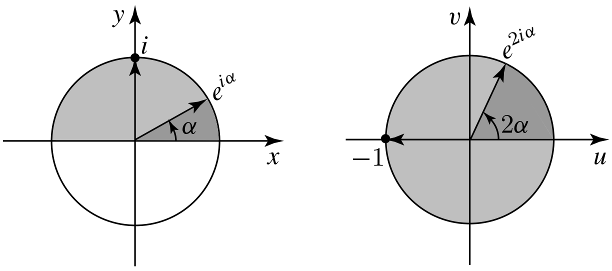

Figure 2.2: Mapping under $w = z^2$, $|z| \leq 1$.

vertex at 0 of α radians in the z-plane is mapped into an angular region of 2α radians in the w-plane, under this mapping (see Figure 2.2). The fact that $f(z) = z^2$ is not univalent on $\overline{\Delta}$ can be seen as follows: let $z = re^{i\theta}$ $(0 < r \leq 1)$, be the polar representation of the points on $\overline{\Delta}$. Then

$$w = r^2 e^{2i\theta} = \rho e^{i\phi} \quad \text{with } \rho = r^2 \text{ and } \phi = 2\theta,\ 0 \leq \phi \leq 4\pi.$$

So, $z_1 = re^{i\theta}$, $z_2 = re^{i(\theta+\pi)} = -z_1$ are such that

$$z_1^2 = r^2 e^{2i\theta} = r^2 e^{2i\theta + 2i\pi} = r^2 e^{2i(\theta+\pi)} = z_2^2.$$

This shows that the function is not univalent. In fact, it maps $\overline{\Delta}$ onto $\overline{\Delta}$ twice.

Next, we consider $w = f(z) = z^2$, $z \in D = \{z : |z| \leq 1,\ 0 \leq \arg z < \pi\}$. Then the range is $\overline{\Delta}$. Thus, if $z = re^{i\theta}$ then $w = r^2 e^{2i\theta} = \rho e^{i\phi}$ with

$$\sqrt{\rho} = r\ (0 < \rho < 1) \ \text{ and } \ \phi = 2\theta,\ 0 \leq \phi < 2\pi.$$

If the two pre-images (why not more?) for $w = \rho e^{i\phi}$, where $\rho = r^2$ and $\phi = 2\theta$ by $f(z) = z^2$, are $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+\pi)}$, then only one of z_1 or z_2 can lie in D since any two elements of D have their arguments differing by less than π. This shows that the function is univalent in D.

Again we consider the same function $f(z) = z^2$, but this time without indicating the domain of definition. Then, we have

$$f(z_1) = f(z_2) \implies z_1^2 - z_2^2 = 0 \implies \text{ either } \ z_1 = z_2 \ \text{ or } \ z_1 = -z_2.$$

The points z_1 and z_2 such that $z_1 = -z_2$ are symmetric with respect to the origin, i.e. lie at the same distance from the origin on the same straight line through the origin. This shows that $f(z) = z^2$ is univalent in a domain D iff this domain does not have even a single pair of points symmetric with respect to the origin. For instance, $f(z) = z^2$ is univalent in the upper half-plane $U = \{z : \operatorname{Im} z > 0\}$, the lower half-plane $L = \{z : \operatorname{Im} z < 0\}$, respectively. Note also that f maps U and L, into the w-plane with a cut along the negative real semi-axis omitting the origin. ●

2.4. Examples. Consider $w = f(z) = \overline{z}$. Observe that the effect of this mapping on the points of the plane is a reflection on the real axis. This function is one-to-one and the inverse is $z = \overline{w}$. It maps the entire z-plane onto the entire w-plane.

On the other hand, the function $w = g(z) = |z|$ maps the entire z-plane onto the non-negative real axis of the w-plane. In fact, it maps every circle centered at the origin to a point. It is not one-to-one and hence no inverse exists. ●

2.5. Examples. It is easy to see that the function $f(z) = 3z + z^2$ is one-to-one in Δ. Indeed, for z_1, z_2 in Δ,

$$f(z_1) = f(z_2) \implies (z_1 - z_2)(z_1 + z_2 + 3) = 0 \implies z_1 - z_2 = 0$$

($z_1 + z_2 + 3 \neq 0$ in Δ, since $\mathrm{Re}\,(z_1 + z_2 + 3) > -1 - 1 + 3 = 1$).

Similarly, it is easy to see that the function $f(z) = (1+z)^2$ is univalent in Δ. More generally, $f(z) = z + \alpha z^2$ is univalent for $|z| < 1/(2|\alpha|)$ which is in fact the largest disk centered at the origin on which f is one-to-one. This is easily seen using the argument relating to $f(z) = 3z + z^2$. ●

A domain D is said to be *symmetric* with respect to the origin, if for every $z \in D$, the point $-z \in D$. A function f defined on the domain D which is symmetric with respect to origin is said to be *even* if $f(z) = f(-z)$ is valid for every $z \in D$. For example, $f(z) = z^{2n}$ ($n \in \mathbb{N}$) is even in $\mathbb{C}$. A function f is said to be *odd* on D if $f(z) = -f(-z)$ for every $z \in D$. For example, $f(z) = z^{2n-1}$ ($n \in \mathbb{N}$) is an odd function in $\mathbb{C}$.

A domain D is said to be starlike with respect to a point $z_0 \in D$ if for each $z \in D$ the line segment, $[z_0, z]$, from z_0 to z lies entirely in D. The point z_0 is called a *star center* of D. A *starlike domain* is a domain which is starlike with respect to the origin. A domain D is said to be *convex* if for each pair of points $\zeta, z \in D$, the line segment, $[\zeta, z]$, joining z and ζ lies entirely in D. Obviously, a convex domain is starlike with respect to any of its points.

An open disk, half-planes such as $\mathrm{Re}\, z > 0$, an open ellipse and an open rectangle are examples of convex domain. An example of a starlike domain but non-convex is the domain $\mathbb{C}\backslash\{z = -x : x \geq 1\}$, i.e. the plane minus the negative real axis from -1. How about the set $\mathbb{C}\backslash\{x + iy : x = 0, |y| \geq 1\}$?

2.2 Concepts of Limit and Continuity

The definitions of limit, continuity and uniform continuity are analogous to those in Real Analysis. Suppose that a complex-valued function f is defined on $D \subseteq \mathbb{C}$ and $z_0 \in \overline{D}$. Then f is said to have a *limit* ℓ as $z \to z_0$ and we write

$$\lim_{z \to z_0} f(z) = \ell \quad \text{or} \quad f(z) \to \ell \quad \text{as} \quad z \to z_0$$

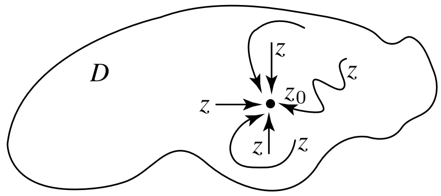

Figure 2.3: $z \to z_0$ in $\mathbb{C}$.

iff for any given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, z_0) > 0$ such that

$$|f(z) - \ell| < \epsilon \text{ whenever } z \in D \text{ and } 0 < |z - z_0| < \delta;$$

i.e. iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that (see 1.6)

$$f(z) \in \Delta(\ell;\epsilon) \quad \text{whenever} \quad z \in [\Delta(z_0;\delta) \setminus \{z_0\}] \cap D.$$

It is straightforward to state

$$\lim_{z \to z_0} f(z) = l \iff \lim_{z \to z_0} |f(z) - \ell| = 0.$$

Less precisely stated, this means that if z is near z_0, then $f(z)$ is close to ℓ.

First, it should be noted that the function need not be defined at z_0 in order to have a limit at z_0. Secondly, it is the punctured disk $\Delta(z_0;\delta) \setminus \{z_0\}$ which is involved in D, i.e. z_0 need not be in D. Thirdly, even if the condition that $z_0 \in D$ holds, we may have $f(z_0) \neq \ell$. In real variable theory, we do not have the freedom which a complex variable produces for, if $z_0 = x_0 \in \mathbb{R}$, a neighboring point $z = x \to x_0$ has only two possible ways either from the left or from the right. In the complex case, z can approach z_0 in any manner in the complex plane (see Figure 2.3).

As in Real Analysis, if the limit exists then it must be unique: Suppose that

$$\lim_{z \to z_0} f(z) = \ell \quad \text{and} \quad \lim_{z \to z_0} f(z) = \ell' \quad \text{with} \quad \ell \neq \ell'.$$

Then for a given $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$|f(z) - \ell| < \epsilon \text{ whenever } z \in D \text{ and } 0 < |z - z_0| < \delta_1$$

and

$$|f(z) - \ell'| < \epsilon \text{ whenever } z \in D \text{ and } 0 < |z - z_0| < \delta_2.$$

Therefore, whenever $z \in D$ and $0 < |z - z_0| < \delta = \min\{\delta_1, \delta_2\}$,

$$|\ell - \ell'| = |(f(z) - \ell') - (f(z) - \ell)| \leq |f(z) - \ell'| + |f(z) - \ell| < \epsilon + \epsilon = 2\epsilon.$$

Both the left and the right side of the above inequality are independent of δ. As ϵ is arbitrary, the inequality can hold iff $\ell = \ell'$.

Suppose that f, g are functions defined on $D \subseteq \mathbb{C}$. Then $f + g$ is the function defined on D by

$$(f+g)(z) = f(z) + g(z), \quad z \in D.$$

Similarly, we define $(fg)(z) = f(z)g(z)$ for $z \in D$, and

$$\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}, \quad z \in D \text{ when } g(z) \neq 0 \text{ in } D.$$

The following theorem gives the "complex" theory of limits from the "real" theory, and conversely.

2.6. Theorem. *Let $f(z) = u(z) + iv(z)$, where $u(z) = u(x, y)$ and $v(z) = v(x, y)$ are real-valued functions, be defined on D except possibly at z_0. Then for ℓ and $\ell' \in \mathbb{R}$,*

$$\lim_{z \to z_0} f(z) = \ell_1 + i\ell_2 \tag{2.7}$$

iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \ell_1 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = \ell_2. \tag{2.8}$$

Proof. The proof of this theorem follows immediately from Remark 1.42. However, we include the details here. Suppose that the limit (2.7) exists. Using the triangle inequality it follows that the inequalities

$$|u(x, y) - \ell_1|, \ |v(x, y) - \ell_2| \leq |f(z) - (\ell_1 + i\ell_2)|$$

and $|x - x_0|, \ |y - y_0| \leq |z - z_0|$ are satisfied.

Now if we allow $z \to z_0$, i.e. $(x, y) \to (x_0, y_0)$, it is evident that the two limits (2.8) exist. Conversely, suppose that the two limits (2.8) exist. Then, for a given $\epsilon > 0$, there exist δ_1 and δ_2 such that

$$|u(x, y) - \ell_1| < \epsilon/2 \ \text{ whenever } \ 0 < |(x - x_0) + i(y - y_0)| < \delta_1$$

and

$$|v(x, y) - \ell_2| < \epsilon/2 \ \text{ whenever } \ 0 < |(x - x_0) + i(y - y_0)| < \delta_2.$$

By the triangle inequality

$$\begin{aligned} |f(z) - \ell| &= |(u(x, y) - \ell_1) + i(v(x, y) - \ell_2)| \\ &\leq |u(x, y) - \ell_1| + |v(x, y) - \ell_2| \\ &< (\epsilon/2) + (\epsilon/2) = \epsilon \end{aligned}$$

whenever $0 < |z - z_0| < \delta = \min\{\delta_1, \delta_2\}$ and the proof is complete. ■

2.9. Example. For $z \neq 0$, consider

$$f(z) = \frac{\overline{z}}{z} = \frac{(\overline{z})^2}{|z|^2} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}.$$

If we set $f(x+iy) = u(x,y) + iv(x,y)$, $u(x,y), v(x,y)$ being real, then

$$u(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad v(x,y) = -\frac{2xy}{x^2 + y^2}.$$

For each point $z_0 \neq 0$, it is clear that $\lim_{z \to z_0} f(z)$ exists and equals $f(z_0)$. We shall see whether $\lim_{z \to 0} f(z)$ exists or not, by examining the limits as $z \to 0$ in many ways. Let m be any real number and allow $z \to 0$ along the line $y = mx$. Then

$$f(x + mxi) = \left(\frac{1 - m^2}{1 + m^2}\right) - i\left(\frac{2m}{1 + m^2}\right)$$

which clearly shows that $\lim_{z \to 0} f(z)$ does not exist. ●

2.10. Theorem. *Let f and g be defined in a neighborhood of z_0 except possibly at z_0. Given $\lim\limits_{z \to z_0} f(z) = \ell$ and $\lim\limits_{z \to z_0} g(z) = \ell'$, we have*

(a) $\lim\limits_{z \to z_0} [f(z) + g(z)] = \ell + \ell'$

(b) $\lim\limits_{z \to z_0} [f(z)g(z)] = \ell\ell'$

(c) $\lim\limits_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\ell}{\ell'}$ *if* $\ell' \neq 0$.

In particular, For a, b complex constants, $\lim\limits_{z \to z_0} (az + b) = az_0 + b$.

Proof. By hypothesis, given $\epsilon_1 > 0$, there exists $\delta_f = \delta_f(z_0, \epsilon_1) > 0$ such that

(2.11) $$0 < |z - z_0| < \delta_f \Longrightarrow |f(z) - \ell| < \epsilon_1.$$

Similarly, given $\epsilon_2 > 0$, there exists $\delta_g = \delta_g(z_0, \epsilon_2) > 0$ such that

(2.12) $$0 < |z - z_0| < \delta_g \Longrightarrow |g(z) - \ell'| < \epsilon_2.$$

(a) Then for all z with $0 < |z - z_0| < \delta = \min\{\delta_f, \delta_g\}$ it follows from the triangle inequality that

$$|f(z) + g(z) - (\ell + \ell')| \leq |f(z) - \ell| + |g(z) - \ell'| < \epsilon_1 + \epsilon_2.$$

Given $\epsilon > 0$, choosing ϵ_1 and ϵ_2 so that $\epsilon_1 + \epsilon_2 = \epsilon$, we see that (a) follows from the last inequality.

(b) Now for all z such that $0 < |z - z_0| < \delta$, it follows from the triangle inequality that

$$\begin{aligned} |f(z)g(z) - \ell\ell'| &= |g(z)(f(z) - \ell) + \ell(g(z) - \ell')| \\ &\leq |g(z)|\,|f(z) - \ell| + |\ell|\,|g(z) - \ell'| \\ &\leq [|g(z) - \ell'| + |\ell'|]|f(z) - \ell| + |\ell|\,|g(z) - \ell'| \\ &\leq [\epsilon_2 + |\ell'|]\epsilon_1 + |\ell|\epsilon_2 \end{aligned}$$

and the result follows as ϵ_1 and ϵ_2 are arbitrary.

(c) If we choose $\epsilon_2 = |\ell'|/2$, as $-|g(z)| + \ell' \leq |g(z) - \ell'|$, (2.12) shows that

(2.13) $$0 < |z - z_0| < \delta_g \Longrightarrow |g(z)| > |\ell'|/2.$$

So $g(z) \neq 0$ in the deleted neighborhood $\Delta(z_0; \delta_g) \backslash \{z_0\}$. Now for all z such that $0 < |z - z_0| < \delta$, it follows that for $\epsilon_2 \leq |\ell'|/2$

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - \frac{\ell}{\ell'} \right| &= \left| \frac{f(z)\ell' - g(z)\ell}{g(z)\ell'} \right| \\ &\leq \left| \frac{(f(z) - \ell)\ell' - \ell(g(z) - \ell')}{g(z)\ell'} \right| \\ &\leq [\epsilon_1|\ell'| + \epsilon_2|\ell|] \left[\frac{2}{|\ell'|} \cdot \frac{1}{|\ell'|} \right], \quad \text{by} \quad (2.13), \\ &= \frac{2(\epsilon_1|\ell'| + \epsilon_2|\ell|)}{|\ell'|^2}. \end{aligned}$$

Again the result follows as ϵ_1 and ϵ_2 are arbitrary. ∎

A function $f : D \to \mathbb{C}$ is *continuous* at $z_0 \in D$ iff $\lim_{z \to z_0} f(z)$ exists and equals the function value $f(z_0)$. We say that f is continuous on D or $f : D \to \mathbb{C}$ is continuous when f is continuous at all points of D. Note that f is continuous at z_0 iff the following three conditions hold:

$$f(z_0) \text{ is defined, } \lim_{z \to z_0} f(z) \text{ exists, and } \lim_{z \to z_0} f(z) = f(z_0).$$

In terms of our earlier notation, the definition of continuity is that for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } z \in D \text{ and } |z - z_0| < \delta,$$

or equivalently,

(2.14) $$f(z) \in \Delta(f(z_0); \epsilon) \quad \text{whenever} \quad z \in \Delta(z_0; \delta) \cap D.$$

A function $f : D \to \mathbb{C}$ is discontinuous (or has a discontinuity) at a point z_0 if $z_0 \in D$, yet f is not continuous at z_0.

2.15. Remark. If the domain D of f is such that $z_0 \in D$ and $\Delta(z_0;\delta) \subset D$ for some $\delta > 0$, then all the points in $\Delta(z_0;\delta)$ for this δ or a smaller δ' are to satisfy (2.14). It might happen that $z_0 \in D$ is such that there exists a $\delta > 0$ with only z_0 in $\Delta(z_0;\delta) \cap D$; i.e. z_0 is an isolated point of D. In this case (2.14) is trivially satisfied and f is continuous at z_0. Avoiding this trivial case we shall, henceforth assume that when we consider $z_0 \in D$ for continuity of $f : D \to \mathbb{C}$, z_0 is an interior point of D. If we are concerned with continuity of f on D we shall assume D to be open (since D is open, there exists an r such that $z_0 + h \in D$ with $|h| < r$). •

In the definition of f at z_0, the number δ depends on z_0 and ϵ. When, given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, independent of z_0, satisfying (2.14) for all $z_0 \in D$ then we say that f is *uniformly continuous* on D. Clearly every uniformly continuous function on D is continuous on D, but the converse is not true in general (see Examples 2.28).

2.16. Example. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 0 & \text{if } z = -(1+2i) \\ \dfrac{z^2 - 3z - 10i}{z + (1+2i)} & \text{if } z \neq -(1+2i). \end{cases}$$

As $z^2 - 3z - 10i = [z - (4+2i)][z + (1+2i)]$, we may rewrite

$$f(z) = \begin{cases} 0 & \text{if } z = -(1+2i) \\ z - (4+2i) & \text{if } z \neq -(1+2i), \end{cases}$$

so that

$$\lim_{z \to -(1+2i)} f(z) = -(5+4i) \neq f(-1-2i).$$

To verify this using '$\epsilon - \delta$' notation, let $\epsilon > 0$ be given. For $z \neq -1-2i$, we obtain that $|f(z) + 5 + 4i| = |z + (1+2i)|$. Therefore,

$$|f(z) - (-5-4i)| < \epsilon \quad \text{whenever} \quad 0 < |z - (-1-2i)| < \delta = \epsilon$$

so that $\lim_{z \to -(1+2i)} f(z) = -(5+4i)$. •

In the definition of limit, if $\ell \neq f(z_0)$, then f is said to have a *removable discontinuity* at $z_0 \in D$ (i.e. if the value of f is "corrected" at the point z_0, it becomes continuous there). This can also be done when $f(z_0)$ is not defined there. For instance in the above example if we define $f(-1-2i) = -5-4i$, then f becomes continuous at $z = -1-2i$ and at all other points in $\mathbb{C}$. As an immediate consequence of Theorem 2.6 we have

2.17. Theorem. *The function $f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0). In other words,* $\operatorname{Re} f$, $\operatorname{Im} f$ *are both continuous iff f is continuous.*

Consider $f(z) = |z|^2$. As the real and imaginary parts of f are continuous functions of x and y for all $(x, y) \in \mathbb{R}^2$, $f(z)$ is continuous on $\mathbb{C}$.

As a result of the properties of limits (see Theorem 2.10), we obtain the following theorem.

2.18. Theorem. *If $f, g : D \to \mathbb{C}$ are continuous at $z_0 \in D$, then their sum $f + g$, product fg, quotient f/g where $g(z_0) \neq 0$, and $|f|$ are also continuous at z_0. In particular, every polynomial $a_0 + a_1 z + \cdots + a_n z^n$ is continuous for every z in $\mathbb{C}$.*

Here are some examples of functions whose continuity follow from Theorem 2.18. The function f defined by

$$f(z) = \frac{z^3 + 3}{z^2 + 4}$$

is continuous on $\mathbb{C} \backslash \{2i, -2i\}$. More generally, Theorem 2.18 shows that any rational function $p(z)/q(z)$, where p and q are polynomials, is continuous on $\mathbb{C} \backslash \{z : q(z) = 0\}$. For example, a function of the form

$$\frac{a_{-n}}{z^n} + \frac{a_{-(n-1)}}{z^{n-1}} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots + a_n z^n$$

is continuous on the punctured plane $\mathbb{C} \backslash \{0\}$.

2.19. Theorem. *If $\lim_{z \to z_0} f(z) = w_0$ and g is a function which is continuous at the point w_0, then $\lim_{z \to z_0} (g \circ f)(z) = g(w_0)$.*

Proof. Continuity of g at w_0 implies that for a given $\epsilon > 0$, there exists a $\delta_g > 0$ such that

$$|g(w) - g(w_0)| < \epsilon \quad \text{whenever } |w - w_0| < \delta_g. \tag{2.20}$$

Further, by the condition on f, for this $\delta_g > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \delta_g \quad \text{whenever} \quad z \in \Delta(z_0; \delta) \backslash \{z_0\}.$$

Now if we let $w = f(z)$ in (2.20) we see that for all $z \in \Delta(z_0; \delta) \backslash \{z_0\}$,

$$|(g \circ f)(z) - g(w_0)| = |g(f(z)) - g(w_0)| < \epsilon$$

from which we obtain the required conclusion. ■

2.21. Corollary. *The composition of two continuous functions is continuous; i.e. if $f : D_1 \to D_2$ is continuous at $z_0 \in D_1$ and if $g : D_2 \to \mathbb{C}$ is continuous at $w_0 = f(z_0)$, then $g \circ f$ defined by $(g \circ f)(z) = g(f(z))$ is continuous at z_0.*

Proof. The proof is a consequence of Theorem 2.19, see Figure 2.4. ■

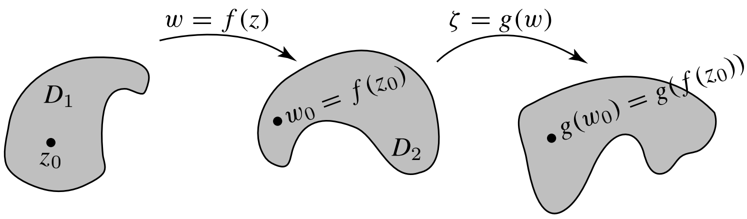

Figure 2.4: Description for a composite map.

An alternative and useful characterization of a continuous function in terms of sequences is the following.

2.22. Theorem. *A function f is continuous at a point $z_0 \in D$ iff $f(z_0) = \lim_{n\to\infty} f(z_n)$ for every sequence $\{z_n\}$ such that $z_n \in D$ for $n = 1, 2, \ldots$ and $z_n \to z_0$ as $n \to \infty$.*

Proof. $\Longrightarrow$: Consider a sequence $\{z_n\}$ in D such that $z_n \to z_0$ as $n \to \infty$. Continuity of f at z_0 implies that, for a given $\epsilon > 0$ there exists a δ with

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

For this δ, since $z_n \to z_0$, there exists an N such that for all $n \geq N$, $|z_n - z_0| < \delta$. Thus,

$$|f(z_n) - f(z_0)| < \epsilon \quad \text{for all} \quad n \geq N$$

from which it follows that $f(z_n) \to f(z_0)$, as desired.

$\Longleftarrow$: Suppose that the converse part is not true. Then, for some $\epsilon > 0$, for every $\delta > 0$ there corresponds a point ζ such that

$$|\zeta - z_0| < \delta \quad \text{and} \quad |f(\zeta) - f(z_0)| \geq \epsilon.$$

Fix such an ϵ. Then for each $n \in \mathbb{N}$ there exists $\zeta \in D \cap \Delta(z_0; 1/n)$, denoted by ζ_n,

$$|\zeta_n - z_0| < \frac{1}{n} \quad \text{and} \quad |f(\zeta_n) - f(z_0)| \geq \epsilon.$$

So, $\zeta_n \to z_0$ but $f(\zeta_n) \not\to f(z_0)$ as $n \to \infty$. This is a contradiction to our assumption, and so the converse part is proved. ∎

Consider the function f defined by

$$f(z) = \frac{\overline{z}}{z}, \quad z \in \mathbb{C}\setminus\{0\}.$$

Note that the function itself is not defined at 0 and so, it is not continuous at 0. If $z_n = 1/n$ and $z_n' = i/n$, then $f(z_n) = 1$ and $f(z_n') = -1$. Note

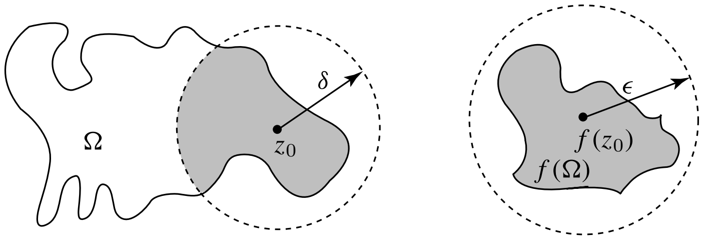

Figure 2.5: Description for continuous mapping.

that $z_n \to 0$ and $z_n' \to 0$. Can this function be made continuous at 0, by defining its value suitably at the origin?

2.23. Theorem. *Let $f : D \to \mathbb{C}$ be a function. Then f is continuous on D iff for every open set $O \subseteq \mathbb{C}$, $f^{-1}(O) = \{z \in D : f(z) \in O\}$ is open in D.*

Proof. $\Longleftarrow$: Suppose that for each open set $O \subseteq \mathbb{C}, f^{-1}(O)$ is open. Let $z_0 \in D$ and $\epsilon > 0$ be given. Then $A = \Delta(f(z_0); \epsilon)$ is an open set in $\mathbb{C}$ and so by hypothesis $f^{-1}(A)$ is open in D. Since $f^{-1}(A)$ is open in D and $z_0 \in f^{-1}(A)$, there exists an open disk $\Delta(z_0; \delta)$ such that $\Delta(z_0; \delta) \cap D \subset f^{-1}(A)$. Hence (see Figure 2.5),

$$f(\Delta(z_0; \delta) \cap D) \subseteq A = \Delta(f(z_0); \epsilon).$$

Continuity of f at z_0 is thus established.

$\Longrightarrow$: Suppose that f is continuous on D and let O be any open set in $\mathbb{C}$. If $f^{-1}(O) = \emptyset$, then it is open. Otherwise, let z_0 be any point of $f^{-1}(O)$. Then $z_0 \in D$ and $f(z_0) \in O$. As O is open, there exists an open disk $\Delta(f(z_0); \epsilon) \subseteq O$. Continuity of f at z_0 then implies that

$$f(z) \in \Delta(f(z_0); \epsilon) \subseteq O \quad \text{whenever } z \in \Delta(z_0; \delta) \cap D.$$

Thus, $\Delta(z_0; \delta) \cap D \subseteq f^{-1}(O)$ and this shows that $f^{-1}(O)$ is open in D. ■

One of the important results concerning continuous functions is that they preserve connectedness.

2.24. Theorem. *Let $f : D \to \mathbb{C}$ be a continuous function and let D be a connected set. Then $f(D)$ is a connected set.*

Proof. Suppose that the theorem is not true. Then there exist two non-empty disjoint open subsets V_1, V_2 of $f(D)$ such that $f(D) = V_1 \cup V_2$. By Theorem 2.23, $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are both non-empty open subsets of D such that $U_1 \cap U_2 = \emptyset$ and $D = U_1 \cup U_2$, contradicting the connectedness of D. ■

Our next theorem is helpful particularly in constructing uniformly continuous functions.

2.25. Theorem. *A continuous function on a compact set D is uniformly continuous therein.*

Proof. Suppose that f is not uniformly continuous on D. Then, proceeding as in the converse part of Theorem 2.22, there exists an $\epsilon > 0$, and two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in D such that for every $n \in \mathbb{N}$,

$$|\zeta_n - \eta_n| < \frac{1}{n} \text{ and } |f(\zeta_n) - f(\eta_n)| \geq \epsilon. \tag{2.26}$$

Since D is compact, $\{\zeta_n\}$ contains a subsequence $\{\zeta_{n_k}\}$ converging to a point z_0, say, where z_0 is a point of D; i.e. $\zeta_{n_k} \to z_0$. Let $\{\eta_{n_k}\}$ be the corresponding subsequence of $\{\eta_n\}$. Then $\eta_{n_k} \to z_0$. For, the triangle inequality gives

$$|\eta_{n_k} - z_0| \leq |\eta_{n_k} - \zeta_{n_k}| + |\zeta_{n_k} - z_0|$$

and so we have $\eta_{n_k} \to z_0$. Therefore, for the subsequences $\{\zeta_{n_k}\}$ and $\{\eta_{n_k}\}$, (2.26) gives

$$|\zeta_{n_k} - \eta_{n_k}| < \frac{1}{n_k} \text{ and } |f(\zeta_{n_k}) - f(\eta_{n_k})| \geq \epsilon \tag{2.27}$$

for every k. However, as f is continuous at z_0, we have

$$f(\zeta_{n_k}) \to f(z_0) \text{ and } f(\eta_{n_k}) \to f(z_0) \text{ as } k \to \infty,$$

contradicting (2.27). Hence, f is uniformly continuous. ■

2.28. Example. There are uniformly continuous functions on a set which is not compact. For instance, the identity function z is obviously uniformly continuous on $\mathbb{C}$. But $f(z) = z^2$ is not uniformly continuous on $\mathbb{C}$ whereas it is uniformly continuous on $\Delta_R = \{z : |z| < R\}$ (also $\overline{\Delta}_R \setminus \{0\}$), where R is a fixed positive number. To verify this, choose two points z_0 and z on Δ_R. Then

$$|z^2 - z_0^2| = |z - z_0|\,|z + z_0| \leq |z - z_0|(|z| + |z_0|) \leq 2R|z - z_0|.$$

Therefore given any $\epsilon > 0$ there exists a $\delta = \epsilon/(2R)$ such that

$$|z - z_0| < \delta \text{ implies } |z^2 - z_0^2| < \epsilon,$$

thus proving the uniform continuity on Δ_R.

Suppose that z is not restricted to Δ_R and allow z, z_0 to be $\mathbb{C}$. We choose two points of $\mathbb{C}$ to be $z = R + 1/R$, $z_0 = 1/R$ so that $|z - z_0| = R$, where $R > 0$. Then,

$$|z^2 - z_0^2| = \left(R + \frac{2}{R}\right) R = R^2 + 2 > 2$$

and so, $f(z) = z^2$ is not uniformly continuous throughout $\mathbb{C}$. •

2.3 Stereographic Projection

When dealing with the real line, we frequently use the concept of infinity and speak of $+\infty$ and $-\infty$. For instance, the sequence $\{2n\}$ diverges to $+\infty$ whereas $\{-n\}$ diverges to $-\infty$ and $\{x_n\} = \{(-1)^n n\}$ is unbounded above and unbounded below, so

$$\limsup_{n\to\infty} x_n = +\infty, \quad \liminf_{n\to\infty} x_n = -\infty.$$

Therefore, in the case of real-valued functions, limits such as

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to\infty} g(x) = l, \quad \lim_{x\to-\infty} h(x) = l', \quad \lim_{x\to\infty} p(x) = \infty$$

provide valuable information about the behavior of functions near these points, namely $a, \infty, -\infty$. In dealing with the complex plane $\mathbb{C}$, we too speak of infinity and denote it by the usual symbol '∞'. In $\mathbb{C}$, we do not give a sign to the complex infinity. One of the main reasons for this is that $\mathbb{C}$ has no natural ordering as $\mathbb{R}$ does.

In Section 1.5, we have discussed the topological properties of the complex plane $\mathbb{C}$ with the usual Euclidean metric. It turned out that the compact sets were the bounded and closed sets. However, the infinite set, for instance

$$S = \{z \in \mathbb{C} : z = m + in, \ m, n \in \mathbb{N}\}$$

has no limit in $\mathbb{C}$ and is closed but is non-compact. Nevertheless, it will be very much useful to develop the notion of limit points for such unbounded sets. Therefore, we shall extend the complex plane $\mathbb{C}$ by adjoining one extra point called the "point at infinity" which we shall denote by ∞, i.e. we consider the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ as a closed surface having a single point at infinity. We shall then introduce a new metric to analyze the behavior of a complex function at infinity and to map the points in $\mathbb{C}$ into the surface of a sphere. This process will be referred to as *stereographic projection*.

In fact, such an extra point '∞' is defined so as to satisfy the following computational properties (see Figure 2.6): Whatever be $z \in \mathbb{C}$,

$$\frac{z}{\infty} = 0, \quad z + \infty = \infty \ (z \neq \infty), \quad \frac{z}{0} = \infty \ (z \neq 0);$$

$$z \cdot \infty = \infty \ (z \neq 0), \quad \frac{\infty}{z} = \infty \ (z \neq \infty).$$

We do not define (or have no sense to define)

$$\frac{0}{0}, \quad \infty + \infty, \quad \infty - \infty, \quad 0 \cdot \infty, \quad \text{and} \quad \frac{\infty}{\infty}.$$

We define $z^0 = 1$ for $z \in \mathbb{C}_\infty$, and for each $n \in \mathbb{N}$, we set

$$0^{-n} = \infty^n = \infty, \quad 0^n = \infty^{-n} = 0.$$

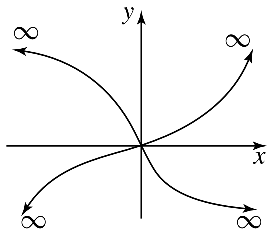

Figure 2.6: Complex '∞'.

We use the following construction due to Riemann. There are two commonly used methods. In one method, a correspondence is set up between the points of $\mathbb{C}$ and those of a sphere of radius $1/2$ with center at $(0,0,1/2)$ tangent to this plane. There is another method of correspondence in which the sphere of radius 1 has center at $(0,0,0)$ and the plane passes through $(0,0,0)$. We shall use the first one.

Let $\mathbb{C}$ be the complex plane. Through the origin construct a line perpendicular to $\mathbb{C}$. Let this be ζ-axis of a 3-dimensional Euclidean space in which a point has coordinates (ξ, η, ζ). Consider the sphere S of radius $1/2$ with center at $(0,0,1/2)$. That is,

$$S = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \zeta^2 + \eta^2 + (\zeta - 1/2)^2 = 1/4\}.$$

It is a common practice to call the points N and O with coordinates $(0,0,1)$ and $(0,0,0)$ the *north pole* and *south pole* of the sphere S, respectively. The great circle in the plane $\zeta = 1/2$ is called the *equator*. The plane $\zeta = 0$ coincides with the complex plane $\mathbb{C}$ and the ξ and η axes are the x and y axes, respectively. Let $Q(x, y, 0)$ be any point in the plane $\mathbb{C}$. Through the points N and Q we draw a straight line NQ intersecting the sphere S at a point, say, $P(\xi, \eta, \zeta)$. Then (ξ, η, ζ) is called the *stereographic projection*, or *image* of $(x, y, 0)$ on the sphere and is considered as the spherical representation of $z = x + iy$. This procedure assigns a unique point on S to every given complex number z so that we are free to think of $\mathbb{C}$ as sitting inside $\mathbb{R}^3$. Conversely, to each point (ξ, η, ζ) on the sphere other than N we can associate the complex number $z = (x, y, 0)$ where the line from (0,0,1) through (ξ, η, ζ) intersects $\mathbb{C}$. Now we immediately see that there is a one-to-one correspondence between $\mathbb{C}$ and the points of S with one exception, namely, the north pole (0,0,1) itself. By assigning to the north pole N of the sphere to correspond to the point at infinity, we obtain a one-to-one correspondence between the points of the sphere S on one hand and the points of the extended complex plane $\mathbb{C}_\infty$ on the other. The sphere is often called *Riemann sphere* or *complex sphere*

It is easy to obtain explicit equations expressing ξ, η and ζ in terms of

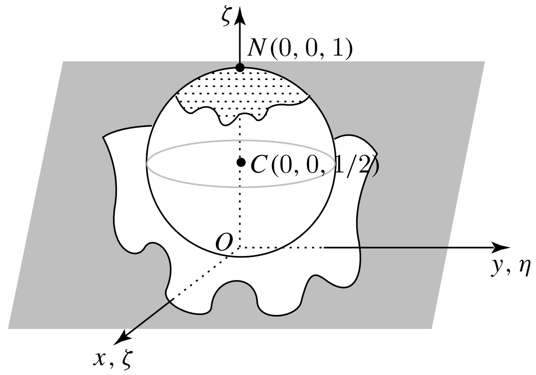

Figure 2.7: Riemann's sphere.

x and y. The line in $\mathbb{R}^3$ passing through $(0,0,1)$ and $(x,y,0)$ is given by

$$\{t(0,0,1)+(1-t)(x,y,0) : t \in \mathbb{R}\} \equiv \{((1-t)x,(1-t)y,t) : t \in \mathbb{R}\}. \tag{2.29}$$

Since this line intersects the sphere S, we must have

$$(1-t)^2x^2 + (1-t)^2y^2 + (t-1/2)^2 = \frac{1}{4}$$

so that $(1-t)^2|z|^2 = t(1-t)$. If $(\xi,\eta,\zeta) \neq (0,0,1)$, then we arrive at

$$t = \frac{|z|^2}{1+|z|^2}, \quad \text{i.e.} \quad 1-t = \frac{1}{1+|z|^2}.$$

Using the fact that the points $(0,0,1)$, (ξ,η,ζ) and $(x,y,0)$ are collinear, (2.29) now yields

$$\begin{cases} \xi = \dfrac{x}{1+x^2+y^2} = \dfrac{z+\overline{z}}{2(1+|z|^2)}; \\ \eta = \dfrac{y}{1+x^2+y^2} = \dfrac{-i(z-\overline{z})}{2(1+|z|^2)}; \\ \zeta = \dfrac{x^2+y^2}{1+x^2+y^2} = \dfrac{|z|^2}{1+|z|^2}; \end{cases} \tag{2.30}$$

that is $z = x+iy \in \mathbb{C}$ corresponds to

$$\left(\frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{x^2+y^2}{1+x^2+y^2}\right) \in S$$

or equivalently to

$$\left(\frac{z+\overline{z}}{2(1+|z|^2)}, \frac{-i(z-\overline{z})}{2(1+|z|^2)}, \frac{|z|^2}{1+|z|^2}\right) \in S.$$

For instance, the images of $1, i, (1-i)/\sqrt{2}$ on the sphere are respectively given by Z_1, Z_2, Z_3, where

$$Z_1 = (1/2, 0, 1/2),\ Z_2 = (0, 1/2, 1/2),\ Z_3 = (\sqrt{2}/4, -\sqrt{2}/4, 1/2).$$

Using the three equations in (2.30) it is easy to see that

$$(2.31) \qquad 1-\zeta = \frac{1}{1+|z|^2}; \quad \text{i.e.} \quad |z|^2 = \frac{\zeta}{1-\zeta}$$

so that

$$(2.32) \qquad x = \frac{\xi}{1-\zeta},\ y = \frac{\eta}{1-\zeta}; \quad \text{or} \quad z = \frac{\xi + i\eta}{1-\zeta}.$$

That is (ξ, η, ζ) corresponds to

$$\left(\frac{\xi}{1-\zeta}\right) + i\left(\frac{\eta}{1-\zeta}\right) \in \mathbb{C}.$$

The map $z \longleftrightarrow (\xi, \eta, \zeta)$ is called *stereographic projection* of $\mathbb{C}$ on $S\setminus\{(0,0,1)\}$ or vice versa. In fact if $\pi : \mathbb{C} \to S\setminus\{(0,0,1)\}$ is the stereographic projection of $\mathbb{C}$ on $S\setminus\{(0,0,1)\}$, then

$$\pi(z) = \left(\frac{z+\overline{z}}{2(1+|z|^2)}, \frac{-i(z-\overline{z})}{2(1+|z|^2)}, \frac{|z|^2}{1+|z|^2}\right).$$

The inverse of π, $\pi^{-1} : S\setminus\{(0,0,1)\} \to \mathbb{C}$, is given by

$$\pi^{-1}(\xi, \eta, \zeta) = \frac{\xi}{1-\zeta} + i\frac{\eta}{1-\zeta}.$$

2.33. Remark. If z is a complex number corresponding to the point $Q(\xi, \eta, \zeta)$ on the punctured sphere $S\setminus\{(0,0,1)\}$, then (2.31) shows that

$$(2.34) \qquad |z| \gtrless R \Longleftrightarrow \zeta \gtrless \frac{R^2}{1+R^2},\ R > 0.$$

In particular, the images of $\{z : |z| < 1\}$ and $\{z : |z| > 1\}$ are the southern hemisphere and the northern hemisphere, respectively. ●

Now suppose that (ξ_n, η_n, ζ_n) is a sequence of points of S which converges to $(0,0,1)$ and let $\{z_n\}$ be the corresponding sequence of points in $\mathbb{C}$. It follows (as is obvious geometrically) from (2.31) that $|z_n|$ becomes very large; i.e. as $(\xi_n, \eta_n, \zeta_n) \to (0,0,1)$, $|z_n| \to \infty$. Conversely, in view of (2.30), if $|z_n| \to \infty$ then $(\xi_n, \eta_n, \zeta_n) \to (0,0,1)$. Thus, it is reasonable to introduce the symbol "∞" to correspond to $(0,0,1) \in S$. We can now think of $\mathbb{C}_\infty$ either as 'a plane + an ideal point' or as a sphere. Correspondingly, as was pointed out earlier, we may think of $\mathbb{C}$ as a plane, or as a sphere

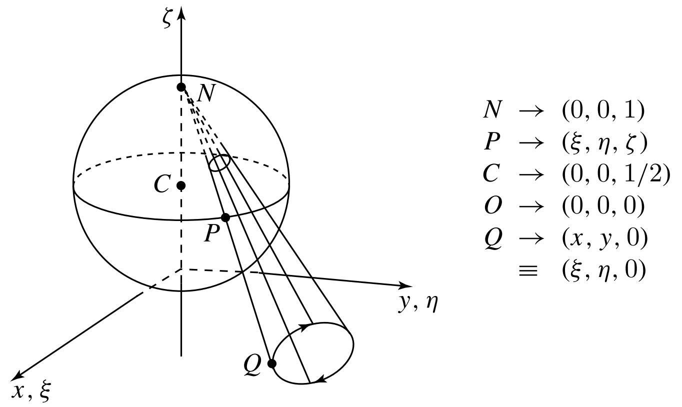

Figure 2.8: Projection of a portion of the Riemann sphere.

without the north pole. Both points of view are useful, depending on the problem on hand.

The notions such as limit point of a sequence and neighborhoods of a point can now be defined for the extended complex plane $\mathbb{C}_\infty$ through this identification with the Riemann sphere S.

2.35. Definition. A sequence $\{z_n\}$ is said to diverge to the limit ∞, written $z_n \to \infty$ or $\lim_{n\to\infty} z_n = \infty$, if for any $R > 0$, there exists a number N such that $|z_n| > R$ for $n > N$.

Since $\{z : |z| > R\}$ corresponds to a 'cap' (see also Figure 2.7) it makes sense to think of $\{z : |z| > R\}$ as a *neighborhood of* ∞. Such a neighborhood becomes smaller as R gets larger and larger. This geometric intuition allows us to define the concept of continuity on $\mathbb{C}_\infty$. We next give a justification for this notion.

We define $\chi(z, z') = d(Z, Z'))$ to be the Euclidean distance between $Z = (\xi, \eta, \zeta)$ and $Z' = (\xi', \eta', \zeta')$ in the three dimensional space which are respectively the pre-images of $z = x + iy$ and $z' = x' + iy'$ (under the stereographic projection of the sphere S onto the complex plane $\mathbb{C}$). Since (ξ, η, ζ) and (ξ', η', ζ') are on the sphere S,

$$\xi^2 + \eta^2 + \zeta^2 = \zeta \quad \text{and} \quad \xi'^2 + \eta'^2 + \zeta'^2 = \zeta'. \tag{2.36}$$

The length of the segment joining Z and Z', known as *chordal distance* of z from z', is defined as $\chi(z, z') = d(Z, Z')$. Therefore, we have

$$\begin{aligned}
\chi(z, z') &= \sqrt{(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2} \\
&= \sqrt{\zeta + \zeta' - 2(\xi\xi' + \eta\eta' + \zeta\zeta')}, \quad \text{by (2.36)}, \\
&= \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}, \quad \text{by (2.30) and (2.36)}.
\end{aligned}$$

In particular, if z' is the point at infinity then $\chi(z,\infty)$, the spatial distance between the images of z and $(0,0,1)$, is given by

$$(2.37)\qquad \begin{cases} \chi(z,\infty) &= \sqrt{\xi^2+\eta^2+(\zeta-1)^2} = \sqrt{1-\zeta}, \\ &= \dfrac{1}{\sqrt{1+|z|^2}}, \text{ by (2.31)}, \\ &= \lim\limits_{z'\to\infty} \chi(z,z'). \end{cases}$$

Further, we note that

$$\begin{aligned} \chi(z,z') = 1 &\iff |z-z'|^2 = (1+|z|^2)(1+|z'|^2) \\ &\iff |1+z\overline{z}'| = 0 \\ &\iff z\overline{z}' = -1. \end{aligned}$$

In other words, the points z and z' in $\mathbb{C}$ represent diametrically opposite (antipodal) points of the Riemann sphere S iff $z\overline{z}' = -1$.

By (2.34) and (2.37), we see that the set $\{z: |z| > R\}$, $R > 0$, corresponds to the set $\{(\xi,\eta,\zeta) \in S\setminus(0,0,1) : (\xi,\eta,\zeta)$ lies within the spherical cap of radius $R/\sqrt{1+R^2}\}$.

Conversely, if $(\xi,\eta,\zeta) \neq (0,0,1)$, as noted above, we have

$$\sqrt{\xi^2+\eta^2+(\zeta-1)^2} = \sqrt{1-\zeta} = \frac{1}{\sqrt{1+|z|^2}}.$$

Note that

$$\frac{1}{\sqrt{1+|z|^2}} < \epsilon \iff |z| > \frac{\sqrt{1-\epsilon^2}}{\epsilon} \quad (\epsilon > 0).$$

Therefore, the ϵ-neighborhood of $(0,0,1)$ on S, namely,

$$\{(\xi,\eta,\zeta) \in S: \sqrt{\xi^2+\eta^2+(\zeta-1)^2} < \epsilon\}$$

is nothing but $N(\infty;\epsilon) = \{z \in \mathbb{C} : |z| > \frac{\sqrt{1-\epsilon^2}}{\epsilon}\} \cup \{\infty\}$, which we call an ϵ-neighborhood of ∞. Clearly,

(a) $\chi(z_1,z_2) \geq 0$

(b) $\chi(z_1,z_2) = 0 \iff z_1 = z_2$

(c) $\chi(z_1,z_2) = \chi(z_2,z_1)$

(d) $\chi(z_1,z_3) \leq \chi(z_1,z_2) + \chi(z_2,z_3)$

(e) $\chi(0,z_1) \leq \chi(0,z_2)$ provided $|z_1| \leq |z_2| \leq \infty$

(f) $\chi(z_1,z_2) \leq |z_1 - z_2| = d(z_1,z_2)$.

In particular, we call χ defined on $\mathbb{C}_\infty$, the *chordal metric* on $\mathbb{C}_\infty$. This allows us to treat the point ∞ like any other point. Thus, we have

2.38. Theorem. *The chordal metric $\chi(.\,,.)$ defined on $\mathbb{C}_\infty$ satisfies the properties of a metric.*

By a circle on the sphere S we mean the intersection of S with some plane $a\xi + b\eta + c\zeta + d = 0$. Let us now find the equation of the circle on the sphere which is the stereographic image of the circle $\{z : |z-a| = R\}$. By (2.32) and $|z-a|^2 = R^2$, the required equation is given by

$$\frac{\xi^2+\eta^2}{(1-\zeta)^2} - a\left(\frac{\xi - i\eta}{1-\zeta}\right) - \overline{a}\left(\frac{\xi + i\eta}{1-\zeta}\right) = R^2 - |a|^2,$$

which upon simplification becomes

$$(a+\overline{a})\xi + i(\overline{a}-a)\eta + (|a|^2 - R^2 - 1)\zeta = 1 + |a|^2 - R^2.$$

2.39. Theorem. *Suppose $T \subset \mathbb{C}_\infty$. Then the corresponding image of T on the Riemann sphere S is*

(a) *a circle in S not containing $(0,0,1)$ if T is a circle;*
(b) *a circle in S minus $(0,0,1)$ if T is a line.*

Proof. First we consider the general equation of a circle in the plane:

(2.40) $$T = \{(x,y) : A(x^2+y^2) + Bx + Cy + D = 0\}.$$

Using stereographic projection, i.e. by (2.32) and (2.31), we have

$$(A-D)\zeta + B\xi + C\eta + D = 0$$

which is the equation of a plane in space when a general point has coordinates (ξ,η,ζ). Note that a plane and a sphere intersect in a circle. Suppose $A = 0$, then (2.40) is a straight line in $\mathbb{C}$. Thus, the corresponding set in the sphere S is given by the intersection of

$$\xi^2 + \eta^2 + \zeta^2 = \zeta,$$

with $B\xi + C\eta - D\zeta + D = 0$ which is a circle minus (0,0,1). If we consider $T \subset \mathbb{C}_\infty$, then the corresponding image set governed by the above equations is a circle passing through the north pole (0,0,1). This proves (a).

Suppose $A \neq 0$ and $D = 0$. Then T reduces to

$$A(x^2+y^2) + Bx + Cy = 0,$$

which is a circle passing through $(0,0)$ and so the corresponding image is given by the intersection of

$$\xi^2 + \eta^2 + \zeta^2 = \zeta$$

with $B\xi + C\eta + A\zeta = 0$. This is in fact a circle passing through the south pole (0,0,0). The general statement (b) is obvious. ∎

The converse of Theorem 2.39 takes the following form.

2.41. Theorem. *If T_S is a circle on the Riemann sphere S and T_I is its stereographic projection on $\mathbb{C}_\infty$, then*

(a) *T_I is a circle if $(0,0,1) \notin T_S$*

(b) *T_I is a line if $(0,0,1) \in T_S$.*

Proof. If T_S is a circle on the sphere S, then

$$T_S = \{\alpha\xi + \beta\eta + \gamma\zeta + \delta = 0\} \cap \{\xi^2 + \eta^2 + \zeta^2 = \zeta\}.$$

Thus, T_S passes through $(0,0,1)$ provided $\gamma + \delta = 0$. It follows from (2.30) that the corresponding set of points of the plane in $\mathbb{C}$ satisfies

$$(\gamma + \delta)(x^2 + y^2) + \alpha x + \beta y + \delta = 0, \ (x, y) \in T_I. \tag{2.42}$$

Clearly, this equation represents the equation of a circle if $\gamma + \delta \neq 0$. If $\gamma + \delta = 0$, then (2.42) is the equation of a line. The conclusion now follows from the fact that $\gamma + \delta = 0 \Longleftrightarrow (0,0,1) \in T_S$. ∎

2.43. Example. Suppose that a cube has its vertices on the Riemann sphere and its edges parallel to the co-ordinate axes. Let us now find the stereographic projections of the vertices. By hypothesis, the vertices are

$$\begin{array}{lll} Z_1 = (\xi, \eta, \zeta), & & Z_2 = (\xi, \eta, 1-\zeta), \\ Z_3 = (\xi, -\eta, 1-\zeta), & & Z_4 = (-\xi, \eta, 1-\zeta), \\ Z_5 = (-\xi, -\eta, 1-\zeta), & & Z_6 = (-\xi, \eta, \zeta), \\ Z_7 = (\xi, -\eta, \zeta), & & Z_8 = (-\xi, -\eta, \zeta). \end{array}$$

Further the length of the sides of the cube gives the relation

$$|\zeta| = |\eta| = |\zeta - 1/2|. \tag{2.44}$$

For convenience, we let $\xi > 0$, $\eta > 0$ and $\zeta > \frac{1}{2}$. As ξ, η, ζ lie on the Riemann sphere

$$\xi^2 + \eta^2 + (\zeta - 1/2)^2 = 1/4,$$

by (2.44), we have $\xi^2 + \xi^2 + \xi^2 = 1/4$, i.e. $3\xi^2 = 1/4$. Thus, we get

$$\xi = \eta = \frac{1}{2\sqrt{3}} = \left(\zeta - \frac{1}{2}\right), \quad \text{i.e.} \quad \zeta = \frac{\sqrt{3}+1}{2\sqrt{3}} \ \left(1 - \zeta = \frac{\sqrt{3}-1}{2\sqrt{3}}\right).$$

Therefore, by (2.32), we have

$$z = \frac{\xi + i\eta}{1-\zeta} = \frac{(1+i)}{\sqrt{3}-1} = \left(\frac{1}{\sqrt{3}-1}\right) + i\left(\frac{1}{\sqrt{3}-1}\right)$$

which is the stereographic projection of

$$\left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right).$$

Similarly the stereographic projections of the other vertices are obtained as follows:

$$\begin{aligned}
Z_2 &= \left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right), & Z_3 &= \left(\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right),\\
Z_4 &= \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right), & Z_5 &= \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right),\\
Z_6 &= \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right), & Z_7 &= \left(\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right),\\
Z_8 &= \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right),
\end{aligned}$$

and

$$\begin{aligned}
Z_2 &\to z_2 = \left(\frac{1}{\sqrt{3}+1}\right) + i\left(\frac{1}{\sqrt{3}+1}\right)\\
Z_3 &\to z_3 = \left(\frac{1}{\sqrt{3}+1}\right) - i\left(\frac{1}{\sqrt{3}+1}\right)\\
Z_4 &\to z_4 = -\left(\frac{1}{\sqrt{3}+1}\right) + i\left(\frac{1}{\sqrt{3}+1}\right)\\
Z_5 &\to z_5 = -\left(\frac{1}{\sqrt{3}+1}\right) - i\left(\frac{1}{\sqrt{3}+1}\right)\\
Z_6 &\to z_6 = -\left(\frac{1}{\sqrt{3}-1}\right) + i\left(\frac{1}{\sqrt{3}-1}\right)\\
Z_7 &\to z_7 = \left(\frac{1}{\sqrt{3}-1}\right) - i\left(\frac{1}{\sqrt{3}-1}\right)\\
Z_8 &\to z_8 = -\left(\frac{1}{\sqrt{3}-1}\right) - i\left(\frac{1}{\sqrt{3}-1}\right).
\end{aligned}$$

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2.45. Example. We wish to prove that the points z and z' in the complex plane will correspond to symmetric points with respect to the equatorial plane (viz., the plane $\zeta = 1/2$) iff $z\overline{z}' = 1$.

To prove this, we first note that z and z' correspond to symmetric points with respect to equatorial plane iff z and z' correspond to (ξ, η, ζ) and $(\xi, \eta, 1-\zeta)$, respectively. This holds if and only if

$$z = \frac{\xi + i\eta}{1-\zeta} \text{ and } z' = \frac{\xi + i\eta}{1-(1-\zeta)}; \text{ i.e. } z\overline{z'} = \frac{\xi^2 + \eta^2}{(1-\zeta)\zeta} = 1. \qquad \bullet$$

We shall now briefly indicate how to extend the limit concepts discussed in the previous section to the extended complex plane.

2.46. Definition. (Limit at infinity) Let f be defined on an unbounded set E. (Then for any $R > 0$, there exists $z \in E$ such that $|z| > R$.) We say that $f(z) \to \ell$ as $z \to \infty$ if for every $\epsilon > 0$, there exists an $R > 0$ such that

$$|f(z) - \ell| < \epsilon \text{ whenever } z \in E \text{ and } |z| > R.$$

In this case, we write

$$\lim_{z\to\infty} f(z) = \ell \text{ or } \lim_{|z|\to\infty} f(z) = \ell.$$

Recall (see Definition 2.35) that a set of the form $\{z : |z| > R\} \cup \{\infty\}$ is called a neighborhood of ∞. For convenience, we may introduce

$$\Delta(\infty; 1/R) := \{z : |z| > R\} \cup \{\infty\} = \mathbb{C}_\infty \setminus \overline{\Delta}(0; R).$$

The closed disk $\overline{\Delta}(\infty; 1/R)$ and the punctured disk $\Delta(\infty; 1/R)\setminus\{\infty\}$ may be defined similarly. For instance, $\Delta(\infty; 1/R)\setminus\{\infty\} := \mathbb{C}\setminus\{z : |z| > R\}$.

Let us show that

(a) $\displaystyle\lim_{z\to\infty} \frac{1}{z} = 0$, (b) $\displaystyle\lim_{z\to\infty} \frac{1}{z^2} = 0$.

To do this, we first note that $1/z$ and $1/z^2$ are defined everywhere in $\mathbb{C}\setminus\{0\}$. Then for every $\epsilon > 0$ there exists a $R = 1/\epsilon$ such that

$$\left|\frac{1}{z}\right| < \epsilon \text{ whenever } |z| > \frac{1}{\epsilon} = R.$$

For the second case, we note that $\left|1/z^2\right| < \epsilon \iff |z| > 1/\sqrt{\epsilon}$ so, in this case, we choose $R = 1/\sqrt{\epsilon}$.

2.47. Definition. (Infinite limit) Let f be defined on D except possibly at z_0 of D. We say that $f(z) \to \infty$ as $z \to z_0$ if for every $R > 0$, there exists a $\delta > 0$ such that

$$|f(z)| > R \text{ whenever } z \in D \cap \Delta(z_0; \delta)\setminus\{z_0\}.$$

In this case, we write,

$$\lim_{z\to z_0} f(z) = \infty \quad \text{or} \quad \lim_{z\to z_0} |f(z)| = \infty.$$

Note. Considering the real-valued function $|f|$ on D or $D\setminus\{z_0\}$ defined by $|f|(z) = |f(z)|$, $z \in D$, we observe that

$$\lim_{z\to z_0} f(z) = \infty \;\text{ iff }\; \lim_{z\to z_0} |f|(z) := \lim_{z\to z_0} |f(z)| = \infty.$$

2.48. Example. We have (a) $\lim\limits_{z\to 1} \dfrac{1}{|z^2-1|} = \infty$, (b) $\lim\limits_{z\to 0} \dfrac{1}{z^2} = \infty$. Note that the function f defined by

$$f(z) = \frac{1}{z^2-1}$$

is defined for all $z \in \mathbb{C}\setminus\{-1,1\}$. Let $R > 0$ be given. Then we must show that we can find a $\delta > 0$ such that

$$|f(z)| = \left|\frac{1}{z^2-1}\right| > R \;\text{ whenever }\; 0 < |z-1| < \delta.$$

Note that $|f(z)| > R \Longleftrightarrow 0 < |z^2-1| < 1/R$. Now $0 < |z-1| < \delta$ implies that

$$|z^2-1| = |z-1|\,|z-1+2| \le |z-1|[|z-1|+2] < \delta(\delta+2) = (\delta+1)^2 - 1$$

and therefore $|z^2-1| < 1/R$ if $\delta = \sqrt{1+R^{-1}} - 1$. Hence

$$|f(z)| > R \;\text{ whenever }\; z \in [\mathbb{C}\setminus\{-1,1\}] \cap [\Delta(1;\sqrt{1+R^{-1}}-1)\setminus\{1\}]$$

and the conclusion follows. The second case is clear, because

$$\left|\frac{1}{z^2}\right| = \frac{1}{|z|^2} > R \;\text{ whenever }\; |z| = |z-0| < \delta = \frac{1}{\sqrt{R}}.$$ •

2.49. Definition. Let f be defined on an unbounded set E. If for every $R > 0$ there exists $K > 0$ such that $|f(z)| > R$ for $|z| > K$ and $z \in E$, then we say that $f(z) \to \infty$ as $z \to \infty$ and write $\lim\limits_{z\to\infty} f(z) = \infty$.

For instance, if $f(z) \to \infty$ as $z \to \infty$, then $|f(z)| \to \infty$ as $z \to \infty$. In particular, $\lim\limits_{z\to\infty} z = \infty$.

2.50. Remark. Using the chordal metric χ on $\mathbb{C} \cup \{\infty\}$, it is easy to see that the following are true:

(a) $z_n \to z_0 \Longleftrightarrow \chi(z_n, z_0) \to 0$ (b) $z_n \to \infty \Longleftrightarrow \chi(z_n, \infty) \to 0$.

Part (b), in particular, helps us to give an alternate meaning to a statement such as $|f(z)| > R$ for $|z - z_0| < \delta$ in the form

$$\chi(f(z), \infty) < \frac{1}{\sqrt{1+R^2}} \quad \text{for} \quad \chi(z, z_0) < |z - z_0| < \delta. \qquad \bullet$$

Let E be an unbounded set. Then $f : E \to \mathbb{C}_\infty$ is *continuous* at $z_0 \in E$ iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } z \in \Delta(z_0; \delta) \cap E,$$

or equivalently,

$$f(\Delta(z_0; \delta) \cap E) \subseteq \Delta(f(z_0); \epsilon).$$

In this definition, with the help of the above notation, we now admit the cases where $z_0 = \infty$ and $f(z_0) = \infty$. We say f is continuous on E iff f is continuous at all points of E. For instance, a rational function in z is a continuous function from $\mathbb{C}_\infty$ into $\mathbb{C}_\infty$. Many other basic notions may also extended to the extended complex plane. We postpone our discussion on certain aspects of functions on the extended complex plane to a later chapter.

2.4 Sequences and Series of Functions

Next we discuss the uniform convergence of sequences and series of functions. Consider a sequence of complex-valued functions $\{f_n(z)\}$, $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ and $n \in \mathbb{N}$. For a fixed $z_0 \in D$, $\{f_n(z_0)\}$ is an ordinary sequence of complex numbers and so the convergence of the sequence of these functions for each $z_0 \in D$ is as in the definition of convergence of a sequence of complex numbers in Section 1.6.

2.51. Definition. A sequence $\{f_n\}$ of functions is said to be convergent at $z_0 \in D$ if the sequence $\{f_n(z_0)\}$ converges. We shall call this limit $f(z_0)$. The sequence $\{f_n(z)\}$ of functions is said to converge '*pointwise*' to $f(z)$ in D if $\{f_n(z_0)\}$ converges to $f(z_0)$ at each point $z_0 \in D$.

Since the limit of a sequence, when it exists, is unique, in the case of 'pointwise' convergence, we have a uniquely defined function f from D into $\mathbb{C}$ and we call f, the pointwise limit, or simply the limit function of the sequence $\{f_n(z)\}$ and write $f(z) = \lim_{n\to\infty} f_n(z)$, $z \in D$. An equivalent formulation of the above discussion is given by the following:

2.52. Definition. Let f_n and f be functions from D into $\mathbb{C}$. We say that $f_n(z) \to f(z)$ on D iff for every $\epsilon > 0$ and every $z \in D$, there exists $N = N(\epsilon, z)$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$. This convergence is said to be *uniform* if it is possible that $N(\epsilon, z)$ can be chosen independent of $z \in D$. That is, one $N(\epsilon)$ works for all $z \in D$.

We often write $f_n \to f$ uniformly on D, or $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on D to denote the uniform convergence of f_n to f on D. Also note that the uniform convergence on D implies pointwise convergence. The converse is false, for example consider $f_n(z) = z^n$ for $z \in \Delta$ (see also Example 2.55). Then the sequence $\{z^n\}$ converges uniformly on $|z| \leq r$ $(0 < r < 1)$; for if $\epsilon > 0$ is given, then

$$|z^n| \leq r^n < \epsilon \quad \text{whenever } n > (\ln \epsilon)/\ln r.$$

Note that

$$f(z) = \lim_{n\to\infty} z^n = \begin{cases} 0 & \text{if } |z| < 1 \\ 1 & \text{if } z = 1. \end{cases}$$

However, the convergence is not uniform on $|z| < 1$. Indeed if $0 < \epsilon < 1$, then for any positive integer n, it is possible to choose z so that

$$1 > |z| \geq \epsilon^{1/n}.$$

Then $|z^n| \geq \epsilon$ and so the choice of N depends on z. For instance, given any positive integer n, one can exhibit points z with $|z|^n \geq 1/3$ (note that the choice $z = \exp(-(1/n)\ln 3)$ will do when $\epsilon = 1/3$).

Alternatively, it suffices to observe that (since $|z| < 1$),

$$|z|^n < \epsilon \quad \text{whenever } n \ln|z| < \ln \epsilon \text{ or } n > (\ln \epsilon)/\ln|z|.$$

Since $\ln|z|$ approaches zero as $|z| \to 1$, the maximum value of n is infinite and so there cannot exist an N independent of z satisfying the condition that is required for uniform convergence.

We are frequently concerned with series of the form $\sum_{n\geq 1} f_n(z)$ for z in some subset of $\mathbb{C}$. In situations like this, as in the case of series of complex terms, we have

2.53. Definition. The series $\sum_{k\geq 1} f_k$ converges 'pointwise' in D to a function f in D if the corresponding sequence of partial sums $s_n = \sum_{k=1}^{n} f_k$ converges to f 'pointwise' in D. Then we write

$$f = \sum_{k=1}^{\infty} f_k$$

and say that the series $\sum_{n\geq 1} f_n$ is pointwise convergent in D with sum f. This series is said to converge to f uniformly in D iff s_n converges to f uniformly in D. The series is said to converge absolutely or said to be *absolutely convergent* in D if the series $\sum_{k\geq 1} |f_k|$ is convergent in D.

2.54. Example. Consider the geometric series $\sum_{k\geq 1} z^{k-1}$. If $z \neq 1$, then the n-th partial sum is

$$s_n = 1 + z + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}.$$

Since $\{z^n\}$ is a null sequence for $|z| < 1$, it follows that

$$\sum_{k\geq 1} z^{k-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-z^n}{1-z} = \frac{1}{1-z}, \quad |z| < 1.$$

In fact,

$$\left| s_n - \frac{1}{1-z} \right| = \frac{|z|^n}{|1-z|} < \epsilon \quad \text{whenever} \quad n \ln|z| < \ln(\epsilon|1-z|).$$

If $0 < |z| < 1$, then $\ln|z| < 0$ and so we have

$$\left| s_n - \frac{1}{1-z} \right| < \epsilon \quad \text{whenever} \quad n > N \geq \frac{\ln(\epsilon|1-z|)}{n \ln|z|}.$$

Thus, the geometric series is pointwise convergent in Δ with sum $f(z) = 1/(1-z)$.

If $|z| \geq 1$, $|z|^n \geq 1$ and since $|z|^n \not\to 0$, the series diverges in this case. These facts can also be verified with the help of Corollary 1.49 and Theorem 1.51.

Next we wish to show that the geometric series is not uniformly convergent in $|z| < 1$. To do this, we need to show that s_n does not converge uniformly in $|z| < 1$. Clearly, from the expression for s_n, it suffices to show that $\{S_n\}$, where $S_n(z) = z^n/(1-z)$, is not uniformly convergent in $|z| < 1$. Let $\epsilon > 0$ be given. If $\{S_n\}$ were uniformly convergent for $|z| < 1$, then there would exist an N such that

$$\left| \frac{z^n}{1-z} \right| < \epsilon \quad \text{for all } n > N \text{ and all } z \in \Delta.$$

Now, for a given $\epsilon > 0$ we choose $z_0 = \frac{N+\epsilon}{N+\epsilon+1}$ in Δ. Then for this point, we note that

$$\frac{z_0^{N+1}}{1-z_0} = \frac{[1-1/(N+\epsilon+1)]^{N+1}}{1/(N+\epsilon+1)} > \frac{1-(N+1)/(N+\epsilon+1)}{1/(N+\epsilon+1)} = \epsilon$$

(since $(1-x)^n > 1 - nx$ if $0 < x < 1$ and n is a positive integer ≥ 1); so the convergence is not uniform for $|z| < 1$. ●

2.55. Example. Consider the series

$$S(z) = -z + z(1-z) + z^2(1-z) + \cdots + z^{n-1}(1-z) + \cdots$$

Then the n-th partial sum is

$$s_n(z) = -z + (1-z)\sum_{k=1}^{n-1} z^k = -z^n$$

so that the sequence $\{s_n\}$ (and, so the series) converges to zero for $|z| < 1$. We have already shown that the sequence $\{z^n\}$ does not converge uniformly for $|z| < 1$. Consequently, the given series is not uniformly convergent for $|z| < 1$. ●

Many results about the convergence of numerical sequences in $\mathbb{C}$ can be carried over easily to the convergence of a sequence of functions at a point and so, for pointwise convergence. We list a few of them.

2.56. Theorem. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions defined on a set $D \subseteq \mathbb{C}$.*

(a) *$f_n \to f$ pointwise in $D \iff$ for each $\epsilon > 0$ and each $z \in D$ there exists an $N(\epsilon, z)$ such that $|f_n(z) - f_m(z)| < \epsilon$ for all $n, m > N(\epsilon, z)$.*
(b) *If $f_n \to f$ and $g_n \to g$ both pointwise in D, then $f_n \pm g_n \to f \pm g$ and $f_n g_n \to fg$ pointwise on D.*
(c) *If $g_n(z) \neq 0$ and $g(z) \neq 0$ for each z in D and $n \in \mathbb{N}$, then $\dfrac{f_n}{g_n} \to \dfrac{f}{g}$ pointwise in D.*
(d) *$\{f_n(z)\}$ is uniformly convergent in D iff for every $\epsilon > 0$ and all $z \in D$ there exists an $N = N(\epsilon)$ such that $|f_n(z) - f_m(z)| < \epsilon$ for all $n, m > N(\epsilon)$.*
(e) *If $f_n \to f$ and $g_n \to g$ both uniformly in a common region D then $f_n \pm g_n \to f \pm g$ uniformly in D and for a complex constant α, $\alpha f_n \to \alpha f$ uniformly in D.*

Theorem 2.56(d) is called the Cauchy criterion for uniform convergence. Uniform convergence does not carry over to the product of functions, in general. To see this, consider

$$f_n(z) = g_n(z) = \frac{1}{z} + \frac{1}{n} \quad \text{and} \quad f(z) = g(z) = \frac{1}{z} \quad \text{in } \Delta \setminus \{0\}.$$

Then, $f_n \to f$ and $g_n \to g$ both uniformly in $\Delta \setminus \{0\}$. Now

$$f_n(z)g_n(z) - f(z)g(z) = \frac{1}{n^2} + \frac{2}{n} \cdot \frac{1}{z}.$$

In particular, for $z = n^{-1}$, or n^{-2} in Δ, we find that $(f_n g_n - fg)(z) \geq 2$ for all $n \geq 1$. Hence, $f_n g_n \not\to fg$ uniformly in Δ.

2.57. Theorem. *The limit function of a uniformly convergent sequence of continuous functions is itself continuous.*

Proof. Let each f_n be continuous in D and suppose that $f_n \to f$ uniformly in D. Let $z_0 \in D$ be given. Now, for all $z \in D$ and for all indices n, the triangle inequality gives

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|.$$

The claim now follows from this inequality upon using the hypothesis. ■

2.58. Corollary. *If $\sum_{n\geq 1} f_n$ is uniformly convergent in D and if f_n is continuous in D for each n, then so is the sum $f = \sum_{n\geq 1} f_n$.*

Corollary 2.58 makes no assertion about the sum of a series of continuous functions if the convergence is not uniform.

The Cauchy convergence criterion for series of complex numbers discussed in Chapter 1 takes the following form: "*The series $\sum_{n\geq 1} f_n(z)$ converges uniformly in D iff for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that*

$$\left|\sum_{k=m+1}^{n} f_k(z)\right| = |s_n - s_m| < \epsilon \text{ for all } z \in D \text{ whenever } n > m \geq N\text{''}.$$

The next theorem gives a sufficient condition for the uniform convergence of the series $\sum_{n\geq 1} f_n(z)$. This is one of the frequently used results in complex analysis.

2.59. Theorem. (Weierstrass' M-test) *Let $\sum_{n\geq 1} M_n$ be a convergent series of positive real numbers such that $|f_n(z)| \leq M_n$ for all $n \in \mathbb{N}$ and for all z in a set D. Then the series $\sum_{n\geq 1} f_n(z)$ converges absolutely and uniformly in D.*

Proof. To prove this, we note that, for all $n > N$ and $z \in D$,

$$\left|\sum_{k=N+1}^{n} f_k(z)\right| \leq \sum_{k=N+1}^{n} |f_k(z)| \leq \sum_{k=N+1}^{n} M_k. \tag{2.60}$$

The Cauchy criterion (see Theorem 2.56(a) and Definition 2.53) applied to the series $\sum_{n\geq 1} M_n$ shows that for every $\epsilon > 0$, there exists an $N(\epsilon)$ such that $\sum_{N+1}^{n} M_k < \epsilon$ for all $n > N = N(\epsilon)$ This observation, by (2.60), implies that $\sum_{n\geq 1} f_n(z)$ satisfies the Cauchy criterion (see Theorem 2.56(a)) for uniform convergence on D and so the assertion follows. ■

We illustrate the application of Weierstrass' M-test with an example which is already familiar for us. The geometric series $\sum_{n\geq 1} z^{n-1}$ converges uniformly for $|z| \leq r$, where $0 < r < 1$. Finally, we end this section with two more examples.

2.61. Example. Define

$$f_n(z) = \frac{1+z^n}{1-z^n} = \frac{2}{1-z^n} - 1, \quad n \in \mathbb{N}.$$

Note that if $|z| < 1$, then $|z|^n \to 0$ as $n \to \infty$; and if $|z| > 1$, then $|z|^n \to \infty$. Therefore, if $|z| > 1$ then for sufficiently large values of n, we have $|1 - z^n| \geq |z|^n - 1 \to \infty$ as $n \to \infty$. Hence, for $|z| \neq 1$, $\lim_{n\to\infty} f_n(z)$ exists and

$$f(z) = \lim_{n\to\infty} f_n(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ -1 & \text{if } |z| > 1. \end{cases}$$

From this, we also conclude that it is not possible to define the limit function $f(z)$ on $|z| = 1$ so that f becomes a continuous function in $\mathbb{C}$. ●

2.62. Example. Let us discuss the convergence of

$$\sum_{k=1}^{\infty} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} \quad (|z| \neq 1).$$

For $|z| \neq 1$, the partial sums takes the form

$$\begin{aligned} S_n(z) &= \sum_{k=1}^{n} \frac{1}{1-z}\left[\frac{z^{k-1}}{1-z^k} - \frac{z^k}{1-z^{k+1}}\right] \\ &= \frac{1}{1-z}\left[\frac{1}{1-z} - \frac{z^n}{1-z^{n+1}}\right]. \end{aligned}$$

Note that $S_n(0) = 1$. For $0 < |z| < 1$, we have

$$\left|\frac{z^n}{1-z^{n+1}}\right| \leq \frac{|z|^n}{1-|z|^{n+1}} = \to 0 \quad \text{as} \quad n \to \infty$$

and for $|z| > 1$, we get

$$\frac{z^n}{1-z^{n+1}} = \frac{1}{z^{-n} - z} \to -\frac{1}{z} \quad \text{as} \quad n \to \infty.$$

It follows that, as $n \to \infty$,

$$S_n(z) \to f(z) = \begin{cases} \dfrac{1}{(1-z)^2} & \text{if } |z| < 1 \\ \dfrac{1}{1-z}\left[\dfrac{1}{1-z} + \dfrac{1}{z}\right] & \text{if } |z| > 1. \end{cases}$$ ●

2.5 Exercises

2.63. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The function $f(z) = \alpha z + 1/(\alpha z)$ is univalent in $\{z : |z| > |\alpha|^{-1}\}$, $\alpha \neq 0$.

(b) The function $f(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$, is univalent in $\mathbb{C}\backslash\{-d/c\}$.

(c) The function $f(z) = (1+z)^3$ is not univalent in Δ.

(d) For every positive integer n, the function $f(z) = nz + z^n$ is univalent in Δ.

(e) If n is a positive integer, then $|z| < 1/n$ is the largest disk centered at 0 on which the function $f(z) = z + z^n$ is univalent.

(f) The Koebe function $k(z) = z/(1-z)^2$ is univalent in Δ.

(g) The range of the Koebe function $k(z) = z/(1-z)^2$ is $(-\infty, -1/4]$.

(h) The series $\sum a_n$ converges to $a \iff \sum \overline{a}_n$ converges to $\overline{a}$.

(i) The function $f(z) = \overline{z}^2$ is uniformly continuous on Δ_R (also $\Delta_R \backslash \{0\}$), where R is a fixed positive number, but not in $\mathbb{C}$.

(j) Each of the limits $\lim_{z\to 0} \dfrac{\mathrm{Re}\, z^2}{|z|^2}$ and $\lim_{z\to 0} \dfrac{\mathrm{Im}\, z^2}{|z|^2}$ does not exist.

(k) For $f(z) = [\mathrm{Re}\,(z) + \mathrm{Im}\,(z)]/|z|^2$, $\lim_{z\to 0} f(z)$ does not exist.

(l) The function $f(z) = (\mathrm{Re}\, z)/(\mathrm{Im}\, z)$ is continuous for all z with $\mathrm{Im}\, z \neq 0$ and discontinuous for all z with $\mathrm{Im}\, z = 0$. Also, no discontinuity of $f(z)$ is removable.

(m) The function $f(z) = (2+z)\mathrm{Arg}\, z$ does not have removable discontinuities.

(n) The function $f(z) = (\mathrm{Arg}\, z)^2$ is continuous on the punctured plane $\mathbb{C}\backslash\{0\}$.

(o) The function $F(z,h) = [(z+h)^n - z^n]/h - nz^{n-1}$ $(0 \neq h, z \in \mathbb{C})$ satisfies the inequality $|F(z,h)| \leq F(|z|,|h|)$.

(p) The sequence $\{f_n(z)\}$, where $f_n(z) = 1/(1+nz)$, does not converge to $f(z) = 0$ uniformly in any closed region containing the origin.

(q) The sequence $\{\frac{1}{nz}\}_{n\geq 1}$ converges for $0 < |z| < 1$, but not uniformly. If $r > 0$ is fixed, the convergence is uniform for $r < |z| < 1$.

(r) The sequence $\{f_n(z)\}_{n\geq 1}$, for $|z| < 1$, where $f_n(z) = z^3 - z/n$, converges uniformly to the limit function $f(z) = z^3$ for $z \in \Delta$.

(s) The sequence $\{z^n/n\}_{n\geq 1}$ converges uniformly to 0.

(t) The series $\sum_{n\geq 0} 3^{-n} z^n$ converges uniformly for $|z| \leq r$, $0 < r < 3$.

(u) The series $\sum_{n=0}^{\infty} z^2/(1+z^2)^n$ converges for all z exterior to the lemniscate $|z^2+1| = 1$.

(v) The transformation $w = 2^{-1}[z + \alpha^2 z^{-1}]$ $(\alpha \in \mathbb{R})$, maps the circle $|z| = r$ $(r \neq \alpha)$ into an ellipse in the w-plane.

(w) The series $\sum_{n\geq 0} z^{2n}/(1-z^{2n})$ converges uniformly for $|z| \leq r$, $0 < r < 1$.

(x) The mapping π defined for stereographic projection is a homeomorphism (i.e. bijective and bicontinuous).

2.64. Discuss the continuity of the following complex-valued functions at $z = 0$:

(a) $f(z) = \begin{cases} \dfrac{(\mathrm{Re}\, z)\,(\mathrm{Im}\, z)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

(b) $f(z) = \begin{cases} \dfrac{\mathrm{Im}\, z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

(c) $f(z) = (\mathrm{Re}\, z)/(1 + |z|)$ for $z \in \mathbb{C}$.

(d) $f(z) = |\mathrm{Re}\,(z)\,\mathrm{Im}\,(z)|$ for $z \in \mathbb{C}$.

(e) $f(z) = \begin{cases} \dfrac{\sin |z|}{|z|} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$

(f) $f(z) = \begin{cases} \dfrac{1 - \exp(-|z|^2)}{|z|^2} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$

2.65. Describe the position of the following points in relation to z in the complex plane, as viewed on the Riemann sphere:

$$z,\ \overline{z},\ 1/z,\ (z + \overline{z})/2,\ \ \pm iz,\ \frac{z - \overline{z}}{2},\ \frac{z}{\overline{z}},\ \frac{z}{|z|}.$$

Also, for each of the following points in $\mathbb{C}$, give the corresponding points on the Riemann sphere: $0,\ 1+i,\ 1-i,\ 2+3i,\ 2-3i$.

2.66. Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$. Check whether the following statements are equivalent or not:

(a) f is not uniformly continuous in D

(b) there exists an $\epsilon > 0$ such that for every $\delta > 0$ there are points ζ_δ and η_δ in D such that $|\zeta_\delta - \eta_\delta| < \delta$ and $|f(\zeta_\delta) - f(\eta_\delta)| \geq \epsilon$

(c) there exists an $\epsilon > 0$, and two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in D such that for every $n \in \mathbb{N}$, $|\zeta_n - \eta_n| < 1/n$ and $|f(\zeta_n) - f(\eta_n)| \geq \epsilon$.

2.67. Show that the field of 2×2 matrices $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is isomorphic to the field of complex numbers $x + iy$.

Chapter 3

Analytic Functions and Power Series

In Section 3.1 we discuss the fundamental difference between the derivative of a function of a real and that of a complex variable, by providing necessary and sufficient conditions for differentiability. As a consequence, a continuously differentiable function (briefly C^1-function) f defined in an open set D is analytic if $f_{\overline{z}}(z) = 0$ on D. There is an interesting relationship between functions that are analytic in a domain and real-valued functions that are harmonic in that domain. In Section 3.2 we discuss this relationship at an introductory level, especially in finding harmonic conjugates. In addition, we discuss the polar form of the Cauchy-Riemann equations and the Laplace equation which have many implications in various branches of applied mathematics. In Section 3.3, we investigate 'infinite polynomials'-a generalization of 'polynomials' and the infinite series of complex numbers. The discussion allow us to characterize the behavior of the power series in a very natural fashion. Therefore, the aim of this section is to study the convergence of the power series. In Section 3.4, we examine various known properties of the so-called standard or elementary functions from a real variable to a complex variable. Section 3.5 illustrates the relationship between $w = \log z$ $(z \neq 0)$ and $z = e^w$ and considers its local properties. Several standard functions are associated with exponential functions or logarithmic functions. As a consequence, in Section 3.5, we define the inverse trigonometric and the inverse hyperbolic functions.

3.1 Differentiability and Cauchy-Riemann Equations

Differentiation in $\mathbb{C}$ is set against a background of limits, continuity and so on. To some extent the rules for differentiation of a function of a complex variable are similar to those of differentiation of a function of a real variable. Since $\mathbb{C}$ is merely $\mathbb{R}^2$ with additional structures of addition and multiplication of complex numbers, we can immediately transfer *most* of the concepts of $\mathbb{R}^2$ into those for the complex field $\mathbb{C}$. In fact, we have already done so when we defined the concept of distance (modulus),

$$|z - z'| = \sqrt{(x-x')^2 + (y-y')^2} \quad (z = x+iy \text{ and } z' = x'+iy')$$

which is the same as the *Euclidean distance* between the points $z = (x, y)$ and $z' = (x', y')$ in $\mathbb{R}^2$. We now start with

3.1. Definition. We say that a complex function f defined in a non-empty open set D is *differentiable* (or *complex differentiable*) at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{3.2}$$

exists. When this is the case, the value of the limit, denoted by $f'(z_0)$, is called the derivative of f at z_0. The number $f'(z_0)$ is generally a complex number. The function f is said to be *differentiable in* (*on*) D if it is differentiable at every point of D. A function which is differentiable in the entire complex plane is called an *entire function.*

We observe that "formally" the limit definition of the derivative given by (3.2) is identical in form to that of the derivative of a real (or complex)-valued function of a real variable. In terms of '$\epsilon - \delta$' notation, the limit in (3.2) exists iff given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, z_0) > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Letting

$$\eta(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{for } 0 < |z - z_0| < \delta \\ 0 & \text{for } z = z_0, \end{cases}$$

we have $\lim_{z \to z_0} \eta(z) = 0 = \eta(z_0)$. Therefore η is continuous at z_0 and we get an explicit expression for $f(z)$ in the form

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z) \tag{3.3}$$

for $|z - z_0| < \delta$. In conclusion, we have

3.4. Proposition. *Let $D \subseteq \mathbb{C}$ be open, $f : D \to \mathbb{C}$ and $z_0 \in D$. Then $f'(z_0)$ exists iff there exists a function $\eta : D \to \mathbb{C}$ which is continuous at z_0 and satisfies* (3.3) *for all $z \in D$. Equivalently, f is differentiable at z_0 iff*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z) \tag{3.5}$$

where E is a function defined in a neighborhood of a such that

$$\lim_{z \to z_0} |E(z)/(z - z_0)| = 0.$$

By (3.5), we obtain a linear function $L(z) = f(z_0) + f'(z_0)(z - z_0)$ which approximates $f(z)$ up to an "error" term $E(z)$, which, for z close to z_0, is small, in absolute value in comparison with $|z - z_0|$. This may be treated as an alternate and geometric description of differentiability at a point. At this place it is necessary to make an important point to the reader. Since it is not possible to graph a complex functions in the usual way as we do with real-valued functions, it is not meaningful to visualize $f'(z_0)$ as a 'slope' of some curve as we do in the real case.

It is immediate that the derivative of a constant function is zero; for if $f(z) \equiv c$, then for each $z_0 \in \mathbb{C}$ we have

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

3.6. Theorem. *A real-valued function of a complex variable either has derivative zero or the derivative does not exist.*

Proof. Suppose that $f(z)$ is a real-valued function of complex variable whose derivative exists at a point a. Then

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If we take the limit $h \to 0$ along the real axis, then $f'(a)$ is real. If we take the limit $h \to 0$ along the imaginary axis, then $f'(a)$ becomes a purely imaginary number. So we must have $f'(a) = 0$. ∎

There are several other immediate consequences of Definition 3.1 which are worth mentioning. First along this line is to allow $z \to z_0$ in (3.3) and obtain

3.7. Theorem. *If f has derivative at z_0, then it is continuous at z_0.*

As in the real case, continuity of f does not necessarily imply differentiability of f in complex case also. The function $f(z) = |z|$ demonstrates this. For if $h = h_1 + ih_2$ (h_1, h_2 real), then

$$\frac{f(h) - f(0)}{h} = \frac{|h|}{h} \longrightarrow \begin{cases} 1 & \text{for } h = h_1 + i \cdot 0,\ 0 < h_1 \to 0 \\ -1 & \text{for } h = h_1 + i \cdot 0,\ 0 > h_1 \to 0 \\ -i & \text{for } h = 0 + ih_2,\ 0 < h_2 \to 0 \\ i & \text{for } h = 0 + ih_2,\ 0 > h_2 \to 0. \end{cases} \tag{3.8}$$

Thus, $|z|$ is not differentiable at $z = 0$ but is continuous at 0. Is f differentiable at other points in $\mathbb{C}$?

Note that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is (real) differentiable on $\mathbb{R} \backslash \{0\}$ and continuous on $\mathbb{R}$ whereas its complex analog map $f : \mathbb{C} \to \mathbb{C}$

defined by $f(z) = |z|$ is nowhere differentiable although it is continuous in $\mathbb{C}$ (see Example 3.10).

Next we give an example of a function which is differentiable at a single point in $\mathbb{C}$ and nowhere else. For example, if $f(z) = |z|^2$ then f is clearly continuous in $\mathbb{C}$. On the other hand (with $h \neq 0$), we have

$$\frac{f(h) - f(0)}{h} = \frac{h\overline{h}}{h} = \overline{h} \to 0 \quad \text{as } h \to 0$$

so that f is differentiable at the origin with derivative $f'(0) = 0$. Is f differentiable at any other point? Now let $z_0 \neq 0$ be an arbitrary point in $\mathbb{C} \setminus \{0\}$. With $0 \neq h$ as neighboring variable point of 0, we have

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{|z_0 + h|^2 - |z_0|^2}{h} \\ &= \frac{\overline{z}_0 h + z_0 \overline{h} + h\overline{h}}{h} \\ &= \overline{z}_0 + z_0 \left(\frac{\overline{h}}{h}\right) + \overline{h}. \end{aligned}$$

As $z_0 \neq 0$, (3.8) shows that the expression on the right tends to different values as $h \to 0$. Thus, the function $f(z) = |z|^2$ cannot be differentiable at $z_0 \neq 0$. How about the translated function g defined by $g(z) = f(z - a)$? Clearly, $g(z) = |z - a|^2$ (a is a fixed point in $\mathbb{C}$) is differentiable only at a and nowhere else.

Similarly, it is easy to verify that the function $f(z) = z\mathrm{Re}\, z$ is differentiable at $z = 0$ only. Problems like this can be done also by using the Cauchy-Riemann equations. Thus, there exist complex functions which are differentiable at a given point but not in any neighborhood of that point.

Many formulas for the derivatives of complex functions are the same as those for the real counterparts. For example, it follows that the monomials $f(z) = z^n$ ($n \in \mathbb{N}$) are entire, i.e. differentiable in the entire complex plane. In fact, for each fixed $z_0 \in \mathbb{C}$, we have

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} \\ &= \lim_{z \to z_0} [z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}] = n z_0^{n-1}. \end{aligned}$$

In consonance with Real Analysis we can write this in the form

$$f'(z) = \frac{d}{dz}(z^n) = n z^{n-1}$$

for every $z \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$. By Definition 3.1, we have the following standard theorems on differentiation in complex counterpart.

3.9. Theorem. *If f and g are differentiable at z_0, then their sum $f+g$, difference $f-g$, product fg, quotient f/g (where $g(z_0) \neq 0$) and the scalar multiplication cf, are also differentiable at z_0 and*

$$(f \pm g)' = f' \pm g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad (cf)' = cf',$$

where c is a complex constant.

More generally, a finite linear combinations (of the form $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$, $\alpha_j \in \mathbb{C}$, $j = 1, 2, \ldots n$) and finite products of functions differentiable at z_0 are also differentiable at z_0.

Proof. For $z \neq z_0$, consider the following:

$$\frac{(f \pm g)(z) - (f \pm g)(z_0)}{z - z_0} = \left\{\frac{f(z) - f(z_0)}{z - z_0}\right\} \pm \left\{\frac{g(z) - g(z_0)}{z - z_0}\right\}$$

$$\frac{(fg)(z) - (fg)(z_0)}{z - z_0} = f(z)\left\{\frac{g(z) - g(z_0)}{z - z_0}\right\} + g(z_0)\left\{\frac{f(z) - f(z_0)}{z - z_0}\right\}$$

and, for $g(z_0) \neq 0$,

$$\frac{\left(\dfrac{f}{g}\right)(z) - \left(\dfrac{f}{g}\right)(z_0)}{z - z_0} = \frac{g(z_0)\left(\dfrac{f(z) - f(z_0)}{z - z_0}\right) - f(z_0)\left(\dfrac{g(z) - g(z_0)}{z - z_0}\right)}{g(z)g(z_0)}.$$

The assertions then follow from the above equalities and the properties of the limit by making $z \to z_0$ and noting that if f, g are differentiable at z_0 they are continuous at z_0 so that $f(z) \to f(z_0)$, $g(z) \to g(z_0)$ as $z \to z_0$. ■

Let $f(z) = 1$ and $g(z) = z^n$ ($z \neq 0$, $n \in \mathbb{N}$). Then $g'(z) = nz^{n-1}$ and Theorem 3.9 help us to get $h'(z)$, where $h(z) = f(z)/g(z) = 1/z^n$, $z \neq 0$:

$$h'(z) = \frac{d}{dz}\left(\frac{1}{z^n}\right) = -nz^{-n-1}$$

and so $h(z) = z^{-n}$ is differentiable in $\mathbb{C}\backslash\{0\}$. Further, Theorem 3.9 shows that each polynomial of the form $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where n is a non-negative integer, is differentiable in $\mathbb{C}$ and has the derivative

$$p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}, \quad z \in \mathbb{C},$$

obtained by term-by-term differentiation of the polynomial $p(z)$. In particular, every polynomial is an entire function. This fact and Theorem 3.9 immediately imply that a rational function of the form $R(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials in z, is differentiable at all points except where the denominator vanishes and the formula for $R'(z)$ is obtained by using Theorem 3.9.

3.10. Example. Let $f(z) = \operatorname{Re} z$ and z_0 be an arbitrary fixed point in $\mathbb{C}$. Then, for $h = h_1 + ih_2$ ($\neq 0$),

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\operatorname{Re}(h)}{h} = \begin{cases} 1 & \text{for } h = h_1 + i.0 \in \mathbb{R}\setminus\{0\} \\ 0 & \text{for } h = 0 + ih_2 \in i(\mathbb{R}\setminus\{0\}) \end{cases}$$

so that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

does not exist. Therefore, $f(z) = \operatorname{Re} z$ is nowhere differentiable even though it is continuous in $\mathbb{C}$.

Similarly, we see that $\operatorname{Im} z$, $|z|$, $\overline{z}$ and $\operatorname{Arg} z$ are all nowhere differentiable in $\mathbb{C}$. Is each of the functions listed here continuous in $\mathbb{C}$? Is each of these functions infinitely often real differentiable in $\mathbb{R}^2$? Is a product of two nowhere differentiable functions always nowhere differentiable? How about $g(z) = (\overline{z})^2$? ●

3.11. Definition. A function f is said to be *analytic*, or *holomorphic* at a point $a \in \mathbb{C}$ if it is differentiable at every point of some neighborhood of a. Similarly, f is analytic in (on) an arbitrary set S if it is differentiable at every point of some open set containing S. An *entire function* is then the one which is analytic in (on) the whole complex plane.

Formally, we say that a point at which a function ceases to be analytic is called a *singularity* or a *singular point*. For a precise definition, we refer to Section 7.1. Can we say $f(z) = \overline{z}$ has every point in $\mathbb{C}$ as its singularity? The answer is no, see Example 7.1.

Observe that even though $f(z) = |z-a|^2$ is differentiable at a, it is not analytic at this point because there does not exist a neighborhood of a in which $|z-a|^2$ is differentiable at each point of the neighborhood. Moreover, this function is nowhere analytic in $\mathbb{C}$.

The meaning of sentences such as "f is analytic for $|z| \leq R$, $R > 0$" should now be clear (meaning that f is analytic in some domain containing the closed region $\overline{\Delta}_R$. Further, it is also clear from the definition that if D is an open set then the phrase "analytic in (on) D" means "differentiable at all points of D". The set of all analytic functions (mappings) in the open set D is denoted by $\mathcal{H}(D)$, $\mathcal{H}$ standing for 'holomorphic', an alternative for 'analytic'. As usual, we write

$$f^{(0)} = f, \;\; f^{(1)} = f', \;\; f^{(2)} = f'' \;\text{ and so on.}$$

Because of the inherent two-dimensional ($\mathbb{R}^2$) character of a complex variable ($\mathbb{C}$), the usual rules of the derivative from calculus may be used when differentiating analytic functions. The proofs extend without much change beyond the real case by replacing the real variables x, y by complex variables z, w. As an example, a direct consequence of Definition 3.1 gives the following *chain rule* for differentiation of composite functions.

3.12. Theorem. *Let $f : D_1 \to \mathbb{C}$, $g : D_2 \to \mathbb{C}$ be such that $f(D_1) \subseteq D_2$. If f is differentiable at z_0 and g is differentiable at $w_0 = f(z_0)$, then the composition $(g \circ f)(z) = g(f(z))$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0) = (g' \circ f)(z_0)f'(z_0).$$

Further, if $f \in \mathcal{H}(D_1)$ and $g \in \mathcal{H}(D_2)$ then $g \circ f \in \mathcal{H}(D_1)$.

Proof. Let $w = f(z)$, $z \in D_1$ and assume the hypotheses. By (3.3), since f is differentiable at z_0, there exists a $\delta_1 > 0$ such that

$$f(z) - f(z_0) = (z - z_0)\eta_f(z) \quad \text{for } z \in \Delta(z_0; \delta_1) \subset D_1,$$

where η_f ($\equiv f'(z_0) + \eta(z)$ in (3.3)) is continuous in $\Delta(z_0; \delta_1)$ with

$$\lim_{z \to z_0} \eta_f(z) = f'(z_0).$$

Further, since g is continuous at w_0, there exists a $\delta_2 > 0$ such that

$$g(w) - g(w_0) = (w - w_0)\eta_g(w) \quad \text{for } w \in \Delta(w_0; \delta_2) \subset D_2,$$

where η_g is continuous in $\Delta(w_0; \delta_2)$ with $\lim_{w \to w_0} \eta_g(w) = g'(w_0)$. Now choose $\delta > 0$ such that $\delta < \delta_1$ and

$$|z - z_0| < \delta_1 \Rightarrow |f(z) - f(z_0)| < \delta_2.$$

Then for $z \in \Delta(z_0; \delta)$, we have by substitution

$$\begin{aligned} g(f(z)) - g(f(z_0)) &= (f(z) - f(z_0))\eta_g(f(z)) \\ &= (z - z_0)\eta_f(z) \cdot \eta_g(f(z)) \\ &= (z - z_0)\eta_{g \circ f}(z). \end{aligned}$$

By Corollary 2.21, $\eta_g \circ f$ is continuous at z_0 and its value at z_0 is $\eta_g(w_0) = g'(w_0)$. By Theorem 2.18, the product $\eta_f(z) \cdot \eta_g(f(z))$, i.e. $\eta_{g \circ f}(z)$, is continuous at z_0 and

$$\lim_{z \to z_0} \eta_{g \circ f}(z) = \lim_{z \to z_0} [\eta_f(z) \cdot \eta_g(f(z))] = f'(z_0) \cdot g'(w_0) = f'(z_0)g'(f(z_0)).$$

The assertion now follows from (3.3). ■

Theorem 3.9 leads quickly to the following useful properties of analytic functions.

3.13. Theorem. *Linear combinations and finite products of analytic functions in an open set D are all analytic in D. If f and g are analytic in*

D, then the quotient f/g is analytic in D except for those z in D at which g vanishes.

3.14. Corollary. *If f and g are entire then so are $f \pm g$, fg; and $f \circ g$ is entire when it is defined.*

We know that if $F(z) = U(x,y) + iV(x,y)$ where U and V are real-valued functions defined in a neighborhood of z_0, then the partial derivative $U_x(x,y)$ of U with respect to x at (x,y) (where $z = x + iy$), if it exists, is defined to be

$$U_x(x,y) = \lim_{h \to 0} \frac{U(x+h,y) - U(x,y)}{h}. \tag{3.15}$$

Similarly, the partial derivative $U_y(x,y)$ of U with respect to y at (x,y), if it exists, is defined to be

$$U_y(x,y) = \lim_{h \to 0} \frac{U(x,y+h) - U(x,y)}{h}. \tag{3.16}$$

Sometimes (3.15) and (3.16) are denoted by the Leibnitz notation,

$$\frac{\partial U}{\partial x}(z) = \frac{\partial U}{\partial x}(x,y) \quad \text{and} \quad \frac{\partial U}{\partial y}(z) = \frac{\partial U}{\partial y}(x,y),$$

respectively. Note that h in (3.15) and (3.16) is a non-zero real number near 0. The partial derivatives $F_x(x,y)$ and $F_y(x,y)$ of the complex-valued function of the complex variable $F(z) = U(x,y) + iV(x,y)$ at $z = x + iy$ are defined by

$$F_x(z) = U_x(x,y) + iV_x(x,y) \quad \text{and} \quad F_y(z) = U_y(x,y) + iV_y(x,y),$$

respectively, provided the partial derivatives on the right side of the corresponding equations exist. We simply write these expressions as

$$F_x = U_x + iV_x, \quad \text{and} \quad F_y = U_y + iV_y.$$

If $F_y = iF_x$, then, by equating imaginary and real parts, we have a pair of famous partial differential equations

$$U_x = V_y, \quad U_y = -V_x.$$

Conversely, the later two equations imply $F_y = iF_x$. These two extremely important partial differential equations are called the *Cauchy-Riemann*[6] (briefly we write the C-R) equations. The former is actually referred to as

[6]These two equations are named in honor of French a mathematician, Augustin-Louis Cauchy (1789-1857), who discovered them, and in honor of a German mathematician, Georg Friedrich Bernhard Riemann (1826-1866), who made them fundamental in the development of the theory of complex analysis. Riemann is considered one of the three founders of complex function theory; the others being Cauchy and Weierstrass.

the C-R equations in Cartesian form. We note that the *existence of the derivative* for real-valued functions of single real variable is a mild smooth condition while the same for complex-valued functions of a complex variable leads to the above pair of partial differential equations. We usually write

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) = \operatorname{Re} f(z) + i\operatorname{Im} f(z)$$

or $f(z) = u(z)+iv(z)$ instead. However, we may abuse the notation slightly by writing $f(z) = f(x,y)$, but when we write it like this, we actually identify $u(x,y) + iv(x,y) \in \mathbb{C}$ with $(u(x,y), v(x,y)) \in \mathbb{R}^2$ and use results from two variable calculus for our investigation.

If $\Omega \subseteq \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ is a continuous function, then u is called a C^1- (or *continuously differentiable*) function in Ω if u_x and u_y exist and are continuous in Ω. More generally, if $k \in \mathbb{N}$ then u is said to be in $C^k(\Omega)$ (or simply a C^k function or k-times continuously differentiable function) if all the partial derivatives of u up to and including order k exist and are continuous in Ω. We indicate this by writing $u \in C^k$. Note that $C^0(\Omega)$ denotes the set of all continuous functions in Ω. A function $f : \Omega \to \mathbb{C}$ is said to belong to $C^k(\Omega)$, or simply call it a C^k-function, if both u and v belong to $C^k(\Omega)$.

From the inspection of functions such as $|z|$, $|z|^2$ and $z\operatorname{Re} z$, we conclude that it is not necessarily an easy task to determine whether a given function does or does not have a derivative. Let us start deriving a simple criterion which helps us to handle this problem.

3.17. Theorem. *If $f(z) = u(x,y) + iv(x,y)$ is differentiable at z_0, then the C-R equations hold at $z_0 = x_0 + iy_0$:*

$$if_x(z_0) = f_y(z_0);$$

or equivalently,

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Proof. Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is a neighborhood of z_0. If $f'(z_0)$ exists for some point $z_0 = x_0 + iy_0$, then the limit

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists and is independent of the path along which $h = h_1 + ih_2 \to 0$. In particular, we have

$$f'(z_0) = \lim_{h_1\to 0} \frac{f(x_0+h_1, y_0) - f(x_0,y_0)}{h_1 + i0} = \frac{\partial f}{\partial x}(z_0),$$

and

$$f'(z_0) = \lim_{h_2\to 0} \frac{f(x_0, y_0+h_2) - f(x_0,y_0)}{0 + ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

A comparison of the two expressions for $f'(z_0)$ shows that the complex differentiability of f at z_0 implies that not only the partial derivatives of f (with respect to x and y) exist at z_0, but also that they satisfy the C-R equations–in complex form

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0), \quad \text{i.e. } \; i f_x(z_0) = f_y(z_0). \tag{3.18}$$

Equating the real and imaginary parts yields the C-R equations–in Cartesian form $u_x(z_0) = v_y(z_0)$ and $u_y(z_0) = -v_x(z_0)$. ■

Another convenient notation is to treat the pair of conjugate complex variables z and $\overline{z}$ as two independent variables by writing

$$x = \frac{z+\overline{z}}{2}, \;\; y = -i\left(\frac{z-\overline{z}}{2}\right).$$

Now we introduce the following differential operators:

$$\frac{\partial}{\partial z} := \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It follows that if $f = u + iv$, then

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left([u_x + v_y] + i[v_x - u_y]\right) \tag{3.19}$$

and similarly,

$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left([u_x - v_y] + i[u_y + v_x]\right). \tag{3.20}$$

Therefore, the C-R equations (3.18) are exactly equivalent to $f_{\overline{z}}(z_0) = 0$ which is also referred to as the complex form of the C-R equations. With this notation, we have the following alternate form of Theorem 3.17.

3.21. Theorem. *A necessary condition for a complex-valued function $f = u + iv$ to be differentiable at z_0 is that $f_{\overline{z}}(z_0) = 0$. In particular, if $f \in \mathcal{H}(D)$, then C-R equations hold at every point $z \in D$.*

The C-R equations are more helpful in proving non-differentiability. For example, consider $f(z) = \overline{z}$ and $g(z) = \operatorname{Re} z$. Then $f_{\overline{z}}(z) = 1 \neq 0$ and writing g as

$$g(z) = \frac{z+\overline{z}}{2},$$

it follows that $g_{\overline{z}}(z) = 1/2 \neq 0$. Thus, both f and g are nowhere differentiable.

On the other hand, the converse of Theorem 3.17 (equivalently, Theorem 3.21) is not true. We can demonstrate this by a number of examples (for instance, see Examples 3.23 and 3.24). That is, the necessary condition stated in Theorem 3.17 for differentiability at a point is not generally sufficient for differentiability at that point.

3.22. Remark. Each of the functions

$$f_1(z) = |z|, \quad f_2(z) = \operatorname{Re} z \quad \text{and} \quad f_3(z) = \operatorname{Im} z,$$

is a non-constant real-valued function defined in $\mathbb{C}$ (see Theorem 3.6). Each of them is nowhere analytic. If we rewrite these functions as

$$f_1(x,y) = \sqrt{x^2+y^2}, \quad f_2(x,y) = x, \quad f_3(x,y) = y,$$

then, except f_1, each of these functions are real differentiable in $\mathbb{R}^2$. •

3.23. Example. It is easy to see that the function f defined by

$$f(z) = |\operatorname{Re} z \operatorname{Im} z|^{1/2}$$

satisfies the C-R equations at the origin, but is not differentiable at this point. To see this, we may rewrite the given function as

$$f(z) = \frac{|z^2 - \overline{z}^2|^{1/2}}{2} \quad \text{or } f = u + iv \text{ with } u(x,y) = |xy|^{1/2} \text{ and } v(x,y) = 0.$$

Note that f is identically zero on the real and imaginary axes. Therefore, it is trivial to see that $u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$. For example,

$$u_x(0,0) = \lim_{s\to 0} \frac{u(s,0) - u(0,0)}{s} = 0.$$

Thus, the C-R equations hold at $z = 0$. However, taking $h = re^{i\theta} \neq 0$ with $r \to 0$, we find that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{r\to 0} \frac{|r^2 \cos\theta \sin\theta|^{1/2}}{r(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}|\sin 2\theta|^{1/2}}{\sqrt{2}}$$

which is clearly depending upon θ (e.g. take $\theta = 0$ and $\theta = \pi/4$). We conclude that f is not differentiable at $z = 0$ even though f satisfies the C-R equations at the origin. Here, since $v(x,y) = 0$, v is a C^∞-function in $\mathbb{R}^2$. *Are the partial derivatives u_x and u_y continuous at the origin?* How about the functions

$$f(z) = |\operatorname{Re} z \operatorname{Im} z|^{1/3} \quad \text{and} \quad f(z) = |\operatorname{Re} z \operatorname{Im} z|^{1/4}?$$ •

3.24. Examples. Consider

$$f(z) = u + iv = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } x = y = 0. \end{cases}$$

For this function the corresponding u and v are continuous at the origin and therefore, f is continuous at the origin. Using the identification

$$f(x, y) = (u(x, y), v(x, y)) \longleftrightarrow f(z) = u(z) + iv(z),$$

for $0 \neq h \in \mathbb{R}$ and $0 \neq k \in \mathbb{R}$, we have

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{(h^3/h^2 + ih^3/h^2) - 0}{h} = 1 + i, \quad \text{i.e. } f_x(0, 0) = 1 + i$$

and

$$\frac{f(0, h) - f(0, 0)}{h} = \frac{(-h^3/h^2 + ih^3/h^2) - 0}{h} = -1 + i, \quad \text{i.e. } f_y(0, 0) = -1 + i$$

which shows that $if_x(0, 0) = f_y(0, 0)$. Thus we see that the C-R equations are certainly satisfied at the origin. But f is not differentiable at the origin, because for $h = h_1 + ih_1$, $h_1 \in \mathbb{R}$,

$$\frac{f(0 + h) - f(0)}{h} = \begin{cases} \dfrac{ih_1}{h_1 + ih_1} = \dfrac{1 + i}{2} & \text{for } h = h_1 + ih_1 \\ \dfrac{h_1 + ih_1}{h_1} = 1 + i & \text{for } h = h_1 + i \cdot 0. \end{cases}$$

This observation shows that the partial derivatives exist and satisfy the C-R equations at the origin even though the function is not differentiable there. A similar conclusion continues to hold for the following two functions:

$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases} = \begin{cases} \dfrac{z^3}{\overline{z}^2} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0, \end{cases}$$

and

$$g(z) = u + iv = \begin{cases} \dfrac{\operatorname{Im}(z^2)}{|z|^2} & \text{for } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

We shall provide details for the first case and leave the second case as an exercise. As before, using the notation $f(x, y) = (u(x, y), v(x, y))$, we see that

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{(h^5/|h|^4) - 0}{h} = 1, \quad \text{i.e. } f_x(0, 0) = 1,$$

and

$$\frac{f(0,h)-f(0,0)}{h} = \frac{((ih)^5/|h|^4)-0}{h} = i, \quad \text{i.e.} \quad f_y(0,0) = i,$$

so that $if_x(0,0) = f_y(0,0)$ and the C-R equations hold at the origin. Then, taking $h = re^{i\theta} \neq 0$ with $r \neq 0$, it follows that for $h \neq 0$,

$$\frac{f(h)-f(0)}{h} = \frac{h^4}{|h|^4} = e^{i4\theta}$$

and therefore, as $h \to 0$ along different paths, the difference quotient does not yield a unique value. Thus, f is not differentiable at the origin. Again, can f be differentiable at other points in $\mathbb{C}$? For $z_0 \neq 0$, we observe that

$$f_{\overline{z}}(z_0) = -2\left(\frac{z_0}{\overline{z}_0}\right)^3$$

which gives that $|f_{\overline{z}}(z_0)| = 2$ for $z_0 \neq 0$. Thus, f cannot be differentiable at $z_0 \neq 0$ and hence, it is nowhere differentiable. ●

Next we give sufficient conditions for a complex-valued function to be differentiable. To present this we need the following lemma which is a well-known result from two variable calculus.

3.25. Lemma. *Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, where Ω is an open set containing the point (x_0, y_0). Suppose that*

(i) *u_x and u_y exist at every point in a neighborhood of (x_0, y_0), and*

(ii) *u_x and u_y are continuous at (x_0, y_0).*

Then, for sufficiently small s and t in $\mathbb{R}$,

$$u(x_0+s, y_0+t) - u(x_0, y_0) = su_x(x_0+s^*, y_0+t) + tu_y(x_0, y_0+t^*)$$

with $|s^*| < |s|$ and $|t^*| < |t|$.

Proof. We consider

$$\begin{aligned} u(x_0+s, y_0+t) - u(x_0, y_0) &= [u(x_0+s, y_0+t) - u(x_0, y_0+t)] \\ &\quad + [u(x_0, y_0+t) - u(x_0, y_0)]. \end{aligned}$$

By the mean value theorem of calculus applied to the function

$$\phi(x) = u(x, y_0+t)$$

for x between x_0 and x_0+s, we have

$$u(x_0+s, y_0+t) - u(x_0, y_0+t) = su_x(x_0+s^*, y_0+t)$$

with $|s^*| < |s|$. (Indeed when we restrict u to a horizontal line segment in the disk $\Delta(z_0;\delta)$, then it reduces to a differentiable function of the real variable x with derivative u_x, whilst in the vertical segment in this disk u is a function of y alone and has the derivative u_y. Note that corresponding to (x_0, y_0), there exist points x^* in

$$[x_0, x_0 + s] = \{(1-t)x_0 + t(x_0 + s) : t \in [0,1]\} \subseteq \Delta(z_0;\delta),$$

that is

$$\begin{cases} x_0 < x^* < x_0 + s & \text{if } s > 0 \\ x_0 + s < x^* < x_0 & \text{if } s < 0, \end{cases}$$

so that $|x^* - x_0| < |s|$. Thus, corresponding to z_0, there exist points $x^* \in [x_0, x_0 + s]$ with $|s^*| < |s|$, satisfying the desired equation). Similarly, we have

$$u(x_0, y_0 + t) - u(x_0, y_0) = tu_y(x_0, y_0 + t^*)$$

with $|t^*| < |t|$. Adding this with the previous expression gives the desired result. ∎

3.26. Theorem. *Let $f : \Omega \to \mathbb{C}$, where Ω is an open set in $\mathbb{C}$ containing the point z_0, and $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy \in \Omega$. Suppose that*

(i) *u_x, u_y, v_x, and v_y exist at each point in a neighborhood of z_0 and are continuous at z_0*

(ii) *the C-R equations are valid at z_0.*

Then f is differentiable at z_0 and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

Proof. Let $z_0 = x_0 + iy_0$. We wish to show that $f'(z_0)$ exists. Since Ω is open and $z_0 \in \Omega$, there exists a $\delta > 0$ such that $\Delta(z_0;\delta) \subset \Omega$. Choose $h = s + it \neq 0$ with $0 < |h| < \delta$, so that $z_0 + h \in \Delta(z_0;\delta)$. Now we consider

$$\begin{aligned} f(z_0 + h) - f(z_0) &= f(x_0 + s, y_0 + t) - f(x_0, y_0) \\ &= f(x_0 + s, y_0 + t) - f(x_0, y_0 + t) \\ &\quad + f(x_0, y_0 + t) - f(x_0, y_0). \end{aligned}$$

The mean value theorem applied to both real and imaginary parts of $f(z)$ (see Lemma 3.25) shows that

$$f(z_0 + h) - f(z_0) = sf_x(x_0 + s^*, y_0 + t) + tf_y(x_0, y_0 + t^*) \tag{3.27}$$

with $|s^*| < |s|$ and $|t^*| < |t|$. Remember that

$$h \to 0 \iff s \to 0 \ \text{ and } \ t \to 0 \implies s^* \to 0 \ \text{ and } \ t^* \to 0.$$

By assumption u_x, u_y, v_x and v_y exist in some neighborhood of z_0 and are continuous at z_0 and therefore, f_x and f_y exist in that neighborhood and are continuous at z_0. In particular, as f_x is continuous, we note that

$$f_x(x_0 + s^*, y_0 + t) \to f_x(x_0, y_0) \quad \text{as} \quad h \to 0;$$

or equivalently

$$f_x(x_0 + s^*, y_0 + t) = f_x(x_0, y_0) + \alpha(h), \tag{3.28}$$

where $\alpha(h) \to 0$ as $h \to 0$. Similarly, as f_y is continuous at z_0, we find that

$$f_y(x_0, y_0 + t^*) = f_y(x_0, y_0) + \beta(h),$$

where $\beta(h) \to 0$ as $h \to 0$. Further, since the C-R equations are satisfied at z_0, we have $f_y(x_0, y_0) = if_x(x_0, y_0)$ and so the last equation becomes

$$f_y(x_0, y_0 + t^*) = if_x(x_0, y_0) + \beta(h). \tag{3.29}$$

By (3.28) and (3.29), (3.27) gives that for $h = s + it$ with $0 < |h| < \delta$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{s[f_x(x_0, y_0) + \alpha(h)]}{s + it} + \frac{t[if_x(x_0, y_0) + \beta(h)]}{s + it} \\ &= f_x(x_0, y_0) + \alpha(h)\left(\frac{s}{s + it}\right) + \beta(h)\left(\frac{t}{s + it}\right). \end{aligned}$$

Note that $s/(s + it)$ and $t/(s + it)$ are bounded by 1, and each of $\alpha(h)$ and $\beta(h)$ approaches zero as $h \to 0$. Consequently, taking the limit as $h \to 0$ in the above equation, we see that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and is equal to $f_x(x_0, y_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. ∎

If the hypotheses of Theorem 3.26 *hold at each point of a neighborhood* $\Delta(z_0; \delta)$ *of* z_0*, then* f *is analytic in* $\Delta(z_0; \delta)$. Further, to minimize the notational difficulties, we could assume $z_0 = 0$ in Theorem 3.26, because the result for $z_0 \neq 0$ can be retrieved from the case $z_0 = 0$ by merely applying the latter to the function $g(z) = f(z + z_0)$ which exhibits the same behavior at the origin that the function $f(z)$ does for $z_0 \neq 0$.

3.30. Example. Consider $f = u + iv$, where $u = x^2y^2$ and $v = 2x^2y^2$. Then all the partial derivatives exist and are continuous in $\mathbb{C}$. However, the C-R equations are satisfied only when $2xy^2 = 4x^2y$ and $2x^2y = -4xy^2$; that is when

$$xy(y - 2x) = 0 \quad \text{and} \quad xy(x + 2y) = 0.$$

These two equations show that the C-R equations hold only along the lines $x = 0, y = 0$ and nowhere else. Thus, $f = u + iv$ is differentiable only along points on the real and imaginary axes and nowhere else. Note that, every neighborhood of every point on the line $x = 0$ (and $y = 0$, respectively) will contain points off the line at which the C-R equations does not hold. It follows that $f = u + iv$ is nowhere analytic. ●

If f is constant in some domain D, then $f'(z) = 0$ in D. As a consequence of the C-R equations, we shall now establish the converse part of it (see also Corollaries 4.21 and 12.3 for a proof). For more about the basic properties concerning the range of non-constant analytic functions, we refer to Section 12.6.

3.31. Theorem. *Suppose that f is analytic in a domain D. We have*

(i) *if $f'(z) = 0$ in D, then f is constant*

(ii) *if one of $|f|$, $\mathrm{Re}\, f$, $\mathrm{Im}\, f$, $\mathrm{Arg}\, f$ is constant in D, then f is constant. Equivalently, we say that if the range $f(D)$ lies in either a circle or a vertical line or horizontal line or any line with constant slope, then f is constant.*

Proof. (i) For $z = x + iy \in D$ and $f = u + iv$, we have

$$f' = u_x + iv_x = v_y - iu_y.$$

Suppose $f'(z) = 0$ in D. Then all the partial derivatives of u and v on D are zero, i.e. $u_x(z) = u_y(z) = v_x(z) = v_y(z) = 0$ for $z \in D$. We know from Real Analysis that if ϕ is a real-valued differentiable function of a real variable and such that $\phi'(x) = 0$ on (a, b), then ϕ is a constant on (a, b). Let $z_0 = x_0 + iy_0$ be an arbitrary point in D and for $a < b$, let

$$L = \{x + iy_0 : x \in (a, b)\}.$$

If the horizontal line segment L lies in D, then ϕ defined by

$$\phi(x) = u(x, y_0)$$

satisfies $\phi'(x) = u_x(x, y_0) = 0$ on L and so u is constant on L. It follows that u is constant on each horizontal line segment in D. Likewise u is constant on each vertical line segment in D. Since D is open and connected, each pair of points in D can be connected by a polygonal step path consisting entirely of such line segments. Therefore, u is constant in D. Similarly, v is constant in D. Consequently, f is constant throughout D (see also Corollary 4.21).

(ii) Now suppose that $|f(z)| = k$, so that $u^2 + v^2 = k^2$. If $k = 0$ it is obvious that $f = 0$ in D. So, let $k \neq 0$. Differentiating partially with respect to x and y, we see that

$$uu_x + vv_x = 0, \;\; uu_y + vv_y = 0.$$

By the C-R equations, these become

$$uu_x - vu_y = 0, \ \ uu_y + vu_x = 0.$$

Squaring and adding, we obtain

$$0 = (u^2 + v^2)(u_x^2 + u_y^2) = k^2|f'(z)|^2$$

so that $f'(z) = 0$ in D. Therefore from (i) we deduce that f is constant.

Suppose u is constant. Then $u_x = u_y = 0$; therefore, by the C-R equations, $v_x = v_y = 0$ so that $f'(z) = 0$ in D. Hence f is constant.

If $\operatorname{Arg} f = \alpha$, a constant, then $g(z) = e^{-i\alpha} f(z)$ is a real-valued analytic function and so, by part (i), $g(z)$ is constant. ∎

Suppose f and g are analytic in a domain D such that $f'(z) = g'(z)$ in D then applying part (i) of Theorem 3.31 to $f - g$ we conclude that f and g differ from each other by a constant.

The hypothesis that D is connected in Theorem 3.31 is not superfluous. For example, if $D = \mathbb{C}\backslash\{z : 1 \le |z| \le 3\}$ and if $f : D \to \mathbb{C}$ is defined by

$$f(z) = \begin{cases} 0 & \text{for } |z| < 1 \\ i & \text{for } |z| > 3, \end{cases}$$

then $f \in \mathcal{H}(D)$ and $f'(z) = 0$ in D, yet f is non-constant in D.

Here is another example. Define

$$f(z) = \begin{cases} 0 & \text{for } \operatorname{Re} z > a \\ i & \text{for } \operatorname{Re} z < a, \end{cases}$$

where $a \in \mathbb{R}$ is fixed. Then $f \in \mathcal{H}(D)$, where $D = \mathbb{C}\backslash\{z : \operatorname{Re} z = a\}$, yet f is not constant in D. Note that, in each of these two examples, $\operatorname{Re} f(z) = u(x, y) = 0$ in D but $\operatorname{Im} f(z) = v(x, y)$ is not constant.

3.2 Harmonic Functions

We begin with the formal definition of a harmonic function. A real-valued function $\phi = \phi(x, y)$ of real variables x and y is said to be *harmonic* in an open subset Ω of $\mathbb{C}$ if it has continuous partial derivatives of second order and satisfies the Laplace's equation[7] in two variables $\nabla^2\phi = 0$ throughout Ω, where ∇^2 is the second order differential operator given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \phi_{xx} + \phi_{yy}.$$

[7]This equation that bears the name of the French mathematician, Pierre Simon de Laplace (1749-1827), had been found by Leonhard Euler in 1752 in connection with Euler's studies on hydrodynamics. Laplace contributed significantly to the fields of celestial mechanics and probability theory and his parents were farmers.

The operator $\phi \mapsto \nabla^2\phi$ is called *Laplace operator* or simply a *Laplacian.* A function v is called a conjugate harmonic function (or, more briefly, a *harmonic conjugate*) for a harmonic function u in Ω whenever $f = u + iv$ is analytic in Ω. Note that the word conjugate here is not the same as conjugate of a complex number z. Further, the harmonic conjugate v is unique, up to an addition of a real constant. Indeed, if v_1 is another harmonic conjugate of u, so that $F = u + iv_1$ is also analytic in Ω, then the difference $F - f = i(v_1 - v)$ becomes analytic in Ω. But then by Theorem 3.31(ii), $v_1 - v$ is a constant.

Since $-if = v + i(-u)$, we also observe that $-u$ is a harmonic conjugate of v whenever v is a harmonic conjugate of u.

3.32. Remark. As can be observed from the C-R equations, we cannot choose two arbitrary harmonic functions u and v and claim that the resulting function $f = u + iv$ is analytic. For example, $u(x, y) \equiv x$ and $v(x, y) \equiv -y$ are harmonic functions in $\mathbb{C}$, but $f = u + iv = x + i(-y) = \overline{z}$ is nowhere analytic. On the other hand,

$$v + iu = -y + ix = i(x + iy) = iz$$

which is analytic in $\mathbb{C}$. Further, it would be appropriate to have the Laplace equation to be satisfied not just for any set of points but for an open set or a domain or more importantly a simply connected domain. For example, if $u = x^3 - y^3$ then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x - y) = 0 \quad \text{only when } y = x$$

and the set $\{z = x + iy : y = x\}$ is not open in $\mathbb{C}$ and so, u cannot be treated as a harmonic function in any open subset of $\mathbb{C}$. ●

Let us now discuss some facts about this topic and construct simple examples of harmonic functions. We first introduce a complex Laplacian. Suppose that we are given a harmonic function

$$u(x, y) = u\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right).$$

Then, formally treating z and $\overline{z}$ as independent variables, as before

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}[u_x - iu_y]$$

so that

$$\begin{aligned}\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{1}{2}\left[\frac{\partial}{\partial x}(u_x - iu_y)\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}(u_x - iu_y)\cdot\frac{\partial y}{\partial \overline{z}}\right] \\ &= \frac{1}{4}\left[(u_{xx} + u_{yy}) + i(u_{xy} - u_{yx})\right],\end{aligned}$$

where we have used the notation

$$u_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \; u_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \; u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \; u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right).$$

Therefore any harmonic function u satisfies the differential equation

$$\nabla_c^2 u = \frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

The operator ∇_c^2 is sometimes called the *complex Laplacian*. Thus if u and its first and second partial derivatives are continuous in an open subset $\Omega \subseteq \mathbb{C}$, then

$$\nabla_c^2 u = (1/4)\, \nabla^2 u.$$

Hence, the Laplacian (operator) in complex form is given by

$$\nabla^2 = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial z}\right). \tag{3.33}$$

For example, if $f = u + iv$ is a complex-valued function in a domain Ω then

$$\nabla^2 (zf) = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial z}(zf)\right) = 4\frac{\partial}{\partial \overline{z}}\left(f + \frac{\partial f}{\partial z}\right) = 4 f_{\overline{z}} + 4\, \nabla^2 f$$

which shows that if f and zf are harmonic in Ω, then $f_{\overline{z}} = 0$ in Ω and hence f is analytic in Ω, by Theorem 3.26. *We say that a complex-valued function is harmonic in an open set if both its real and imaginary parts are harmonic thereat.*

Harmonic functions play an important role in both mathematics and physics. There are (at least) two important reasons why harmonic functions coupled with the C-R equations are discussed as an important part of complex analysis as demonstrated in the next two theorems. First, we recall that if $f = u + iv$ is analytic in an open set Ω, then the C-R equations hold throughout Ω:

$$u_x = v_y, \;\; u_y = -v_x. \tag{3.34}$$

The C-R equations have some interesting consequences. For instance, if the second partial derivatives exist and are continuous (in fact, we shall later see that the derivative of an analytic function in a domain is itself analytic there−more generally infinitely differentiable, and so u and v both have continuous partial derivatives of all orders) then by differentiating the first equation with respect to x and second with respect to y, we get

$$u_{xx} = v_{xy}, \;\; u_{yy} = -v_{yx}.$$

The continuity of these partial derivatives implies that the mixed derivatives are equal (which is an important fact from two variable calculus) and, in particular, $v_{xy} = v_{yx}$ and therefore,

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0.$$

In a similar way, differentiating the first equation in (3.34) with respect to y and the second in (3.34) with respect to x, we find that v satisfies the Laplace equation $\nabla^2 v = 0$. In conclusion we have

3.35. Theorem. *Let Ω be an open subset of $\mathbb{C}$. Then the real and imaginary parts of an analytic function in Ω are harmonic in Ω.*

At this point it should be noted that the mixed second partial derivatives do not coincide in general. For instance, for the real-valued function $u(x,y)$ defined by

$$u(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x = y = 0, \end{cases}$$

it is easily seen that $u_{xy}(0,0) \neq u_{yx}(0,0)$.

3.36. Example. If $\phi(x,y)$ is harmonic in a domain D, then, by Theorem 3.26 with $u\,(=\phi_x)$ and $v\,(=-\phi_y)$, we find that $f = \phi_x - i\phi_y \in \mathcal{H}(D)$. Similarly, if $\psi(x,y)$ is harmonic in D then $g = \psi_x - i\psi_y \in \mathcal{H}(D)$. In particular, we have

$$\begin{aligned} f \pm g &= (\phi_x \pm \psi_x) - i(\phi_y \pm \psi_y) \in \mathcal{H}(D) \\ f \pm ig &= (\phi_x \pm \psi_y) - i(\phi_y \mp \psi_x) \in \mathcal{H}(D) \\ if \mp g &= (\phi_y \mp \psi_x) + i(\phi_x \pm \psi_y) \in \mathcal{H}(D). \end{aligned}$$

This example gives a way of obtaining analytic functions from harmonic functions. ●

Theorem 3.35 provides numerous examples of harmonic functions as we shall soon see. However, the function u defined by $u(x,y) = x^2 + y^2$ cannot be a real part of an analytic function since $\nabla^2 u = 4$. We raise the question: *Given a real-valued harmonic function u in an open set Ω, can it be written as a real or imaginary part of an analytic function $f : \Omega \to \mathbb{C}$? If so, under what condition this is possible.* We need some preparation to answer this question.

3.37. Definition. A domain D in $\mathbb{C}$ is *simply connected* if its complement with respect to $\mathbb{C}_\infty$ (i.e. $\mathbb{C}_\infty \setminus D$) is a connected subset of $\mathbb{C}_\infty$.

Topologically, a simply connected domain D in $\mathbb{C}$ can be continuously shrunk to a point in D. Note that the punctured unit disk $A = \{z : 0 < |z| < 1\}$ can be shrunk to an arbitrarily small domain, but not to a point in A. Intuitively, a simply connected domain in $\mathbb{C}$ means that it does not contain any "holes". Consequently, according to Definition 3.37, exterior of circles (i.e. complement of closed disks in $\mathbb{C}$) are not simply connected in $\mathbb{C}$. Note that, in the extended plane, the exterior of a circle is simply connected because it can be "shrunk" to the point at ∞. Some simple examples of simply connected domains in $\mathbb{C}$ are

(i) disks $\Delta(a;r)$

(ii) half-planes $\{z : \mathrm{Re}\,(e^{i\theta}z) > \beta\}$

(iii) convex domains such as in (i), (ii) and infinite parallel strips, such as $D = \{z : \alpha < \mathrm{Re}\,(e^{i\theta}z) < \beta\}$ for some α and β with $\alpha < \beta$

(iv) domains that are starlike with respect to the origin such as $\mathbb{C}\backslash[0,\infty)$ and $\mathbb{C}\backslash\{iy : |y| > 1\}$

(v) the whole complex plane.

The annular domain $E = \{z : 1 < |z| < 2\}$ is not simply connected, whereas the domain E with $\{z : 1 < \mathrm{Re}\, z < 2,\ \mathrm{Im}\, z = 0\}$ removed from it (i.e. $E\backslash\{(1,2)\}$), is simply connected.

We note that there are many equivalent definitions of simply connectedness. One such equivalent form of it is the following (see also Definition 4.50):

3.38. Definition. Let D be a domain in $\mathbb{C}$ and D_∞ be the set corresponding to D on the Riemann sphere S. Then D is called simply connected if $D_\infty^c = S\backslash D_\infty$ is connected and contains the north pole.

It would have been nice if every harmonic function is a real part of some analytic function. But this is not true in general. However, it is indeed true locally, but globally provided Ω is a simply connected domain. More precisely, we have

3.39. Theorem. *Let Ω be a simply connected domain and let ϕ be harmonic in Ω. Then ϕ has a harmonic conjugate in Ω.*

It is convenient to supply a proof of Theorem 3.39 in Chapter 4 since a motivation for an explicit construction for the harmonic conjugate comes from (complex) integration. On the other hand, one can give a simple and a direct proof of Theorem 3.39 especially when

(i) $\phi(x,y)$ is a real-valued harmonic polynomial in $\mathbb{C}$

(ii) $\phi(x,y)$ is a real valued harmonic function in Ω where Ω is either an open disk or open rectangle, see Corollary 3.57.

3.40. Example. Consider $u(x,y) = 4xy - x^3 + 3xy^2$. Then

$$u_x = 4y - 3x^2 + 3y^2 \quad \text{and} \quad u_y = 4x + 6xy$$

from which it is easy to see that u is harmonic in $\mathbb{R}^2$. To find the harmonic conjugate $v(x,y)$ in $\mathbb{C}$ for $u(x,y)$, we may proceed as follows. If $f = u + iv$ is the corresponding analytic function, then

$$\begin{aligned} f'(z) &= u_x - iu_y \\ &= 4y - 3x^2 + 3y^2 - i(4x + 6xy) \\ &= -3[x^2 - y^2 + 2ixy] - 4i(x + iy) \\ &= -3z^2 - 4iz \end{aligned}$$

which gives $f(z) = -z^3 - 2iz^2 + ik$, where k is a real constant. The harmonic conjugate $v(x,y)$ may be obtained by taking the imaginary part of the last expression.

Alternately, use the C-R equations to obtain

$$v_y = u_x = 4y - 3x^2 + 3y^2. \tag{3.41}$$

Integrate (3.41) with respect to y to obtain

$$v = \int v_y dy + \phi(x) = 2y^2 - 3x^2y + y^3 + \phi(x), \tag{3.42}$$

where ϕ is a function of x. If ϕ in (3.42) is differentiable with respect to x the equation $v_x = -6xy + \phi'(x)\ \ (= -u_y)$ is obtained which, together with the C-R equation $u_y = -v_x$, gives

$$\phi'(x) = 6xy - u_y = 6xy - (4x + 6xy) = -4x$$

so that $\phi(x) = -2x^2 + k$, where k is some real constant. Hence, by (3.42), the harmonic conjugate function $v(x,y)$ is given by

$$v = (2y^2 - 3x^2y + y^3) + (-2x^2 + k).$$

Now we can use $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$ to write f as a function of z. ●

3.43. Theorem. *If u and v are harmonic conjugates to each other in some domain then u and v must be constant there.*

Proof. By the definition and the hypotheses, $f = u + iv$ and $g = v + iu$ are analytic in D. But

$$f - ig = 2u, \quad \text{and} \quad f + ig = 2iv.$$

By Theorem 3.9, these imply that the real-valued functions of a complex variable, namely u and v are analytic in D. Therefore, by Theorem 3.6, $f'(z) = 0$ in D. Since D is a domain, it follows that f and g are constants and hence, u and v are constants. ■

3.44. Polar form of the C-R equations and Laplacian. There are ways to obtain the C-R equations in polar form. Here is a direct method. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be differentiable at a point z. Then $f'(z)$ exists and equals

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

Set $z = re^{i\theta}$. The nearby points of z along the radial direction may be given by $z + h = (r + \Delta r)e^{i\theta}$. Therefore, for $h = \Delta r e^{i\theta} \neq 0$,

$$\begin{aligned}\frac{f(z+h) - f(z)}{h} &= \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta)}{\Delta r e^{i\theta}} - \frac{u(r,\theta) + iv(r,\theta)}{\Delta r e^{i\theta}} \\ &= \frac{1}{e^{i\theta}}\left\{\frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r}\right\} \\ &\quad + \frac{i}{e^{i\theta}}\left\{\frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r}\right\}.\end{aligned}$$

Allowing $\Delta r \to 0$ shows that

$$f'(z) = \frac{1}{e^{i\theta}}[u_r + iv_r]. \tag{3.45}$$

Next choose nearby points of z along the circular path through the point z so that

$$z + h = re^{i(\theta+\Delta\theta)} = re^{i\theta}e^{i\Delta\theta}, \quad \text{i.e.} \quad h = re^{i\theta}(e^{i\Delta\theta} - 1)$$

and for $\Delta\theta \neq 0$,

$$\begin{aligned}\frac{f(z+h) - f(z)}{h} &= \frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{re^{i\theta}(e^{i\Delta\theta} - 1)} + i\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{re^{i\theta}(e^{i\Delta\theta} - 1)} \\ &= \frac{1}{ire^{i\theta}}\left\{\left(\frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{\Delta\theta}\right)\right. \\ &\quad \left. + i\left(\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{\Delta\theta}\right)\right\}\frac{i\Delta\theta}{e^{i\Delta\theta} - 1}.\end{aligned}$$

Now allowing $\Delta\theta \to 0$, it follows that

$$f'(z) = \frac{1}{ire^{i\theta}}[u_\theta + iv_\theta] = \frac{1}{re^{i\theta}}[v_\theta - iu_\theta]. \tag{3.46}$$

Comparing (3.45) and (3.46) produces the C-R equations in polar form:

$$u_r = \frac{v_\theta}{r} \quad \text{and} \quad v_r = -\frac{u_\theta}{r}. \tag{3.47}$$

Alternatively, let $f(z) = f(re^{i\theta})$, $r \neq 0$, be differentiable at a point z_0. Then both

$$f'(z_0) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\bigg|_{z=z_0} = (u_r + iv_r)e^{-i\theta}\big|_{z_0}$$

and

$$f'(z_0) = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}\bigg|_{z=z_0} = (u_\theta + iv_\theta)\frac{e^{-i\theta}}{ir}\bigg|_{z_0}$$

must be the same. Therefore, equating the right hand side of the last two equations yields (3.47).

If we differentiate the first equation in (3.47) with respect to r and the second in (3.47) with respect to θ we get

$$\frac{\partial}{\partial r}(v_\theta) = u_r + ru_{rr}, \quad \frac{\partial}{\partial \theta}(v_r) = -\frac{1}{r}u_{\theta\theta}$$

and so, using the continuity of second partial derivatives, these two equations give

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

which is the polar form of the Laplace equation. If $u(r,\theta)$ depends on r alone, then the above Laplace equation becomes

$$u_{rr} + \frac{1}{r}u_r = 0.$$

For example, we have

- $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic for any positive integer n
- $\ln r = \ln|z|$ is harmonic in the punctured disk $\mathbb{C}\backslash\{0\}$. We note that $\ln|z|$ has no harmonic conjugate in $\mathbb{C}\backslash\{0\}$, though it does have in $\mathbb{C}\backslash[0,\infty)$, see Section 3.5.

Next we attempt to develop methods for finding an analytic function f whose real part is a given harmonic function $u(x,y)$ which is a rational function in x and y. For this we start with

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

so that

$$\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y),$$

where $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/(2i)$. Adding the above two equations we find that

$$f(x+iy) + \overline{f(x+iy)} = 2u(x,y).$$

Recall that analytic functions are completely characterized by the condition:

$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = 0. \tag{3.48}$$

For example, if we substitute z for f then we see that $f_{\overline{z}} = 0$ and $f_z = 1$. Thus we remark that $\overline{f(z)}$, the conjugate of an analytic function $f(z)$, has the derivative zero with respect to z and so $\overline{f(z)}$ may be considered as a function of $\overline{z}$ alone. Therefore, we can denote $\overline{f(z)}$ simply by $\overline{f}(\overline{z})$. That is

$$\overline{f(x+iy)} = \overline{f}(x-iy), \quad \text{and} \quad 2u(x,y) = f(x+iy) + \overline{f}(x-iy).$$

In particular, the last equation yields

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \overline{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \overline{f}(0)]$$

which gives

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \overline{f}(0).$$

Also we observe that

$$u(0,0) = (1/2)[f(0) + \overline{f}(0)] = \operatorname{Re} f(0)$$

and so

$$\overline{f(0)} = \operatorname{Re} f(0) - i \operatorname{Im} f(0) = u(0,0) - ik,$$

where $k = \operatorname{Im} f(0)$ is a real number. Consequently, the function f may be computed by the formula

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ik.$$

Similarly, it follows that in a neighborhood of z_0, the analytic function $f = u + iv$ associated with the harmonic function $u(x,y)$ is given by

$$f(z) = 2u\left(\frac{z+\overline{z}_0}{2}, \frac{z-\overline{z}_0}{2i}\right) - u(x_0,y_0) + ik,$$

where k is a real constant. We shall illustrate the application of this method with two examples.

3.49. Example. Consider $u(x,y) = x^3 - 3xy^2$. Then u is defined in $\mathbb{R}^2$ and u satisfies Laplace's equation for all points in $\mathbb{R}^2$. Using the above method, the corresponding analytic function f is given by

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ik = 2\left[\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2\right] + ik = z^3 + ik,$$

where k is a real number. ●

3.50. Example. We wish to find the most general cubic form

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \quad (a,b,c,d\text{-real}),$$

which satisfies Laplace's equation, and to determine an analytic function which has u as its real part. To do this, for $(x,y) \in \mathbb{R}^2$, we compute

$$u_{xx} + u_{yy} = 2[(3a+c)x + (b+3d)y]$$

and so, $\bigtriangledown^2 u = 0$ in $\mathbb{R}^2$ provided $3a + c = 0 = b + 3d = 0$. Thus, the most general harmonic polynomial of degree three takes the form

$$u = ax^3 - 3dx^2y - 3axy^2 + dy^3.$$

Further, using the above method of construction of analytic function, we have

$$\begin{aligned} f(z) &= 2u\left(\frac{z}{2}, \frac{z}{2i}\right) + ik \\ &= 2\left[a\left(\frac{z}{2}\right)^3 - 3d\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - 3a\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2 + d\left(\frac{z}{2i}\right)^3\right] + ik \\ &= (a + id)z^3 + ik \end{aligned}$$

which is the required analytic function having u as its real part, where k is a real constant. ●

3.51. Discussion on finding harmonic conjugates. Let us carefully look at the problem of whether a real-valued harmonic function u, a C^2-solution of Laplace's equation in some domain Ω, is a real or imaginary part of some analytic function. For the time being we suppose that a harmonic function u is given and we have found an analytic function $F(z)$ in Ω such that $F(z) = u + iv$, $z \in \Omega$. Then, the C-R equations give $v_x = -u_y$ and $v_y = u_x$ which show that, v_x and v_y are completely determined from the given function u. Therefore, v may be found up to an additive constant. Again, in view of the C-R equations, we see that

$$u_{xx} + u_{yy} = 0 \iff \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial y}(-u_y) \iff \frac{\partial}{\partial x}(v_y) = \frac{\partial}{\partial y}(v_x).$$

Set $f = -u_y$ and $g = u_x$. This amounts to rephrasing our problem as follow.

3.52. Problem. Given $f, g \in C^1(\Omega)$ with $f_y = g_x$, can we find a function $v \in C^2(\Omega)$ such that $v_x = f$ and $v_y = g$ in Ω? If so under what conditions on Ω, is this possible?

First, we aim at giving a partial solution to this problem by discussing a special case when Ω is an open rectangle (and the proof is similar when Ω is an open disk).

3.53. Theorem. *Let $\Omega = \{(x, y) \in \mathbb{R}^2 : |x - a| < \delta, |y - b| < \delta'\}$ be an open rectangle in $\mathbb{R}^2$. Suppose that $f, g \in C^1(\Omega)$ such that*

$$f_y = g_x \quad \text{in } \Omega. \tag{3.54}$$

Then there exists a function $v \in C^2(\Omega)$ satisfying the conditions

$$v_x = f \quad \text{and} \quad v_y = g \tag{3.55}$$

in Ω. (*We may take* v *as real-valued whenever* f *and* g *are also*).

Proof. Choose an arbitrary point $(x, y) \in \Omega$ and define

$$v(x,y) = \int_a^x f(s,b)\,ds + \int_b^y g(x,t)\,dt.$$

Observe that any other v may differ from the above merely by a constant. What motivates us to define $v(x, y)$ in this way? (see Remark 3.56). We need to show that v has the desired properties. By the fundamental theorem of calculus for C^1-functions,

$$v_y(x,y) = g(x,y).$$

Next, once again using the fundamental theorem of calculus for the first integral and the theorem on differentiation under the integral sign, it follows that

$$\begin{aligned} v_x(x,y) &= f(x,b) + \frac{\partial}{\partial x}\left(\int_b^y g(x,t)\,dt\right) \\ &= f(x,b) + \int_b^y \frac{\partial}{\partial x} g(x,t)\,dt \\ &= f(x,b) + \int_b^y \frac{\partial}{\partial y} f(x,t)\,dt \quad (\text{since } g_x = f_y) \\ &= f(x,b) + f(x,y) - f(x,b) \end{aligned}$$

so that $v_x(x, y) = f(x, y)$. Thus, we have found a function v with $v_y = g \in C^1(\Omega)$ and $v_x = f \in C^1(\Omega)$ which means that $v \in C^2(\Omega)$. ∎

3.56. Remark. The function $v(x, y)$ that we need to define must satisfy (3.55). Indeed, integrating $v_y(x, y) = g(x, y)$ with respect to y, we get

$$v(x,y) = \int_b^y g(x,t)\,dt + \phi(x)$$

where ϕ is some C^1-function of x. Since we also need $v_x = f$, by the first equation in (3.55), we compute that

$$v_x(x,y) = \int_b^y \frac{\partial}{\partial x} g(x,t)\,dt + \phi'(x) = \int_b^y \frac{\partial}{\partial y} f(x,t)\,dt + \phi'(x)$$

which gives

$$f(x,y) = f(x,y) - f(x,b) + \phi'(x); \quad \text{i.e.} \quad \phi'(x) = f(x,b).$$

This has the solution

$$\phi(x) = \int_a^x f(s,b)\,ds + c,$$

where c is some constant. Hence, $v(x, y)$ must be of the form

$$v(x,y) = \int_a^x f(s,b)\,ds + \int_b^y g(x,t)\,dt + c. \qquad \bullet$$

3.57. Corollary. *If Ω is either an open rectangle (with sides parallel to the axes) or open disk and if u is a real-valued harmonic function in Ω, then there exists an analytic function F in Ω such that $u = \operatorname{Re} F$.*

Proof. Set $f = -u_y$ and $g = u_x$. Then $f, g \in C^1(\Omega)$ and, since $\nabla^2 u = 0$ in Ω, we have $f_y = g_x$ in Ω. By Theorem 3.53, there exists a real-valued function $v \in C^2(\Omega)$ such that

$$v_x = f = -u_y \quad \text{and} \quad v_y = g = u_x \quad \text{in} \quad \Omega.$$

By the theorem on sufficient conditions for analytic functions (see Theorem 3.26), we conclude that $F = u + iv$ is analytic in Ω. ∎

Theorem 3.53 is called an antiderivative theorem (for real-valued functions). Now it is natural to raise the following

3.58. Problem. Given an analytic function F in Ω, can we find an analytic function G such that $G'(z) = F(z)$?

If the answer to this problem is yes, then we call the function G a *primitive* or *anti-derivative* for F. Any other anti-derivative of F would differ from G by a constant. Indeed, if both G_1 and G_2 are primitives of a function F, then

$$(G_1(z) - G_2(z))' = G_1'(z) - G_2'(z) = F(z) - F(z) = 0$$

and so by Theorem 3.31, we see that $G_1(z) - G_2(z)$ is a constant.

Here is a simple illustration. The primitive of the polynomial function p defined by $p(z) = \sum_{k=0}^{n} a_k z^k$ is $P(z) = \sum_{k=0}^{n} (a_k/(k+1)) z^{k+1} + K$, where K is an arbitrary constant.

Is there any restriction on Ω for the existence of primitives? First we give an affirmative answer when Ω is an open rectangle or convex or open disk in $\mathbb{C}$. However, we shall later discuss a more general theorem which provides an affirmative answer to this problem whenever Ω is a simply connected domain.

3.59. Theorem. (Antiderivative Theorem) *Let Ω be either an open rectangle (with sides parallel to the axes) or an open disk. Then every analytic function $F(z)$ in Ω possesses a primitive in Ω.*

Proof. Set $F(z) = u(z) + iv(z)$. Since F is analytic in Ω,

$$u_y = -v_x \Longleftrightarrow f_y = g_x,$$

where $u = f \in C^1(\Omega)$ and $-v = g \in C^1(\Omega)$. By Theorem 3.53, there exists $U \in C^2(\Omega)$ such that

$$U_x = f\ (= u) \ \text{ and } \ U_y = g\ (= -v) \ \text{ in } \Omega.$$

As F is analytic, we have $v_y = u_x$ where $v, u \in C^1(\Omega)$. Again, by Theorem 3.53, it follows that there exists $V \in C^2(\Omega)$ such that

$$V_x = v \ \text{ and } \ V_y = u \ \text{ in } \Omega.$$

Finally, define $G(z) = U + iV$. Then, $U, V \in C^2(\Omega)$ and the C-R equations are satisfied in Ω. Hence, by Theorem 3.26, G is analytic in Ω. Note that $G'(z) = U_x + iV_x = u + iv = F(z)$. ■

3.3 Power Series as an Analytic Function

In our earlier sections we have seen that the polynomial of degree $n \geq 1$ given by

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (a_n \neq 0) \tag{3.60}$$

can be differentiated term-by-term to get

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}. \tag{3.61}$$

Consider a series of functions of the form $\sum_{n\geq 0} a_n(\zeta - \zeta_0)^n$, where ζ is a complex variable and ζ_0, a_n, $n = 0, 1, 2, \ldots$, are fixed constants. Such a series will be called a (formal) *power series* with center ζ_0 and coefficients a_n. The substitution $z = \zeta - \zeta_0$ transforms the above series into the power series

$$\sum_{n\geq 0} a_n z^n. \tag{3.62}$$

So, for our discussion it is enough to consider power series with center 0 as the results about general power series can be obtained by translation. Now the polynomial (3.60) may be thought of as a power series at 0 with coefficients $a_k = 0$ for all $k > n$ and so we say that the family of polynomials is contained in the family of power series. Further, we also note that a power series defined by (3.62) is a special case of the limit of infinite sequences of functions, namely, $\{f_n(z)\}$, where $f_n(z) = \sum_{k=0}^{n-1} a_k z^k$. In any case, (3.61) suggests that for the power series (3.62) with sum $f(z)$, we should have

$$f'(z) = \sum_{n\geq 1} n a_n z^{n-1}$$

and in that case we call the R.H.S the *derived series* of $\sum_{n\geq 0} a_n z^n$. This fact of course requires justification and before giving a proof, we discuss some important facts about the convergence of (3.62) to appreciate fully the ideas involved.

Now, we consider the polynomials

$$\begin{aligned} p_n(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!}, \\ q_n(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ r_n(z) &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}. \end{aligned}$$

We know that these are entire. As $n \to \infty$, we obtain that

$$p_n(z) \to \cos z, \quad q_n(z) \to \sin z, \quad r_n(z) \to e^z$$

and in all three cases, the corresponding limit functions are also entire. How about the polynomial $1 + z + z^2 + \cdots + z^n$ when $n \to \infty$? How about $1 + rz + r^2z^2 + \cdots + r^n z^n$ $(r > 1)$ as $n \to \infty$?

We start with some interesting facts due to Abel: Suppose $0 \neq z_1 \in \mathbb{C}$ is such that the power series $\sum_{n\geq 0} a_n z_1^n$ converges. The terms of the series are then bounded (see Section 1.6). Indeed, as $|a_n z_1^n| \to 0$ as $n \to \infty$, there exists an $M > 0$ such that $|a_n z_1^n| < M$ for all $n \geq 0$. We have then, for all z with $|z| < |z_1|$,

$$|a_n z^n| = |a_n z_1^n| \left|\frac{z}{z_1}\right|^n < M \left|\frac{z}{z_1}\right|^n \quad \text{for all} \quad n.$$

Since $|z/z_1| < 1$, the geometric series $\sum_{n\geq 0} |z/z_1|^n$ converges, so that, by the comparison test, $\sum_{n\geq 0} |a_n z^n|$ converges for all z with $|z| < |z_1|$. For example, if $f(z) = \sum_{n\geq 1} z^n/n$ then the series converges for $|z| < 1$, since $f(-1) = \sum_{n\geq 1} (-1)^n/n$ converges.

If the series $\sum_{n\geq 0} a_n z_2^n$ diverges, then, for all z with $|z| > |z_2|$, we have

$$|a_n z^n| \geq |a_n z_2^n| \quad \text{for each } n \geq 0.$$

So, by the comparison test, the series $\sum_{n\geq 0} a_n z^n$ does not converge. For example, $f(z) = \sum_{n\geq 1} z^n/n$ diverges for $|z| > 1$, since $f(1) = \sum_{n\geq 1} 1/n$ diverges.

The above facts allow us to characterize the behavior of the power series (3.62) in a very natural fashion. Indeed, translating the series about '0' into a series about $a \in \mathbb{C}$, the above discussion gives (see Figure 3.1)

3.63. Theorem. *If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges at some point z_1 $(\neq a)$, then the series converges (absolutely) at all points in the disk $\Delta(a; |z_1 - a|)$. If the series diverges at z_2 $(\neq a)$, then it diverges for all z with $|z - a| > |z_2 - a|$.*

Because of the interesting information about the convergence from Theorem 3.63, it is natural to ask: *what is the largest disk about a on which*

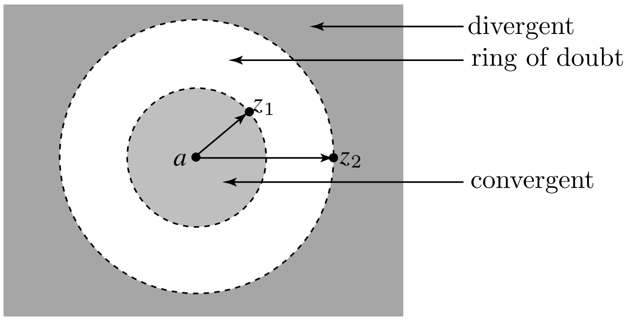


Figure 3.1: Description for Abel's test.

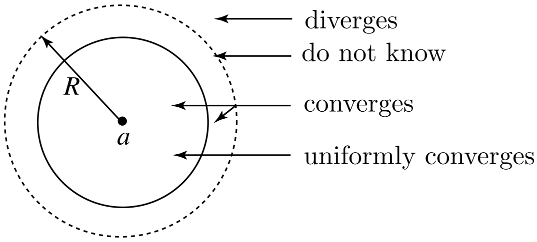


Figure 3.2: Illustration for the disk of convergence.

the series $\sum_{n\geq 0} a_n(z-a)^n$ *converges*? We now make this issue more precise in the form of a definition for a series about the origin. The *radius of convergence* R of the given power series (3.62) is defined by

$$R = \sup\{\rho : \sum_{n\geq 0} a_n z^n \text{ converges for all } z \text{ satisfying } |z| \leq \rho\}.$$

Note that $R = 0$ if $\sum_{n\geq 0} a_n z^n$ converges only for $z = 0$. If $R = \infty$, then the series $\sum_{n\geq 0} a_n z^n$ converges for all $z \in \mathbb{C}$. Thus, by definition, if $0 < R < \infty$, $\sum |a_n z^n|$ converges for all z such that $|z| < R$ and diverges for all z such that $|z| > R$. The series may converge for some or all points on the circle $|z| = R$. The circle $|z| = R$ is then called the *circle of convergence* because this is the greatest circle about $a = 0$ inside which $\sum_{n\geq 0} |a_n z^n|$ converges at each point (see Figure 3.2).

The conventions $0^{-1} = \infty$ and $\infty^{-1} = 0$ are observed so that R is the unique number in $[0, \infty]$.

3.64. Theorem. (Root Test) *Let* $L^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. *Then*

(i) *if* $L = \infty$, *the series* $\sum_{n\geq 0} a_n z^n$ *converges absolutely for all finite* z *and uniformly in any bounded set*

(ii) *if $L = 0$, the series converges only at $z = 0$ and diverges at all other points other than 0*

(iii) *if $0 < L < \infty$, the series converges absolutely for $|z| < L$, uniformly for $|z| \le r$, $r < L$ and diverges for $|z| > L$*

(iv) $L = R$.

Proof. As the series converges for $z = 0$, we need to consider only the case $z \neq 0$. So, for $z \neq 0$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{L}.$$

By Theorem 1.51, the series $\sum_{n\ge 0} a_n z^n$ converges absolutely for $|z| < L$ and diverges for $|z| > L$. The uniform convergence of the series for $|z| \le r$ ($r < L$) follows from the Weierstrass M-test (see Theorem 2.59). Thus, (i) to (iii) follows. We leave (iv) as a simple exercise. ∎

3.65. Remark. Note that if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

In this case, the radius of convergence R is simply determined from

$$R^{-1} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$ •

3.66. Example. For each fixed $z \neq 0$, we have $\lim_{n\to\infty} |z|^{1/n} = 1$. This can be proved as follows: Let $\delta_n = n^{1/n} - 1$. Then for each $n > 1$, δ_n is a positive real number. Now, for $n \ge 2$, we have

$$n = (1+\delta_n)^n = \sum_{k=0}^{n} \binom{n}{k} \delta_n^k > 1 + \frac{n(n-1)}{2}\delta_n^2 > \frac{n(n-1)}{2}\delta_n^2 \ge \frac{n^2}{4}\delta_n^2.$$

This implies that $0 \le \delta_n^2 < \frac{2}{n}$ and so we have $\delta_n \to 0$ as $n \to \infty$; i.e. $\lim_{n\to\infty} n^{1/n} = 1$. Applying this result for $|z| \ge 1$, we get $|z|^{1/n} \to 1$, since

$$1 \le |z|^{1/n} \le n^{1/n} \text{ for sufficiently large } n.$$

If $|z| < 1$, apply the preceding argument for $1/z$. •

3.67. Remark. The above example shows that $\{z_n\}$, where $z_n = n^{1/n} - 1$, is a null sequence. Since $n^{1/n} \to 1$, we also have

$$\frac{1}{n}\left[1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}\right] \to 1 \text{ as } n \to \infty.$$ •

The determination of the radius of convergence for the series (3.62) will not always require an application of Theorem 3.64, i.e. the *Root test*. In most of the cases, the following theorem called the *Ratio test* will serve the purpose instead.

3.68. Theorem. (Ratio Test) *If $a_n \neq 0$ for all but finitely many values of n, then the radius of convergence R of* (3.62) *is related by*

$$l := \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1}{R} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =: L.$$

In particular, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ *exists either as finite or* $+\infty$ *then*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Proof. Proof of this theorem is an immediate consequence of Theorem 1.48. Again it suffices to consider the case $z \neq 0$. In this event,

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = |z|L \quad \text{and} \quad \liminf_{n\to\infty} \left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = |z|l.$$

By Theorem 1.48, the series $\sum_{n\geq 0} a_n z^n$ converges absolutely for $|z| < 1/L$ and diverges for $|z| > 1/l$. Thus, R must be at least $1/L$ and at most $1/l$; that is R is related by $1/L \leq R \leq 1/l$. Consequently, if $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists then $L = l$ and hence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} |a_n|^{1/n} = \liminf_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}.$$

This completes the proof. ∎

Note that if the sequence $\{|a_{n+1}|/|a_n|\}$ oscillates, the limit does not exist and therefore the Ratio test becomes of no use. For instance, for series such as

$$\sum_{n\geq 0} a_n z^n = 3 + z + 3z^2 + z^3 + 3z^4 + \cdots + 3z^{2k} + z^{2k+1} + \cdots$$

the ratio $|a_{n+1}|/|a_n|$ is alternately $1/3$ and 3, so the limit does not exist. In this particular series, even though the Ratio test indicates that the value of R is lying between $1/3$ and 3, it does not yield the exact value of R. However, as $|a_{2k}| = 3$ and $|a_{2k+1}| = 1$, Example 3.66 gives $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n} = 1$.

Thus in the above two examples *the Ratio test* gives no information whereas the *Root test* gives the radius of convergence. We look now at a string of examples.

(i) Since (see Example 3.66) $n^{1/n} \to 1$ as $n \to \infty$, the series $\sum_{n\geq 1} nz^n$ converges for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$, $|nz^n| = n \to \infty$ and so the series diverges for $|z| = 1$. In this case we note that the series diverges at all points on the circle of convergence. Also, we observe that the Root test is clearly applicable.

(ii) Consider the series $\sum_{n\geq 0}(-1)^n z^{pn} = 1 - z^p + z^{2p} - \cdots$, where $p \geq$ is a fixed positive integer. Then

$$a_{np} = \begin{cases} -1 & \text{if } n = 1, 3, \ldots \\ 1 & \text{if } n = 0, 2, \ldots \end{cases}, \quad \text{i.e.} \quad a_m = \begin{cases} (-1)^{m/k} & \text{if } m = 0, k, 2k, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. Hence, $R = 1$ and the series converges absolutely for $|z| < 1$, uniformly for $|z| \leq \rho < 1$, and diverges for $|z| > 1$. Since the sequence of terms of the series does not approach zero when $|z| = 1$, it follows that the series diverges for $|z| = 1$. Note that the Ratio test is not applicable directly, but we could obtain the radius of convergence by translating the given series into a new series with a new variable ζ, $\zeta = z^p$.

(iii) By means of the Ratio test we at once see that each of the series

$$\sum_{n\geq 1}(-1)^n n^k z^n, \quad \sum_{n\geq 1}(-1)^{n-1}\frac{z^n}{n^k}, \quad \sum_{n\geq 1} n^k z^n \quad (k = 1, 2, \ldots)$$

has 1 as the radius of convergence. For $k \geq 2$, the second of the above converges absolutely on $|z| = 1$ since, on $|z| = 1$,

$$\sum_{n\geq 1}\left|\frac{(-1)^n z^n}{n^k}\right| = \sum_{n\geq 1}\frac{1}{n^k}$$

converges for $k \geq 2$. For $k = 1$, the series becomes

$$\sum_{n\geq 1}(-1)^{n-1}\frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1.$$

This series is called the *Logarithmic series* (see Section 3.5). On the other hand, the series

$$\sum_{n\geq 1}\frac{z^n}{n} \tag{3.69}$$

at $z = -1$ becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ and so converges for $z = -1$, but non-absolutely. At $z = 1$, this series is $\sum_{n\geq 1}\frac{1}{n}$ which is the harmonic series and we know that this is divergent. What happens on the rest of $\partial\Delta$? To see this, we let $z = e^{i\theta}$ ($\theta \neq 0, 2\pi$) and $a_k = z^k = e^{ik\theta}$. Then

$$\left|\sum_{k=1}^{n} z^k\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - e^{i\theta}|}$$

showing that the partial sums are bounded provided $\theta \neq 0, 2\pi$. Hence, by Theorem 1.52(b) with $b_n = n^{-1}$, the series (3.69) converges for $|z| = 1$ except at $z = 1$. Consequently, the logarithmic series $\sum_{n\geq 1}(-1)^{n-1}z^n/n$ converges for $|z| = 1$ except at $z = -1$.

(iv) Consider a series $\sum_{n\geq 0} q^n z^{kn}$, where $k \geq$ is a fixed positive integer and $q \neq 0$, independent of n. Note that

$$a_n = \begin{cases} q^{n/k} & \text{if } n = 0, k, 2k, 3k, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then, according to the Root test, one has $\limsup_{n\to\infty} |a_n|^{1/n} = |q|^{1/k}$ and so, the power series has the radius of convergence $R = |q|^{-1/k}$. The series converges absolutely for $|z| < |q|^{-1/k}$, uniformly for $|z| \leq \rho < |q|^{-1/k}$, and diverges for $|z| > |q|^{-1/k}$.

For instance, if $q = -1$ then $a_n = (-1)^n$ for $n \geq 0$ and the corresponding series $\sum_{n\geq 0}(-1)^n z^{kn}$ converges absolutely for $|z| < 1$, uniformly for $|z| \leq \rho < 1$, and diverges for $|z| > 1$. In this special case, the terms of the series does not approach zero when $|z| = 1$, and therefore, the series diverges for $|z| = 1$.

If we set $q = 1/2$ then we get the series $\sum_{n\geq 0} 2^{-n} z^{kn}$ which converges absolutely for $|z| < 2^{1/k}$, uniformly for $|z| \leq \rho < 2^{1/k}$, and diverges for $|z| > 2^{1/k}$. For $|z| = 2^{1/k}$, $|2^{-n}z^{nk}| = 1$, showing that the series diverges. In particular, for $k = 2$ the power series is

$$\sum_{n\geq 0} 2^{-n} z^{2n}, \quad \text{with} \quad a_n = \begin{cases} 0 & \text{if } n = 1, 3, 5, \ldots \\ 2^{-n/2} & \text{if } n = 0, 2, 4, \ldots \end{cases},$$

and so $R = \sqrt{2}$ is the radius of convergence for this series.

If we set $q = 5$, then the corresponding series

$$\sum_{n\geq 0} 5^n z^{kn} = \sum_{n\geq 0} (5z^k)^n$$

converges absolutely for $|z| < 5^{-1/k}$, uniformly for $|z| \leq \rho < 5^{-1/k}$, and diverges for $|z| > 5^{-1/k}$. For $|z| = 5^{-1/k}$, $|5^n z^{nk}| = 1$, showing that the series diverges. In particular, for $k = 3$ the above discussion gives the power series

$$\sum_{n\geq 0} 5^n z^{3n}, \quad \text{with} \quad a_n = \begin{cases} 5^{n/3} & \text{if } n = 0, 3, 6, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

and so $R = 5^{-1/3}$ is the radius of convergence for this series.

3.70. Theorem. *A power series $\sum_{n\geq 0} a_n z^n$ and the k-times derived series defined by $\sum_{n\geq k} n(n-1)\cdots(n-k+1)a_n z^{n-k}$ have the same radius of convergence.*

Proof. Let $A_n = n(n-1)\cdots(n-k+1)a_n = n!a_n/(n-k)!$, $k \geq 1$. Then

$$|A_n|^{1/n} = \left|\frac{n!}{(n-k)!}a_n\right|^{1/n} = \left|\frac{n!}{(n-k)!}\right|^{1/n} \cdot |a_n|^{1/n}.$$

Using the particular case of Theorem 3.68 (see also Example 3.66) we have

$$\begin{aligned}\limsup_{n\to\infty}\left[\frac{n!}{(n-k)!}\right]^{1/n} &= \lim_{n\to\infty}\left[\frac{(n+1)!/(n+1-k)!}{n!/(n-k)!}\right] \\ &= \lim_{n\to\infty}\frac{n+1}{n+1-k} = 1.\end{aligned}$$

Therefore, $\limsup_{n\to\infty}|A_n|^{1/n} = \limsup_{n\to\infty}|a_n|^{1/n}$ which proves our theorem. ∎

Since a power series of the form (3.62) with non-zero radius of convergence R converges (absolutely) for $|z| < R$, we can study its behavior as a function f defined by the sum $f(z) = \sum_{n\geq 0} a_n z^n$. The power series obtained by differentiating this series term-by-term gives

$$f'(z) = \sum_{n\geq 1} n a_n z^{n-1}, \quad |z| < R.$$

Next we have

3.71. Theorem. *If $\sum_{n\geq 0} a_n z^n$ has radius of convergence $R > 0$, then $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in $|z| < R$, $f^{(k)}(z)$ exists for every $k \in \mathbb{N}$ and*

$$f^{(k)}(z) = k!a_k + \sum_{n\geq k+1}\frac{n!}{(n-k)!}a_n z^{n-k} \quad (|z| < R). \tag{3.72}$$

where $a_k = f^{(k)}(0)/k$.

For example, the geometric series $(1-z)^{-1} = \sum_{n\geq 0} z^n$ which converges for $|z| < 1$, after k-times differentiation yields

$$\frac{1}{(1-z)^{k+1}} = \sum_{n\geq k}\binom{n}{k}z^{n-k} = \sum_{m\geq 0}\frac{(m+k)!}{k!m!}z^m \quad \text{for } |z| < 1.$$

In particular, $z(1-z)^{-2} = \sum_{n\geq 1} nz^n$ for $|z| < 1$.

Proof. Let $f(z) = \sum_{n\geq 0} a_n z^n$ with the radius of convergence R. We have to prove the existence of $f'(z)$ in Δ_R. By Theorem 3.70 with $k = 1$, the series

$$\sum_{n\geq 1} n a_n z^{n-1}$$

converges for $|z| < R$ and defines a function, say $g(z)$, in $|z| < R$. We show that

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = g(z) \quad \text{for all } z \in \Delta_R.$$

Let $z \in \Delta_R$ be fixed and let $h \in \mathbb{C}$, $0 < |h| < (R - |z|)/2$. Then

$$|z + h| \leq |z| + |h| < |z| + \frac{R - |z|}{2} = \frac{|z| + R}{2} < \frac{R + R}{2} = R.$$

Now we consider

$$\frac{f(z+h) - f(z)}{h} = \sum_{n\geq 1} a_n \left(\frac{(z+h)^n - z^n}{h} \right),$$

where z and $z+h$ are such that $\max\{|z|, |z+h|\} \leq r < R$ and $|h|$ is positive. So, we must show that as $h \to 0$,

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| = \left| \sum_{n\geq 2} a_n \left(\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right) \right| \to 0.$$

First we note that, for $\alpha \neq 1$ and $n \geq 2$, we have the identity

$$\frac{1 - \alpha^n}{1 - \alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$$

and differentiating with respect to α shows that

$$\frac{1 - \alpha^n}{1 - \alpha} - n\alpha^{n-1} = (1 - \alpha)[1 + 2\alpha + 3\alpha^2 + \cdots + (n-1)\alpha^{n-2}].$$

Replacing α by z/w $(w \neq z)$ gives

$$\frac{w^n - z^n}{w - z} - nz^{n-1} = (w - z)[w^{n-2} + 2w^{n-3}z + \cdots + (n-1)z^{n-2}]$$

so that for $\max\{|w|, |z|\} \leq r$,

$$\begin{aligned} \left| \frac{w^n - z^n}{w - z} - nz^{n-1} \right| &\leq |w - z|\, [|w|^{n-2} + 2|w|^{n-3}\,|z| + \cdots + (n-1)|z|^{n-2}] \\ &\leq |w - z| r^{n-2}\, [1 + 2 + 3 + \cdots + n - 1] \\ &= |w - z| \frac{n(n-1)}{2} r^{n-2}. \end{aligned}$$

By Theorem 3.70, the derived series $\sum_{n\geq 2} n(n-1)|a_n|r^{n-2}$ is convergent for each r such that $|z| \leq r < R$. Using this we see that as $h \to 0$,

$$\left| \sum_{n\geq 2} a_n \left(\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right) \right| \leq |h| \sum_{n\geq 2} |a_n| \frac{n(n-1)}{2} r^{n-2} \to 0.$$

Consequently, $f'(z)$ exists and equals $g(z)$. Since z is arbitrary, this holds at any interior point in the disk of convergence.

A repeated application of this argument shows that all the derivatives $f', f'', \dots, f^{(k)}, \dots$ exist in $|z| < R$ and (3.72) holds. Putting $z = 0$ in (3.72) we have the formula for the coefficient a_k: $f^{(k)}(0)/k = a_k$ and we are done. ■

Suppose $z_0 \neq 0$ and $f(z) = \sum_{n\geq 0} a_n(z - z_0)^n$, $|z - z_0| < R$. Then consider a simple transformation $w = z - z_0$ so that $z = z_0 + w$. Then by Theorem 3.71 we see that the function ϕ defined by

$$\phi(w) = f(w + z_0) = \sum_{n\geq 0} a_n w^n$$

and the k-times derived series

$$\phi^{(k)}(w) = k!a_k + \sum_{n\geq k+1} \frac{n!}{(n-k)!} a_n w^{n-k},$$

have the same radius of convergence and note that

$$\phi^{(k)}(w) = f^{(k)}(w + z_0)\frac{d(w + z_0)}{dz} = f^{(k)}(z), \quad |w| < R.$$

From this it follows that if R is the radius of convergence of the series $f(z) = \sum_{n\geq 0} a_n(z - z_0)^n$, then

$$f^{(k)}(z) = k!a_k + \sum_{n\geq k+1} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}, \quad |z - z_0| < R,$$

so that $a_k = f^{(k)}(z_0)/k!$. Note that $a_0, a_1, \dots, a_k, \dots$ depend on z_0. Thus, f becomes

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad |z - z_0| < R.$$

This is often called the *Taylor series* expansion for f in $|z - z_0| < R$. In the special case when $z_0 = 0$, it is called the *Maclaurin series* expansion.

One of the remarkable results in Complex Analysis is that the converse of Theorem 3.71 holds. If f is analytic in a domain D, then f can be represented (locally about each point $z_0 \in D$) as a Taylor series expansion about z_0:

$$f(z) = \sum_{n\geq 0} a_n(z - z_0)^n,$$

where z_0 is a center of the largest disk $\Delta(z_0; R) \subseteq D$ and $z \in \Delta(z_0; R)$. For instance if $f(z) = z^{-1}$, then f is analytic in the punctured plane $\mathbb{C}\backslash\{0\}$. If $z_0 \neq 0$, then we have the Taylor expansion

$$f(z) = \frac{1}{z - z_0 + z_0} = \frac{1}{z_0}\sum_{n\geq 0}(-1)^n \frac{(z - z_0)^n}{z_0^n} \quad \text{for} \quad |z - z_0| < |z_0|.$$

However, we shall prove a more general result in Chapter 4 (see Corollary 4.95) which states that for a given analytic function defined in a domain D and for each z_0 such that $\Delta(z_0; r) \subseteq D$ there is always a power series converging in $\Delta(z_0; r)$ whose sum is $f(z)$.

3.73. Corollary. *If f is entire and if $z_0 \in \mathbb{C}$, then $f^{(n)}(z_0)$ exists for $n = 0, 1, \ldots$ and has the power series expansion*

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad \text{for all} \quad z.$$

3.74. Remark. Now it is clear that entire functions are just those functions which are defined by the sums of the series $\sum_{n \geq 0} a_n z^n$ where $\limsup_{n \to \infty} |a_n|^{1/n} = 0$. ●

In the next theorem (called the **Uniqueness Theorem** for power series), we show that a power series representing a given function, when obtained by whatever method, is unique.

3.75. Theorem. *Suppose $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$ converge for $|z| < R_1$ and $|z| < R_2$, respectively. If $f(z_k) = g(z_k)$ for a sequence $\{z_k\}$ of nonzero complex numbers in $0 < |z| < \delta$ such that $z_k \to 0$ as $k \to \infty$, or if $f(z) = g(z)$ for all z with $|z| < \delta$, where $0 < \delta < \min\{R_1, R_2\}$, then $a_n = b_n$ for every $n \in \mathbb{N}_0$.*

Proof. Since f, g are continuous in $|z| \leq \delta$, being analytic thereon,

$$\begin{aligned} z_k \to 0 \text{ and } f(z_k) = g(z_k) &\implies \lim_{k \to \infty} f(z_k) = \lim_{k \to \infty} g(z_k) \\ &\implies f(0) = g(0). \end{aligned}$$

Thus, $a_0 = b_0$. To complete the proof, we use the method of induction. Now suppose that $a_j = b_j$ for $j = 0, 1, \ldots, m - 1$. Then

$$\begin{aligned} f(z_k) = g(z_k) &\implies \sum_{n \geq m} a_n z_k^n = \sum_{n \geq m} b_n z_k^n \\ &\implies \sum_{n \geq m} a_n z_k^{n-m} = \sum_{n \geq m} b_n z_k^{n-m}, \quad \text{since} \quad z_k \neq 0, \\ &\implies F(z_k) = G(z_k), \end{aligned}$$

where

$$F(z) = \sum_{n \geq m} a_n z^{n-m} \quad \text{and} \quad G(z) = \sum_{n \geq m} b_n z^{n-m}.$$

Since the radius of convergence of $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq m} a_n z^{n-m}$ are the same, F is continuous for $|z| \leq \delta$. Similarly, G is continuous for $|z| \leq \delta$. Thus (see Theorem 2.22),

$$F(z_k) = G(z_k) \Longrightarrow \lim_{k\to\infty} F(z_k) = \lim_{k\to\infty} G(z_k) \Longrightarrow a_m = b_m$$

and we conclude that $a_n = b_n$ for every $n \geq 0$. Note that if $f(z) = g(z)$ for $|z| < \delta$, we can find a sequence $\{z_k\}_{k\geq 1} \in \{z : |z| < \delta\}$ such that $z_k \to 0$ as $k \to \infty$ (see Theorem 2.22). The proof is complete. ■

3.76. Corollary. *Suppose $f(z) = \sum_{n\geq 0} a_n z^n$, with the power series having radius of convergence $R > 0$. If 0 is a limit point of the set of zeros of f, then $f \equiv 0$ for $|z| < R$.*

It will be shown, more generally later (Theorem 4.103) that if f, g are two analytic functions which agree at a sequence of points $\{z_k\}$ having a limit point in their common domain of analyticity, then $f \equiv g$ therein.

3.4 Exponential and Trigonometric Functions

In this section we describe the complex analogues of exponential and trigonometric functions of elementary calculus. There are many approaches which lead to definitions of these functions although our approach is intuitive and direct. Let us first recall the following facts from calculus:

(a) Trigonometric functions are defined by means of the ratios of the sides of a right triangle; for instance, $\sin^2 x + \cos^2 x = 1$, $x \in \mathbb{R}$.

(b) $\dfrac{d}{dx}(\sin x) = \cos x$, $\dfrac{d}{dx}(\cos x) = -\sin x$, $\dfrac{d}{dx}(e^x) = e^x$;

(c) Further,

$$\begin{aligned} \sin x &= x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \end{aligned}$$

The properties (b) and (c) show that $\sin x$ and $\cos x$ are the unique solution of the second order ordinary differential equation

$$f''(x) + f(x) = 0$$

subject to the conditions $f(0) = f'(0) - 1 = 0$ and $f(0) = f'(0) - 1 = 1$, respectively. On the other hand e^x is defined to be the solution of

$$f'(x) = f(x), \quad f(0) = 1.$$

In the same way we begin with a function $f(z) = \sum_{n\geq 0} a_n z^n$ satisfying

(i) $f'(z) = f(z)$ for $z \in \mathbb{C}$, i.e. f is analytic in $\mathbb{C}$

(ii) $f(x) = e^x$, $x \in \mathbb{R}$.

Using (i) we find that $a_{n-1} = na_n$. Since $f(0) = e^0 = 1 = a_0$, induction on n yields $a_n = 1/n!$. Thus we define the complex exponential function as

$$f(z) = \exp(z) = e^z = \sum_{n\geq 0} \frac{z^n}{n!}. \tag{3.77}$$

Conversely, if the complex exponential function is given by (3.77) then (i)-(ii) follow easily. By the Ratio test, the series (3.77) converges absolutely for all $z \in \mathbb{C}$ and hence, e^z is an entire function. In the same way one can obtain the series representations of $\sin z$ and $\cos z$.

As $(\exp z)' = \exp z$, we have the following: if $g \in \mathcal{H}(D)$, then so does $F(z) = \exp(g(z))$ and $F'(z) = g'(z)\exp(g(z))$ for $z \in D$.

Next, for a fixed $\zeta \in \mathbb{C}$, we define

$$g_\zeta(z) = e^z e^{\zeta - z} (= f(z)f(\zeta - z)).$$

Then for every $z \in \mathbb{C}$, we have $g'_\zeta(z) = 0$. Therefore, by Theorem 3.31, g_ζ is constant. Since $g_\zeta(0) = e^0 e^\zeta = e^\zeta$, we must have

$$e^z e^{\zeta - z} = e^\zeta.$$

Taking $z = z_1$ and $\zeta = z_1 + z_2$, we see that for every $z_1, z_2 \in \mathbb{C}$,

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}. \tag{3.78}$$

The result (3.78) is called the *"Addition theorem"* for the exponential function which is also a consequence of the Uniqueness theorem (Section 4.11). This property can also be obtained directly using the Cauchy product of series:

$$\begin{aligned} e^{z_1} e^{z_2} &= \sum_{n\geq 0} \frac{z_1^n}{n!} \sum_{m\geq 0} \frac{z_2^m}{m!} = \sum_{k\geq 0} \left[\sum_{p=0}^{k} \frac{z_1^p}{p!} \frac{z_2^{k-p}}{(k-p)!} \right] \\ &= \sum_{k\geq 0} \left[\frac{1}{k!} \sum_{p=0}^{k} \binom{k}{p} z_1^p z_2^{k-p} \right] \\ &= \sum_{k\geq 0} \frac{(z_1 + z_2)^k}{k!} = e^{z_1 + z_2}. \end{aligned}$$

For any positive integer m, by induction, (3.78) with $z_1 = z_2 = z$ shows that

$$e^{mz} = (e^z)^m. \tag{3.79}$$

For $z = iy$, $y \in \mathbb{R}$, (3.77) gives

$$e^{iy} = \sum_{n\geq 0} \frac{i^n y^n}{n!} = \sum_{k\geq 0} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k\geq 0} \frac{(-1)^k y^{2k+1}}{(2k+1)!}.$$

Thus, we have the Euler formula

$$e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}. \tag{3.80}$$

Therefore, for $z_1 = x$ and $z_2 = iy$, $x, y \in \mathbb{R}$, (3.78) shows that

$$e^{x+iy} = e^x(\cos y + i \sin y); \; e^z = e^x e^{iy}. \tag{3.81}$$

Since $e^z e^{-z} = e^{z-z} = e^0 = 1$, $e^z \neq 0$ for each $z \in \mathbb{C}$. Further,

$$|e^z|^2 = e^z \overline{(e^z)} = e^z e^{\overline{z}} = e^{z+\overline{z}} = e^{2\mathrm{Re}\, z}$$

so that $|e^z|^2 = (e^{\mathrm{Re}\, z})^2$, by (3.79). Thus, from (3.81) and the above, we write

$$|e^z| = e^{\mathrm{Re}\, z}; \; \arg(e^z) = \mathrm{Im}\, z \; (\mathrm{mod}\, 2\pi). \tag{3.82}$$

In particular, we have $|e^z| = 1 \iff z \in i\mathbb{R}$. For m, n integers and $k = 0, 1, \dots, n-1$, we have

$$(e^z)^{1/n} = (e^x e^{iy})^{1/n} = e^{x/n}[e^{i(y+2k\pi)/n}] = e^{(z+2k\pi i)/n}$$

and so we write

$$(e^z)^{m/n} = e^{m(z+2k\pi i)/n}. \tag{3.83}$$

From (3.80) we see in particular that, if z is real then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{3.84}$$

Thus the extension of the exponential to the complex plane suggests to make (3.84) as definitions of $\cos z$ and $\sin z$ for any $z \in \mathbb{C}$ although these definitions can be realized as a consequence of the Uniqueness theorem. Hence we make the definition–the famous Euler formula:

$$e^{iz} = \cos z + i \sin z, \quad z \in \mathbb{C}.$$

Now consider $f(z) = \sin z$ for $z \in \mathbb{C}$. Then, by the definition of $\sin z$ and the fact that $(\exp z)' = \exp z$, (3.84) gives

$$f'(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

and so using the definition of $\cos z$,

$$f''(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

Consequently, we have

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k & \text{if } n = 2k+1 \end{cases}, \quad k \in \mathbb{N}_0 .$$

Similarly, if $f(z) = \cos z$, we obtain $f'(z) = -\sin z$ and $f''(z) = -\cos z$ so that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k+1 \\ (-1)^k & \text{if } n = 2k \end{cases}, \quad k \in \mathbb{N}_0 .$$

Now we use the Maclaurin series expansion (see just above Corollary 3.73), namely $f(z) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} z^n$, to get the series expansions for $\cos z$ and $\sin z$. Therefore, for any $z \in \mathbb{C}$, the cosine and sine series are now

(3.85) $$\cos z = \sum_{n\geq 0} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

(3.86) $$\sin z = \sum_{n\geq 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

respectively. Also, note that (3.85) and (3.86) follow from the definition of e^{iz} and (3.84). Each of the series converges absolutely for every $z \in \mathbb{C}$, because

$$\sum_{n\geq 0} \frac{|z|^{2n}}{(2n)!} \quad \text{and} \quad \sum_{n\geq 0} \frac{|z|^{2n+1}}{(2n+1)!}$$

are the subseries (i.e. terms chosen from those of) of the convergent series $\sum_{n\geq 0} [|z|^n/n!]$ for $z \in \mathbb{C}$. The remaining circular functions are then defined by

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z} \quad \left(z \neq \tfrac{(2k+1)\pi}{2}\right); \\ \cot z &= \frac{\cos z}{\sin z}, \quad \csc z = \frac{1}{\sin z} \quad (z \neq k\pi), \end{aligned}$$

where $k \in \mathbb{Z}$. These functions are analytic except at points where the denominators vanish. The hyperbolic functions are defined by

$$\cos(iz) = \cosh z \quad \text{and} \quad \sin(iz) = i \sinh z$$

so that

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \\ \tanh z &= \frac{\sinh z}{\cosh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z} \quad \left(z \neq \frac{(2k+1)\pi i}{2}\right), \\ \coth z &= \frac{\cosh z}{\sinh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z} \quad (z \neq k\pi i), \end{aligned}$$

where $k \in \mathbb{Z}$. Again these functions are analytic except at points where the denominators vanish. For $z_1, z_2 \in \mathbb{C}$, the identity

$$e^{i(z_1 \pm z_2)} = e^{iz_1} e^{\pm iz_2} \tag{3.87}$$

yields the addition formulae (which will be proved later also by using the Uniqueness theorem)

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \tag{3.88}$$

and

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2. \tag{3.89}$$

Putting $z_1 = z_2 = z$ in (3.88) and (3.89), we get

$$\cos^2 z + \sin^2 z = 1; \quad \cos 2z = \cos^2 z - \sin^2 z; \quad \sin 2z = 2 \sin z \cos z.$$

Recall the known facts about the sine and cosine functions of a real variable indicated by the following chart:

Interval	cos	sin
$[0, \pi/2]$	$1 \searrow 0$	$0 \nearrow 1$
$[\pi/2, \pi]$	$0 \searrow -1$	$1 \searrow 0$
$[\pi, 3\pi/2]$	$-1 \nearrow 0$	$0 \searrow -1$
$[3\pi/2, 2\pi]$	$0 \nearrow 1$	$-1 \nearrow 0$

as also the properties $\cos 2k\pi = 1$, $\sin 2k\pi = 0$, $k \in \mathbb{N}_0$. Therefore, we have from (3.88) and (3.89)

$$\cos(z + 2k\pi) = \cos z \quad \text{and} \quad \sin(z + 2k\pi) = \sin z \tag{3.90}$$

where $k \in \mathbb{Z}$. The above facts together with the equation in (3.84) immediately yield the following necessary and sufficient conditions for certain identities to become true:

(a) $\cos z = 1 \iff z = 2\pi k$,
(b) $\sin z = 1 \iff z = (4k+1)\pi/2$,
(c) $\tan z = 1 \iff z = (4k+1)\pi/4$,
(d) $\cos z_1 = \cos z_2 \iff$ either $z_1 + z_2 = 2\pi k$ or $z_1 - z_2 = 2\pi k$,
(e) $\sin z_1 = \sin z_2 \iff$ either $z_1 + z_2 = (2k+1)\pi$ or $z_1 - z_2 = 2\pi k$,

where $k \in \mathbb{Z}$. Note that (d) $\iff$ (e). This fact can be easily seen by taking $z_1 = \zeta_1 - \pi/2$ and $z_2 = \zeta_2 - \pi/2$.

3.91. Definition. A function $f : D \to \mathbb{C}$ is said to be *periodic* if there exists an $\omega \neq 0$ such that $f(z + \omega) = f(z)$ for all $z \in D$. The complex number ω is then called a *period* of f. The function f is called *doubly periodic* in D if there are complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$ such that $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z \in D$.

Since $e^{2\pi ki} = 1$, we find that $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$. Suppose $\omega = u+iv$ is a period for e^z. Then by definition $e^{z+\omega} = e^z$; i.e. $e^\omega = 1$, and $|e^\omega| = e^u = 1$ and so $u = 0$. Hence

$$e^{iv} = 1; \quad \text{i.e. } \cos v = 1 \text{ and } \sin v = 0.$$

By (3.90), this occurs only for $v = 2k\pi$, where k is some integer. This shows that ω is a period of e^z iff $\omega = 2k\pi i$. In particular, e^z is a periodic function with period $2\pi i$. Similarly, we have

$$\sin(z+\omega) = \sin z, \quad \text{or} \quad \cos(z+\omega) = \cos z \iff \omega = 2k\pi, \quad k \in \mathbb{Z}.$$

3.92. Example. Consider the function $f(z) = e^z$. Then we have

$$f(z_1) = f(z_2) \Longrightarrow e^{z_1 - z_2} = 1 \Longrightarrow z_1 - z_2 = 2k\pi i \text{ for some } k \in \mathbb{Z}.$$

This shows that f is one-to-one in a domain D iff D does not have even a single pair of distinct points z_1 and z_2 satisfying

$$z_1 = z_2 + 2k\pi i, \quad k = \pm 1, \pm 2, \ldots .$$

For instance, f is univalent in $|z| < \pi$; but not in $\mathbb{C}$. Similarly, we see that f is univalent within any horizontal strip of the form:

$$b < \operatorname{Im} z < a, \quad 0 < a - b \leq 2\pi.$$

In fact we see that f is univalent in each strip

$$D_k = \{z : 2(k-1)\pi < \operatorname{Im} z < 2k\pi\}, \quad k \in \mathbb{Z},$$

and maps each strip onto the w-plane with a cut along (i.e. omitting) the positive real axis: $0 \leq x < \infty$. •

3.93. Example. Note that for $z = x + iy$, we have

$$e^z = \begin{cases} e^x e^{ib} & \text{if } y = b, \text{ a constant,} \\ e^a e^{iy} & \text{if } x = a, \text{ a constant.} \end{cases}$$

As a further illustration of the exponential function, we have

(i) The exponential function e^z maps each horizontal line $y = b$ onto a ray from the origin to infinity, namely, $\{re^{ib} : r \in (0, \infty)\}$.

(ii) The exponential function e^z maps each vertical line $x = a$ onto a circle centered at the origin and radius e^a. Observe that each point on this circle is the image of infinitely many points on the vertical lines as e^{iy} is periodic. Note that these two families of curves (and also their images) are orthogonal to each other.

(iii) The line $y = b$ and the line $x = a$ intersect at (a, b). How about their images in the w-plane?

(iv) The rectangle $D = \{z : a_1 < \operatorname{Re} z < a_2,\ b_1 < \operatorname{Im} z < b_2\}$ is mapped into $D' = \{w : e^{a_1} < |w| < e^{a_2},\ b_1 < \arg w < b_2\}$. In particular, $\exp z$ maps the infinite strip $\{z : 0 < \operatorname{Im} z < \pi/2\}$ onto the open first quadrant $\{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}$. ●

3.5 Logarithmic Functions

It is natural to think of a logarithmic function as the inverse of the exponential function. Recall that, when x is real,

(a) $e^x > 0$ for all $x \in \mathbb{R}$

(b) $e^x \to \infty$ as $x \to \infty$

(c) $e^{-x} = 1/e^x$ which by (a) gives $e^x \to 0$ as $x \to -\infty$

(d) $\dfrac{d}{dx}(e^x) = e^x$ and so e^x is monotonically increasing for all x.

This shows that e^x defines a strictly increasing differentiable function from $\mathbb{R}$ onto $\mathbb{R}^+$, the set of positive reals. Hence it has a continuous strictly increasing inverse function called the *natural logarithm* (with base e)

$$\ln : \mathbb{R}^+ \to \mathbb{R}$$

with the property that $\ln x = y$ is the solution of $e^y = x$. In particular, for each $x > 0$ there is exactly one y such that $e^y = x$.

We would like to mimic the real variable case and say that a complex number w in $\mathbb{C}$ is called *a logarithm of a complex number* z in $\mathbb{C}$, denoted by $w = \log z$, if $e^w = z$ holds. Since $e^w \neq 0$ for any $w \in \mathbb{C}$, the number 0 has no logarithm; hence, the equation $e^w = z$ has no solution in $\mathbb{C}$ when $z = 0$ and $z = \infty$. Consider an arbitrary fixed $z \neq 0$ in polar form

$$z = |z|e^{i\operatorname{Arg} z} = re^{i\theta} \quad (r = |z| > 0,\ -\pi < \theta \leq \pi).$$

Let us now solve the equation $w = \log z$. If we let $w = u + iv$, u, v real, then $e^w = z$ becomes $e^{u+iv} = re^{i\theta}$ which yields

$$e^u = r,\quad e^{(v-\theta)i} = 1;\quad \text{i.e.}\quad u = \ln r,\quad v = \theta + 2k\pi,\ k \in \mathbb{Z}.$$

Accordingly, we therefore have for $z \neq 0$,

$$w = \log z = \ln |z| + i(\operatorname{Arg} z + 2k\pi),\quad k \in \mathbb{Z}. \tag{3.94}$$

The *principal value* of a logarithm, sometimes denoted by Log , corresponds to the value of the principal value of the argument; that is for $z \neq 0$,

$$\text{Log}\, z = \ln|z| + i\text{Arg}\, z, \quad -\pi < \text{Arg}\, z \leq \pi. \tag{3.95}$$

This agrees with the meaning of '$\text{Log}\, z$' for positive real numbers $z = x$, for which $\text{Arg}\, z = 0$, with which the reader is already familiar. We use the notation $*\log z$ to denote the infinite set of values

$$\{\ln|z| + i(\text{Arg}\, z + 2k\pi) : \; k \in \mathbb{Z}\}.$$

It is often convenient to single out a particular member in this set. Thus, we formulate the following

3.96. Definition. If $w \in \mathbb{C}$ satisfies $e^w = z$ for an arbitrary fixed $z \neq 0$, then w defined by

$$w = *\log z = \begin{cases} \ln|z| + i(\text{Arg}\, z + 2k\pi) \\ \text{(or)} \\ \text{Log}\, z + 2k\pi i \\ \text{(or)} \\ \ln|z| + i\arg z, \end{cases} \tag{3.97}$$

is called the *logarithm* of z, where $k \in \mathbb{Z}$.

Note that $z \mapsto \text{Log}\, z$ is a function defined in $\mathbb{C}\backslash\{0\}$ and every non-zero complex number has infinitely many logarithms which differ from each other by integral multiples of $2\pi i$. This shows that $w = \log z$ is not a function in general and is in fact an infinitely-valued association or relation, with infinitely many values for each $z \neq 0$. For each k, the strip of the type

$$D_k = \{u + iv : \; (2k-1)\pi < v \leq (2k+1)\pi, \;\; k \in \mathbb{Z}\}$$

is called a fundamental region for $\log z$ (Note that a region in the z-plane whose image just covers the w-plane once is called a *fundamental region* for the function $w = \log z$). Since the union is the w-plane, the set $\{D_k\}_{k=-\infty}^{\infty}$ is a complete set of fundamental regions for the logarithm. In particular, this observation proves that for each $k \in \mathbb{Z}$, $\exp(D_k) = \mathbb{C}\backslash\{0\}$. It is also easy to see that for each $k \in \mathbb{Z}$, $\exp : D_k \to \mathbb{C}\backslash\{0\}$ is one-to-one and onto.

We have

(a) $\text{Log}\,(\pm i) = \pm i\pi/2, \; \text{Log}\,(1+i) = \ln\sqrt{2} + i\pi/4$
(b) $\text{Log}\,((1 \pm i)/\sqrt{2}) = \pm i\pi/4, \, \ln(-1) = i\pi$
(c) $\text{Log}\,(2 - 3i) = \ln\sqrt{13} - i\text{Arctan}\,(3/2)$
(d) $\text{Log}\,(-2 + 3i) = \ln\sqrt{13} + i(\pi - \text{Arctan}\,(3/2))$
(e) $\text{Log}\, i^{1/4} = i\pi/8, \; \text{Log}\,(\alpha z) = \ln\alpha + \text{Log}\, z \; (\alpha > 0)$.

From these examples, we can see that it is not even true in general that

$$\mathrm{Log}\,(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2$$

when we limit our attention to principal values. However, for $z_1 \neq 0$ and $z_2 \neq 0$, the statements

$$\begin{aligned}\log(z_1z_2) &= \log z_1 + \log z_2 \pmod{2\pi},\\ \log(z_1/z_2) &= \log z_1 - \log z_2 \pmod{2\pi}\end{aligned}$$

hold. Suppose that $z_1 = |z_1|e^{i\mathrm{Arg}\,z_1}$ and $z_2 = |z_2|e^{i\mathrm{Arg}\,z_2}$. Then, for the complex numbers z_1, z_2, we write

$$z_1z_2 = |z_1z_2|e^{i\mathrm{Arg}\,(z_1z_2)}.$$

There is a $k_1 \in \{-1, 0, 1\}$ (see Exercise 1.54(a)) such that

$$\begin{aligned}\mathrm{Log}\,(z_1z_2) &= \ln|z_1z_2| + i\mathrm{Arg}\,(z_1z_2)\\ &= \ln|z_1| + \ln|z_2| + i(\mathrm{Arg}\,z_1 + \mathrm{Arg}\,z_2 + 2\pi k_1)\\ &= \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 + 2\pi i k_1.\end{aligned}$$

Thus, $\mathrm{Log}\,(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 \iff \mathrm{Arg}\,z_1 + \mathrm{Arg}\,z_2 \in (-\pi, \pi]$. In particular, this statement leads to

$$\mathrm{Log}\,(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2$$

for all $z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}\,z_1 > 0$ and $\mathrm{Re}\,z_2 > 0$. A serious trouble ensues if the conditions $\mathrm{Re}\,z_1 > 0$ and $\mathrm{Re}\,z_2 > 0$ are dropped in the last equation. This point can also be checked on taking $z_1 = i$ and $z_2 = i$. Similarly, we see that the condition $\mathrm{Arg}\,z_1 + \mathrm{Arg}\,z_2 \in (-\pi, \pi]$ is met whenever $\mathrm{Im}\,z_1 < 0$ and $\mathrm{Im}\,z_2 > 0$. This means that

$$\mathrm{Log}\,(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2$$

for all z_1, z_2 with $\mathrm{Im}\,z_1 < 0 < \mathrm{Im}\,z_2$.

In fact, by Exercise 1.54(b), we similarly have

$$\mathrm{Log}\,(z_1/z_2) = \mathrm{Log}\,z_1 - \mathrm{Log}\,z_2 + 2\pi i k_2,$$

where $k_2 = k_2(z_1, z_2)$ is defined in Exercise 1.54(b). For instance, let $z_1 = i$ and $z_2 = -1$. Then

$$\mathrm{Log}\,z_1 = i\pi/2,\ \ \mathrm{Log}\,z_2 = i\pi \ \text{ and } \ \mathrm{Log}\,(z_1z_2) = -i\pi/2.$$

Therefore, $\mathrm{Log}\,(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 - 2\pi i$ since $k_1(i, -1) = -1$. Similarly, we have

(a) $\mathrm{Log}\,1 = \mathrm{Log}\,(-1) + \mathrm{Log}\,(-1) + 2\pi i$, since $k_1(-1, -1) = 1$.

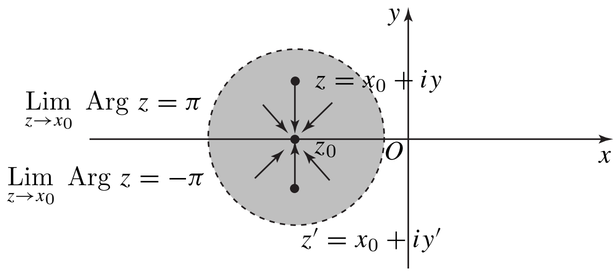


Figure 3.3: Description for continuity of $\mathrm{Arg}\, z$ at $z_0 = x_0 < 0$.

(b) $\mathrm{Log}\,(-1-i) = \mathrm{Log}\, i + \mathrm{Log}\,(-1+i) - 2\pi i$, since $k_1(i, -1+i) = -1$.

(c) $\mathrm{Log}\,(r_1 r_2 e^{i3\pi/2}) = \mathrm{Log}\,(r_1 e^{i3\pi/4}) + \mathrm{Log}\,(r_2 e^{i3\pi/4}) - 2\pi i$.

It is often convenient to single out a particular set in the set $*\log z$. One immediate possibility is to consider the principal value of the logarithm $\mathrm{Log}\, z$ and to study analyticity of the function $\mathrm{Log}\, z$ whose domain of definition is $\mathbb{C}\backslash\{0\}$. For $z \neq 0$, let

$$z = re^{i\theta} \quad \text{and} \quad u + iv = \mathrm{Log}\, z,$$

where $|z| = r$ and $-\pi < \theta = \mathrm{Arg}\, z \leq \pi$. Recall that

$$\mathrm{Log}\, z = \ln|z| + i\mathrm{Arg}\, z.$$

If $\mathrm{Log}\, z = u(x,y) + iv(x,y)$, u, v real, then

$$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta. \tag{3.98}$$

The function $\mathrm{Log}\, z$ is not continuous at points on the negative real axis, since $\mathrm{Arg}\, z$ fails to possess a limit at the points along this axis; for, let $z_0 = x_0 < 0$. Then, for $z = x_0 + iy$ with $y > 0$, we have

$$\lim_{z\to z_0} \mathrm{Arg}\, z = \lim_{\substack{y\to 0\\ y>0}} \mathrm{Arg}\,(x_0 + iy) = \pi$$

and for $z = x_0 + iy$ with $y < 0$, we have

$$\lim_{z\to z_0} \mathrm{Arg}\, z = \lim_{\substack{y\to 0\\ y<0}} \mathrm{Arg}\,(x_0 + iy) = -\pi.$$

Since every neighborhood of z_0 intersects both second and third quadrants making it possible to approach z_0 in two different ways (see Figure 3.3), $\mathrm{Arg}\, z$ cannot converge. However, $\mathrm{Log}\, z$ is single-valued and continuous in the domain D, where $D = \mathbb{C}\backslash\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}$. Now we can compute the derivative of the logarithm. From (3.98), it follows that

$$u_r = \frac{v_\theta}{r} = \frac{1}{r} \quad \text{and} \quad v_r = -\frac{u_\theta}{r} = \theta.$$

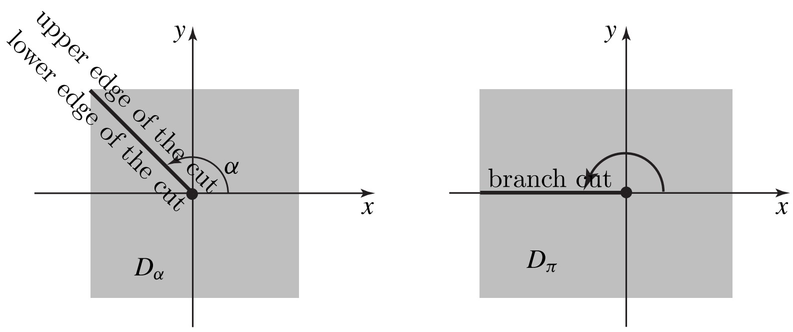


Figure 3.4: Slit plane $D_\alpha = \mathbb{C}\backslash\{Re^{i\alpha} : R > 0, \alpha - 2\pi < \arg_\alpha < \alpha\}$.

Thus u and v satisfy the C-R equations (3.47). Moreover, the partial derivatives u_r, v_r etc., are all continuous in D. This implies the existence of the derivative of $\operatorname{Log} z$ in D and so $\operatorname{Log} z$ is analytic therein. Since

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r},$$

we readily have

$$\frac{d}{dz}(\operatorname{Log} z) = e^{-i\theta}\frac{\partial}{\partial r}(\ln r + i\theta) = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}, \quad z \in D.$$

In the same way, we see that $\arg_\alpha z$ with $z = re^{i\theta}$ $(r > 0,\ \alpha - 2\pi < \theta < \alpha)$, is continuous in the 'slit plane' $D_\alpha = \mathbb{C}\backslash\{Re^{i\alpha} : R \geq 0\}$ where $\alpha \in \mathbb{R}$ is fixed (see Figure 3.4). Thus, in D_α, the function $f_\alpha := \log_\alpha$ defined by

$$f_\alpha(z) := \log_\alpha z = \ln|z| + i\arg_\alpha z, \tag{3.99}$$

where $\arg_\alpha z$ denotes the unique choice of $\arg z$ in the interval $(\alpha - 2\pi, \alpha)$, is continuous with derivative

$$\frac{d}{dz}(f_\alpha(z)) = \frac{1}{z}, \quad z \in D_\alpha.$$

Clearly, f_α is not continuous at points on the ray $\{re^{i\alpha} : 0 \leq r < \infty\}$. Evidently, $\exp(f_\alpha(z)) = z$. For the analytic property of multiple-valued functions, we regard a multiple-valued function as a collection of single-valued functions. More explicitly we have

3.100. Definition. Suppose F is a multiple-valued function defined in S. A *branch* of F is a single-valued analytic function f in some domain $D \subset S$ obtained from F in such a way that at each point of D, f assumes exactly one of the possible values of F.

For example, from our discussion above, for each fixed $\alpha \in \mathbb{R}$, the function f_α defined by (3.99) represents a branch of $\log z$ in the slit plane D_α. We have created a cut along $\theta = \alpha$, so that the restriction on $\theta = \arg_\alpha z$ makes the function f_α single-valued and analytic. This cut corresponding to $\theta = \alpha$, i.e. the semi infinite line $\{Re^{i\alpha} :\ R \geq 0\}$, $\alpha \in \mathbb{R}$ fixed, is regarded as a *branch cut* for the branch f_α of $\log z$. For $\alpha = \pi$, f_α becomes $\operatorname{Log} z$ in D_π and is often called the *principal branch* of $\log z$. The *branch cut* for $\operatorname{Log} z$ is then the negative-real axis from the origin, i.e. $\{-R :\ R \geq 0\}$.

3.101. Definition. A multiple-valued function F defined in S is said to have a *branch point* at $z_0 \in \mathbb{C}$ if, when z describes an arbitrarily small circle about z_0, then for every branch f of F, $f(z)$ does not return to its original value.

For example every $z \neq 0$ is not a branch point of $\log z$, since on a sufficiently small circle enclosing the point z, every branch of $\log z$ returns to its original value.

3.102. Definition. A function $f \in \mathcal{H}(D)$, where D is a domain D, is called an *analytic/holomorphic branch* of $\log z$ in D if f is analytic/holomorphic in D and $\exp(f(z)) = z$ for each $z \in D$.

Let $\phi \in \mathcal{H}(D)$ such that $0 \notin \phi(D)$. We say $f \in \mathcal{H}(D)$ is an *analytic-logarithm* of ϕ if $\exp(f(z)) = \phi(z)$ for each $z \in D$. Clearly, if $\phi(z) = z$ in D then the concept of analytic-logarithm coincides with that of the (analytic) branch of $\log z$. Analytic/holomorphic branches of $\log \phi(z)$ are called *logarithmic functions*.

For instance, in the slit plane D_α, the function f_α defined by (3.99) is an example of a logarithmic function.

3.103. Theorem. *Let D be a domain in $\mathbb{C}$. Then, any two logarithmic functions f, $\widetilde{f} : D \to \mathbb{C}$ are related by*

$$\widetilde{f}(z) = f(z) + 2\pi k i \tag{3.104}$$

for some $k \in \mathbb{Z}$. Conversely, if a logarithmic function $f : D \to \mathbb{C}$ is related by (3.104) *with $\widetilde{f} : D \to \mathbb{C}$, then $\widetilde{f}$ is also a logarithmic function.*

Proof. Let f and $\widetilde{f}$ be two logarithmic functions in D. Then by definition

$$\exp\{\widetilde{f}(z)\} = \exp\{f(z)\}, \quad \text{i.e.} \quad \exp\{\widetilde{f}(z) - f(z)\} = 1 \ \text{ for all } z \in D,$$

which implies that

$$F(z) = \frac{\widetilde{f}(z) - f(z)}{2\pi i} \in \mathbb{Z}.$$

Note that F is an analytic function in D and, in particular, F is continuous in the open and connected set D. But F is an integer-valued and so F maps a connected set onto a connected subset of $\mathbb{Z}$, viz. a single point, k, say. Thus, f and $\widetilde{f}$ are related by (3.104).

Conversely, if f is a logarithmic function in D and $\widetilde{f} : D \to \mathbb{C}$ is related by (3.104) for some $k \in \mathbb{Z}$ then for all $z \in D$, we have

$$\exp\{\widetilde{f}(z)\} = \exp\{f(z)\}\exp\{2\pi ik\} = \exp\{f(z)\} = z$$

and so $\widetilde{f}$ becomes a logarithmic function in D, by definition. ■

Theorem 3.103 shows that, given a branch of logarithm in a domain, one can obtain all the branches of logarithm on that domain. For example, we let $\alpha = 3\pi/2$ and consider (3.99). Then $f_{3\pi/2}$ is an analytic branch of $\log z$ for $z \in D_{3\pi/2}$. Thus to find a numerical value of $f_{3\pi/2}(z_0)$, where $z_0 = -1 - i$, we write

$$f_{3\pi/2}(-1-i) = \ln\sqrt{2} + i\arg_{3\pi/2}(-1-i) = \ln\sqrt{2} + \frac{5\pi i}{4}.$$

From Theorem 3.103, we also note that the value of $f_{3\pi/2}(-1-i)$ for any other branch of $\log z$ which is analytic in the slit plane $D_{3\pi/2}$ is given by

$$\ln\sqrt{2} + \frac{5\pi i}{4} + 2k\pi i, \quad \text{for some } k \in \mathbb{Z}.$$

As another characterization of logarithmic functions, we have

3.105. Theorem. *Let $f : D \to \mathbb{C}$ be analytic in a domain D not containing 0. Then f is a branch of $\log z$ in D iff $f'(z) = 1/z$ for all $z \in D$ and $\exp\{f(a)\} = a$ for at least one $a \in D$.*

Proof. Suppose that f is a branch of $\log z$ in D. Then, $\exp\{f(z)\} = z$ for all $z \in D$ so that

$$f'(z)\exp\{f(z)\} = 1, \quad \text{i.e.} \quad f'(z) = 1/z \quad \text{for all } z \in D.$$

To prove the converse, we need to show that

$$\exp\{f(z)\} = z \quad \text{for all } z \in D. \tag{3.106}$$

To do this, we consider $g(z) = z\exp\{-f(z)\}$. Then g is analytic in D and satisfies

$$g'(z) = [1 - zf'(z)]\exp\{-f(z)\} = 0 \quad \text{for all } z \in D.$$

By Theorem 3.31, g is a constant, say k. Thus, $z\exp\{-f(z)\} = k$ for all $z \in D$. Since $\exp\{f(a)\} = a$, we must have $k = 1$. ■

3.107. Example. We now demonstrate that there can be no branch of logarithm in the domain $\Omega = \mathbb{C} \setminus \{0\}$. This fact can be proved by a number of ways.

Suppose on the contrary that $f(z)$ is a branch of $\log z$ in Ω. The restriction of f to the slit plane $D_\pi = \mathbb{C} \setminus (-\infty, 0]$ is then a branch of logarithm in D_π. As $\operatorname{Log} z$ is the principal branch in D_π, Theorem 3.103 shows that for $z \in D_\pi$, f must be of the form

$$f(z) = \operatorname{Log} z + 2k\pi i$$

for some $k \in \mathbb{Z}$. But then, f being analytic in Ω, f is continuous at $x_0 < 0$ so that

$$f(x_0) = \lim_{\substack{y \to 0 \\ y > 0}} f(x_0 + iy) = \lim_{\substack{y \to 0 \\ y > 0}} \operatorname{Log}(x_0 + iy) + 2k\pi i = (2k+1)\pi i$$

and

$$f(x_0) = \lim_{\substack{y \to 0 \\ y < 0}} f(x_0 + iy) = \lim_{\substack{y \to 0 \\ y < 0}} \operatorname{Log}(x_0 + iy) + 2k\pi i = (2k-1)\pi i$$

which is a contradiction. Thus, no such f can exist. ●

It follows from Theorem 3.105 that every branch f of $\log z$ in D is infinitely differentiable in D. A natural question is whether f can be represented by a power series valid in appropriate disks in D. For example, the following result conveniently stated in a different form shows that

$$\operatorname{Log} z = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (z-1)^n \quad \text{for } |z-1| < 1.$$

3.108. Theorem. *In the unit disk Δ, the power series $\sum_{n \geq 1} (z^n/n)$ represents the logarithmic function $-\operatorname{Log}(1-z)$.*

Proof. Let $f(z) = -\operatorname{Log}(1-z)$. Note that the radius of convergence of the given series is 1 and so by Theorem 3.71, it represents an analytic function, say $g(z)$, in the unit disk $\Delta = \{z : |z| < 1\}$. Then, as we have seen earlier, $f'(z) = (1-z)^{-1}$ and since the power series can be differentiated term-by-term for $z \in \Delta$, by Theorem 3.71, we must have

$$g'(z) = \sum_{n \geq 1} z^{n-1} = \frac{1}{1-z}$$

so that $f'(z) - g'(z) = 0$. Therefore, by Theorem 3.31, $f - g$ is constant in Δ. Since $f(0) = -\operatorname{Log} 1 = 0 = g(0)$, it follows that $f(z) - g(z) = 0$ in Δ. ■

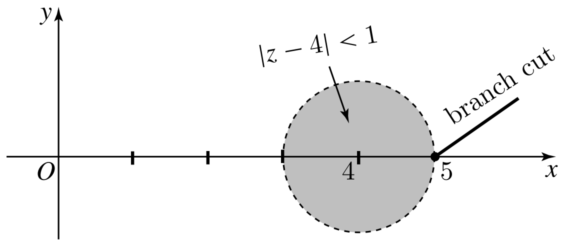


Figure 3.5: Choosing suitable branch.

3.109. Example. Suppose we wish determine the sum of the series $1+\sum_{n=2}^{\infty}(z-4)^n/n$. We know that the series converges for $|z-4|<1$ and so, we may let its sum by $f(z)$:

$$f(z)=1+\sum_{n=2}^{\infty}\frac{1}{n}(z-4)^n, \quad |z-4|<1.$$

The term-by-term differentiation gives

$$f'(z)=\sum_{n=2}^{\infty}(z-4)^{n-1}=\frac{z-4}{1-(z-4)}=-1+\frac{1}{5-z}, \quad |z-4|<1.$$

Antiderivative gives

$$f(z)=-z-\log_{\alpha}(5-z)+c$$

where the branch cut must lie to the right of the disk $|z-4|<1$, i.e. in the half-plane $\mathrm{Re}\, z>5$ (see Figure 3.5). As $f(4)=1$, we see that $c=5+\log_{\alpha}1$ which gives

$$f(z)=-(z-5)-\log_{\alpha}(5-z)+\log_{\alpha}1.$$

For instance if $\alpha=0$, then $\log_0 1=\ln 1+i\mathrm{Arg}\ 1=0$. In this choice, we get

$$f(z)=-(z-5)-\ \mathrm{Log}\,(5-z). \qquad \bullet$$

We conclude this section with a discussion on the generalized 'power functions'. Given $0\neq z\in\mathbb{C}$, the principal value of z^a, i.e. the a-th power of z, is defined by $z^a=\exp(a\,\mathrm{Log}\, z)$ and this is what most mathematicians mean by z^a on most occasions, especially if z is real and positive. The other values of z^a could be obtained from $\exp(a\log z)$ in which case z^a is multi-valued because $\arg z$ (and hence $\log z$) is multiple-valued.

When a is an integer, $\exp(a\log z)=z^a$ is single-valued. For let $a=n$, n an integer. Then, because $\log z=\ln|z|+i(\mathrm{Arg}\, z+2\pi k)$.

$$\begin{aligned} z^a &= e^{n[\ln|z|+i(\theta+2\pi k)]} \quad (\theta=\mathrm{Arg}\, z,\ k\in\mathbb{Z}) \\ &= e^{n\,\mathrm{Log}\, z}e^{i2\pi(kn)}=e^{n\,\mathrm{Log}\, z}, \end{aligned}$$

and hence z^a produces the single point. Note that $\operatorname{Log} z$ is single-valued. Conversely, z^a is single-valued only if a is an integer.

When a is a real rational number with the reduced form m/n, then $\exp(a \log z)$ produces exactly n distinct values and may be represented as $*\sqrt[n]{z^m}$ as in Remark 1.25. For,

$$z^a = e^{(m/n)[\operatorname{Log} z + i2k\pi]} = e^{(m/n)\operatorname{Log} z} e^{i(m/n)2k\pi}, \quad k \in \mathbb{Z}$$

and since $e^{(m/n)\operatorname{Log} z}$ is single-valued and $e^{i(m/n)2k\pi}$ are the n-th roots of unity, we infer that z^a has exactly n values corresponding to $k = 0, 1, 2, \dots, n-1$.

When a is an irrational (real) number or an imaginary number, then z^a is infinite-valued. That is the set of all values of z^a is an infinite set. For example,

$$1^{\sqrt{2}} = \exp\{\sqrt{2}(\operatorname{Log} 1 + i2k\pi)\} = e^{2\sqrt{2}k\pi i}, \quad k \in \mathbb{Z}.$$

The values corresponding to different k's are distinct.

As another example all the values of i^i are obtained from the expression

$$\exp\{i(\operatorname{Log} i + 2k\pi i)\} = \exp\{i((\pi i/2) + 2k\pi i)\} = e^{-(4k+1)\pi/2},$$

where $k \in \mathbb{Z}$, and so $e^{-\pi/2}$ is the principal value of i^i. Similarly, all the values of $(-i)^i$ and $i^{1/\pi}$ are obtained from

$$(-i)^i = e^{-(4k-1)\pi/2} \quad \text{and} \quad i^{1/\pi} = e^{i(4k+1)/2},$$

where $k \in \mathbb{Z}$, respectively.

We emphasize that the principal branch and other branches of z^a are obtained from $\operatorname{Log} z$ and $\log z$, respectively. Different branches of $\log z$ yields different branches of z^a. For instance, $z^a = e^{a \operatorname{Log} z}$ defines a principal branch of z^a which is analytic in $D_\pi = \mathbb{C}\backslash\{z : z = x, x \le 0\}$. More generally, we have

3.110. Theorem. *If we choose f_α defined by (3.99) to be a branch of $\log z$, then, in the cut plane D_α, $(z^a)' = az^{a-1}$ (same branches).*

Proof. By hypothesis, $z^a = \exp(af_\alpha(z))$ is analytic in the cut plane D_α, where $\alpha \in \mathbb{R}$ is fixed. By the Chain rule

$$\frac{d}{dz}(z^a) = e^{af_\alpha(z)}\left(\frac{a}{z}\right) = e^{af_\alpha(z)}\left(\frac{a}{e^{f_\alpha(z)}}\right) = ae^{(a-1)f_\alpha(z)} = az^{a-1}$$

which gives the result. ∎

3.111. Corollary. *For $n \in \mathbb{N}$, $z^{1/n}$ is analytic in the domain of a branch of $\log z$ and $\left(z^{1/n}\right)' = (1/n)z^{1/n-1}$.*

Observe that if $a = 1/n$ $(n \geq 2)$, then

$$\left[\exp\left(\frac{1}{n} f_\alpha(z)\right)\right]^n = \exp(f_\alpha(z)) = z$$

making $\exp((1/n)f_\alpha(z))$ a branch of the n-th root function in D_α, the branch of the n-th root function in D_α associated with $f_\alpha(z)$. A proof similar to Theorem 3.108 gives

3.112. Theorem. **(Binomial Series)** *For* $z \in \Delta = \{z : |z| < 1\}$,

$$(1+z)^a = 1 + \sum_{n \geq 1} \frac{a(a-1)\cdots(a-n+1)}{n!} z^n,$$

where a *is an arbitrary complex number.*

Proof. Since the radius of convergence of the series is 1, the sum converges for all $z \in \Delta$ (and defines an analytic function in Δ). Call it $f(z)$. Now the power series can be differentiated term-by-term. Thus,

$$(1+z)f'(z) = af(z).$$

Further the function $(1+z)^a = \exp\{a \operatorname{Log}(1+z)\}$ is well-defined and analytic in the unit disk Δ. Let

$$g(z) = \frac{f(z)}{(1+z)^a} = f(z)\exp\{-a \operatorname{Log}(1+z)\}.$$

Thus, g is analytic in Δ and

$$\begin{aligned} g'(z) &= f'(z)\exp\{-a \operatorname{Log}(1+z)\} - f(z)\exp\{-a \operatorname{Log}(1+z)\}\frac{a}{1+z} \\ &= \exp\{-a \operatorname{Log}(1+z)\}\left[f'(z) - \frac{a}{1+z} f(z)\right] = 0. \end{aligned}$$

Theorem 3.31(i) gives that $g(z)$ is constant in Δ. So, $f(z) = k(1+z)^a$, where k is some constant. Since $f(0) = 1$ and since $1^a = e^{a \operatorname{Log} 1} = e^0 = 1$, it follows that $k = 1$. This proves the theorem. ■

Note. Unless otherwise stated explicitly, from now onwards, z^a will denote the value of $\exp\{a \operatorname{Log} z\}$. Further, in the particular case $z = e$ the expression $z^a \equiv e^a$ is single-valued for all values of a, since $e^a = e^{\operatorname{Re} a + i \operatorname{Im} a}$ is defined to be

$$e^{a \operatorname{Log} e} = e^{(\operatorname{Re} a + i \operatorname{Im} a) \operatorname{Log} e} = e^{\operatorname{Re} a \operatorname{Log} e}(\cos(\operatorname{Im} a \operatorname{Log} e) + i \sin(\operatorname{Im} a \operatorname{Log} e)),$$

which is clearly a unique complex number. This note is to caution the reader when comparing with Euler's notation (see Equation (3.81)).

3.6 Inverse Functions

The *inverse trigonometric* function $w = \text{trig}^{-1} z$ or $\text{arctrig}\, z$ is defined by the equation

(3.113) $$z = \text{trig}\, w.$$

Similarly we define the *inverse hyperbolic* function $w = \text{hyp}^{-1} z$ or $\text{archyp}\, z$ by the equation

(3.114) $$z = \text{hyp}\, w.$$

Here 'trig w' ("hyp w") denotes any of the trigonometric (hyperbolic) functions $\sin w, \cos w$ etc. ($\sinh w, \cosh w$, etc.). Recall that, as e^z is $2\pi i$-periodic, defining the logarithm function as its inverse led to a multivalued function. As trigonometric and hyperbolic functions are periodic, it is natural to expect a similar situation when we define their inverses. Indeed, we will see that the functions w defined by the equations (3.113) and (3.114) are multiple-valued, since trig and hyp functions are periodic. For instance, the fact that

$$\cos w = \cos(-w)$$

shows that the first and third quadrants, and the second and fourth quadrants map into the same regions by the cosine. First, we wish to show that given a complex number z, there exist infinitely many solutions of $z = \sin w$. To find w, in terms of z, we note that

$$z = \frac{e^{iw} - e^{-iw}}{2i}, \quad \text{i.e.} \quad (e^{iw})^2 - 2ize^{iw} - 1 = 0,$$

which is a quadratic equation in e^{iw}. Note that $e^{iw} \neq 0$. Solving for e^{iw}, we have

$$e^{iw} = iz + Z_j, \quad j = 1, 2,$$

where Z_1 and Z_2 denote the two numbers such that $Z_1^2 = Z_2^2 = 1 - z^2$. Therefore,

$$w = \frac{1}{i} \log(iz + Z_j), \quad j = 1, 2$$

(This also gives that $\sin(\mathbb{C}) = \mathbb{C}$). This function is called the *inverse sine function* and is denoted by $\sin^{-1}(z) := \arcsin z$:

(3.115) $$w = \sin^{-1}(z) = -i \log(iz \pm \sqrt{1 - z^2}).$$

For instance, if $z = 0$, (3.115) becomes $w = -i \log(\pm 1)$. If we select $+1$ and the principal value for $\log 1$, we have $w = 0$ and if we select -1 and principal value for $\log(-1)$, we have $w = -i\pi$. Therefore if Z denotes any of the two values Z_1, Z_2,

$$\begin{aligned}
\sin w &= \sin(-i \log(iz + Z)) \\
&= \frac{\exp(\log(iz + Z)) - \exp(-\log(iz + Z))}{2i} \\
&= \frac{1}{2i} \left[(iz + Z) - \frac{1}{iz + Z} \right] = z.
\end{aligned}$$

Note that $\log 1 = 2k\pi i$ and $\log(-1) = \pi i + 2k\pi i$ ($k \in \mathbb{Z}$) also leads us to $\sin w = z$. This shows that any of the values

$$-i\log(iz + Z_1) + 2k\pi, \quad -i\log(iz + Z_2) + 2k\pi \quad (k \in \mathbb{Z})$$

are the values and the only values of $\sin^{-1} z$. Because of this, for a given z, there are an infinite number of solutions for the equation $z = \sin w$ (and so for (3.113) and (3.114) in general). For instance,

$$\sin^{-1}(i) = -i\log(-1 \pm \sqrt{2}) = -i[\,\mathrm{Log}\,(-1 \pm \sqrt{2}) + 2k\pi i], \quad k \in \mathbb{Z},$$

so that

$$\sin^{-1}(i) = \begin{cases} -i[\ln(1+\sqrt{2}) + (2k+1)\pi i] & \text{corresponding to } - \text{ sign,} \\ -i[\ln(1/(\sqrt{2}+1)) + 2k\pi i] & \text{corresponding to } + \text{ sign.} \end{cases}$$

Therefore (since $\ln(1/|x|) = -\ln|x|$),

$$\sin^{-1}(i) = i(-1)^k \ln(\sqrt{2}+1) + k\pi, \quad k \in \mathbb{Z}.$$

Similarly, all possible values of $w = \sin^{-1}(2)$ are given by

$$w = \frac{1}{i}\left[\ln(2 \pm \sqrt{3}) + i(\pi/2 + 2k\pi)\right] = (2k + 1/2)\pi \pm i\ln(2+\sqrt{3}), \quad k \in \mathbb{Z}$$

(Note that $(2+\sqrt{3})(2-\sqrt{3}) = 1$ and so $\ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$).

The idea used to derive the values of $\sin^{-1} z$ would help us to derive the following:

$$\begin{aligned} \cos^{-1} z &= -i\log(z \pm \sqrt{z^2-1}) = -i\log(z + i\sqrt{1-z^2}) \\ \tan^{-1} z &= \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) \\ \sinh^{-1} z &= \log(z + \sqrt{z^2+1}) \\ \cosh^{-1} z &= \log(z + \sqrt{z^2-1}) \\ \tanh^{-1} z &= \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \end{aligned}$$

where $\sqrt{z^2-1}$, $\sqrt{1-z^2}$ stands for one of the complex numbers Z such that $Z^2 = z^2 - 1$ or $Z^2 = 1 - z^2$ as the case may be. Besides '=' signifies one of the values.

The inverse trig and hyp functions can be made single-valued by choosing a particular branch of the log function (and the particular branch of the square root function if necessary). When the principal branch of log is chosen in the formulas trig $^{-1}$ and hyp $^{-1}$, the resulting single-valued function is called the principal branch of trig $^{-1}$ and hyp $^{-1}$, respectively. For example, with $\sqrt{1-z^2} = \exp((1/2)\,\mathrm{Log}\,(1-z^2))$ the principal square root function, we have

$$\mathrm{Sin}^{-1} z := \mathrm{Arcsin}\, z = -i\,\mathrm{Log}\,(iz + \sqrt{1-z^2}),$$

is called *the principal branch* of $\sin^{-1} z := \arcsin z$. The derivatives of the (single-valued) inverse trig and hyp functions can be obtained from the above formulas or from differentiating $z = \mathrm{trig}\, w$ and $z = \mathrm{hyp}\, w$ implicitly (see Exercise 3.141).

3.7 Exercises

3.116. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The function $f : D \longrightarrow \mathbb{C}$ is analytic at $z_0 \in D$ if and only if there is a $\delta > 0$ such that if $\{z_n\}$ is a sequence in $\Delta(z_0;\delta)\setminus\{z_0\}$ with $z_n \to z_0$, then
$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0).$$

(b) The function $f(x+iy) = x^3 + ax^2y + bxy^2 + cy^3$ is analytic in $\mathbb{C}$ only if $a = 3i$, $b = -3$ and $c = -i$.

(c) If $f = u + iv$ is analytic in a domain D such that $au + bv$ is constant, where a, b are real constants (not both zero), then f is constant.

(d) Let D be a domain in $\mathbb{C}$ and $\overline{D} = \{z \in \mathbb{C} : \overline{z} \in D\}$. Then, $f \in \mathcal{H}(D)$ iff $g \in \mathcal{H}(\overline{D})$, where $g(z) = \overline{f(z)} = \overline{f}(\overline{z})$.

(e) Let f_j $(j = 1, 2, \dots, n)$ be analytic in a domain D such that $\sum_{j=1}^{n} |f_j(z)|^2$ is constant in D. Then each f_j is a constant function.

(f) If u is a real-valued function in a disk Δ_R such that $u^{-1}+iu$ is analytic in Δ_R, then u is constant throughout the disk.

(g) There exists a homeomorphism (bijective and bicontinuous map) between the unit disk Δ and the upper half-plane.

(h) The mean value theorem of a real-valued function of a real variable does not hold in general for complex-valued functions.

(i) A real-valued function u in domain D cannot be analytic in D unless it is a constant function.

(j) There exists a non-constant real-valued function u in an open set D which is analytic in D.

(k) The function $f(x+iy) = x^2 - y^2 + x + i(2xy + y)$ represents an analytic function in $\mathbb{C}$ whereas $g(x+iy) = x^2 + y^2 + x + i(2xy + y)$ is not.

(l) If f and g are functions that satisfy the C-R equations at a point $a \in \mathbb{C}$, then $f+g$ and fg also satisfy the C-R equations at a.

(m) The function f defined by

$$f(z) = u + iv = \begin{cases} \dfrac{\mathrm{Im}\,(z^2)}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

satisfies the C-R equations at the origin, yet it is not differentiable there.

(n) The function f defined by

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \exp(-1/z^4) & \text{if } z \neq 0 \end{cases}$$

is not continuous at the origin but satisfies the C-R equations at the origin.

(o) The function $f(z)$ is analytic in a domain D iff both real and imaginary parts of $f(z)$ and $zf(z)$ are harmonic in D.

(p) If Ω is a domain which is symmetric about the real axis and if f is differentiable at $a \in \Omega$ as well as at $\overline{a} \in \Omega$, then $f(\overline{z})$ is not differentiable at a.

(q) The function $f(x+iy) = x^3 + i(y-1)^3$ is nowhere analytic but is continuous in $\mathbb{C}$ and differentiable on $\{x+iy : x+y=1\} \cup \{x+iy : x-y=-1\}$.

(r) An entire function f such that $f(x+iy) = u(x) + iv(y)$ must be of the form $f(z) = \alpha z + \beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

(s) If $f = u + iv$ and $g = u + iV$ are two analytic functions defined in a domain Ω, then $\mathrm{Im}\,(f-g)$ is necessarily a constant.

(t) If $f(z)$ and $\overline{f(z)}$ are analytic functions in a domain, then f is necessarily a constant.

(u) There exist no analytic functions f such that $\mathrm{Re}\, f(z) = y^2 - 2x$.

(v) There exist no analytic functions f such that $\mathrm{Im}\, f(z) = x^3 - y^3$.

(w) An analytic function $f = u + iv$ in a domain Ω such that $v = u^2$ is necessarily a constant.

(x) A real-valued function $u(x,y)$ is harmonic in D iff $u(x,-y)$ is harmonic in D.

(y) If $f : D \to D'$ is analytic and $u : D' \to \mathbb{R}$ is harmonic, then the composition $u \circ f$ is harmonic in D.
Note: If $u(x,y)$ is harmonic in $\mathbb{R}^2$, then $u(x^2-y^2, 2xy)$ is harmonic in $\mathbb{R}^2$. Similarly, if $f : D \to \mathbb{C}\backslash\{0\}$ is analytic then $\log|f(z)|$ is harmonic in D. In particular, if $f(z) = z$ then $\log|z|$ is harmonic in $\mathbb{C}\backslash\{0\}$.

(z) Let $\{f_n(z)\}$ be a sequence of analytic functions in a domain D such that $f_n \to f$ uniformly in D. Then, $f_n' \to f'$ uniformly in D.

3.117. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) $\lim_{n\to\infty}(n!)^{1/n} = \infty$.

(b) If $a_n \geq 0$, $b_n \geq 0$ and $b_n \to b$, then $\limsup_{n\to\infty} a_n b_n = b \limsup_{n\to\infty} a_n$.

(c) The set of all power series with positive radius of convergence forms a vector space over the field $\mathbb{C}$.

(d) A power series has an anti-derivative everywhere inside its disk of convergence.

(e) If R is the radius of convergence of $\sum_{n\geq 0} a_n z^n$, then the radii of convergence of $\sum_{n\geq 0} a_n z^{2n}$ and $\sum_{n\geq 0} a_n^2 z^n$ are $\sqrt{R}$ and R^2, respectively.

(f) The radius of convergence of $\sum_{n\geq 0}(\operatorname{Re} a_n)z^n$ is always greater than or equal to the radius of convergence of $\sum_{n\geq 0} a_n z^n$.

(g) If $k \geq 1$ is a fixed positive integer, then the series $\sum_{n\geq 1} \frac{z^{kn}}{n}$ converges for $|z| = 1$ except when $z = \omega_j$, $j = 0, 1, \dots, k-1$, the k-th roots of unity.

(h) The radius of convergence of the series $\sum_{n\geq 1} \frac{(-1)^n}{n(n+1)} z^{n(n+1)}$ is 1.

(i) The series $\sum_{n\geq 0}((z+1)/(z-1))^n$ converges for $\operatorname{Re} z \leq 0$ and diverges for $\operatorname{Re} z > 0$.

(j) The series $\sum_{n=0}^{\infty} \left(\frac{z}{1-z}\right)^n$ converges for $\operatorname{Re} z < 1/2$.

(k) The series $\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^4}{z^n}\right)$ converges for $|z| > 1$ and diverges everywhere else.

(l) Each of the series $\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n}$ and $\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$ converges uniformly in the closed disk $|z| \leq r$, $r \in (0,1)$. The two series are actually the same.

(m) If f is analytic in a domain D and if a, b are constants, then $f(z) = ae^{bz}$ and $f'(z) = bf(z)$ are equivalent statements in D.

(n) e^z assumes all values except zero; i.e., the equation $e^z = \omega$ is solvable for any $0 \neq \omega \in \mathbb{C}$.

(o) For $z = x + iy$,

(i) $e^z > 0$ when y is an even multiple of π

(ii) $e^z < 0$ when y is an odd multiple of π

(iii) e^z is purely imaginary when y is an odd multiple of $\pi/2$

(iv) $|e^{iz}| < 1$ when $y > 0$

(v) $|e^{iz}| > 1$ when $y < 0$
(vi) $|e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|}$ for any $z \in \mathbb{C}$
(vii) $|\exp(z^2)| \leq \exp(|z|^2)$
(viii) $|\sin z|^2 + |\cos z|^2 = 1 \Longleftrightarrow z = x$, a real.

(p) For $z = x + iy$,

$$\sinh y \leq \left| \begin{matrix} \cos \\ \sin \end{matrix} (x \pm iy) \right| \leq \cosh y.$$

(q) If f is any one of the six hyperbolic (circular) functions, then, for all $z \in \mathbb{C}$, $\overline{f(z)} = f(\overline{z})$.

(r) Each of the trigonometric, hyperbolic functions $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ takes every value $c \in \mathbb{C}$ countably many times.

(s) For x real, $\cos^{-1}(x) + \sin^{-1}(x) = \pi/2$.

(t) The roots of the equation $\tan z + \cot z = 2$ are at $z = (4k+1)\pi/4$, $k \in \mathbb{Z}$.

(u) If $f(z) = e^{1/z}$, then $|f(z)|$ is constant on $|z - r| = r$ $(r > 0)$.

(v) If $z_1, z_2, \dots, z_n \in \mathbb{C}$ such that $\operatorname{Re}(z_k) > 0$ and $\operatorname{Re}(z_1 \cdots z_k) > 0$ for each $k = 1, 2, \dots, n$, then $\operatorname{Log}\left(\prod_{k=1}^n z_k\right) = \sum_{k=1}^n \operatorname{Log} z_k$.

(w) $\operatorname{Re} z^{1/2} > 0$ for all $z \in \mathbb{C} \setminus \{0\}$.

(x) If $f(z) = z^i = e^{i \operatorname{Log} z}$ (principal branch is chosen), then there exists a constant $C > 0$ such $|f(z)| \leq C$ for all z in the domain of definition of f.
Note: What happens if i in this problem is replaced by ib, where b is finite real number.

(y) There does not exist an analytic function f in a neighborhood of 0 whose square is z.

(z) For $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$, we have $|z^a| \leq |z|^{\operatorname{Re} a} e^{\pi |\operatorname{Im} a|}$.

3.118. Let $f(z) = x^2 + iy^2$. Does it satisfy the C-R equations at the origin? Is f differentiable at the origin? Is f analytic at the origin?

3.119. If $f(z) = |x^2 - y^2| + 2i|xy|$, then show that

$$f(z) = \begin{cases} z^2 & \text{if } 0 < \operatorname{Arg} z < \pi/4 \text{ and } -\pi < \operatorname{Arg} z < -3\pi/4 \\ -z^2 & \text{if } \pi/2 < \operatorname{Arg} z < 3\pi/4 \text{ and } \quad -\pi/2 < \operatorname{Arg} z < -\pi/4. \end{cases}$$

3.120. An entire function $f = u + iv$ has the feature that $u_x v_y - u_y v_x = 1$ in $\mathbb{C}$. Demonstrate that f has the form $f(z) = az + b$, where a and b are some constants with $|a| = 1$.

3.121. Suppose that $f = u + iv$ is entire such that $u_x + v_y = 0$ in $\mathbb{C}$. Demonstrate that f has the form $f(z) = az + b$ where a, b are some constants with $\operatorname{Re} a = 0$.

3.122. Suppose that $f(z)$ is defined in the unit disk Δ such that both $f^2(z)$ and $f^3(z)$ are analytic in Δ. Prove or disprove that $f(z)$ is analytic in Δ.

3.123. If f is analytic in a neighborhood N of the origin such that $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1, z_2 \in N$, then show that $f(z) = az$ for some complex constant a.

3.124. If $a \in \mathbb{R}$, then show that $u(x,y) = e^{-2axy}\cos a(x^2 - y^2)$ is harmonic in $\mathbb{R}^2$. Find all its harmonic conjugates $v(x,y)$ in $\mathbb{R}^2$. Write $f = u + iv$ as a function of z with $f(0) = 1$.

3.125. If $f \in \mathcal{H}(\Omega)$ and $|f|$ is harmonic in Ω, then show that f is a constant function.

3.126. Is it always possible to find a function v which is a harmonic conjugate to u in the same domain where u is harmonic? Justify your answer by giving an example.

3.127. Let R_1 and R_2 be the radius of convergence of the power series $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$, respectively. Find the radius of convergence of $f(z) \pm g(z)$ and $f(z)g(z)$. Also find the radius of convergence of $\sum_{n\geq 0} n^\alpha a_n z^n$, and $\sum_{n\geq 0} \alpha^n a_n z^n$, where $\alpha > 0$.

3.128. Find the region of convergence and uniform convergence of the series:

$$\text{(i)} \quad \sum_{n\geq 0} \left(\frac{z}{1+z}\right)^n \qquad \text{(ii)} \quad \sum_{n\geq 0} \frac{z^n}{1+z^{2n}}.$$

3.129. If $f(z) = z(1-z)^{-2}$, then use the relation $(1-z)^2 f(z) = z$ to compute the Maclaurin series of $f(z)$.

3.130. We know that $f(z) = e^z$ is a solution of the differential equation $f'(z) - f(z) = 0$. Suppose that g is a solution of the first order differential equation $g'(2z) + g(z) = 0$ such that g is analytic at 0. Then show that g is entire.

3.131. Let $a \in \mathbb{R}$ be fixed. Find the set of $z \in \mathbb{C}$ for which $\sum_{n=1}^{\infty} n^{i(z^2+a)}$ represents an analytic function.

3.132. Using the 'ϵ-δ' definition verify the continuity of $f(z) = z^2$ at $z_0 \in \mathbb{C}$. For $z \neq 0$, let $z = re^{i\theta}$ ($r = |z|$, $\theta = \operatorname{Arg} z$). Find the region of continuity for $g_1(z) = \sqrt{r}e^{i\theta/2}$ and $g_2(z) = -i\sqrt{r}e^{i\theta/2}$.

3.133. If z_1 and z_2 satisfy one of the following conditions

(a) $\mathrm{Re}\, z_1 > 0$, $\mathrm{Re}\, z_2 > 0$ (c) $\mathrm{Im}\,(z_1z_2) > 0$, $\mathrm{Im}\,(z_1z_2) \geq 0$

(b) $\mathrm{Im}\, z_1 < 0 < \mathrm{Im}\, z_2$ (d) $\mathrm{Im}\,(z_1z_2) < 0$, $\mathrm{Im}\, z_2 < 0$,

then show that $z_1^a z_2^a = (z_1z_2)^a$ (Assume that all the powers involved have their principal values).

Note: If $z_1 = 1 = z_2 = -1$, then $\sqrt{z_1z_2} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$ whereas $\sqrt{z_1}\sqrt{z_2} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$. Thus, one needs to be careful when performing algebraic manipulations with real/complex powers.

3.134. Let $z \notin (-\infty, 0]$. Using the principal logarithms on each side, show that $\overline{z^a} = (\overline{z})^{\overline{a}}$.

3.135. Suppose $f(z) = z^z$. Using the principal logarithm find $f'(z)$, $f'(i)$, $\mathrm{Re}\, f(z)$ and $\mathrm{Im}\, f(z)$.

3.136. Let a be a constant. Consider the principal value of z^a and a non-principal value of z^{-a}.

(a) In general, is $z^a z^{-a} = 1$ always true?

(b) Suppose the principal values of both z^a and z^{-a} are used. Is $z^a z^{-a} = 1$ true?

3.137. Let $F(z)$ be the branch of the function $(z+i)^{1/4}$ such that $F(1)$ equals the value $-2^{1/8}e^{i\pi/16}$. Find $F(0)$, $F(i)$ and $F(-1)$.

3.138. Find all the values of $\log(z^{1/2})$ and $\log(z^{1/3})$ in Cartesian form.

3.139. At what points is each of the functions $\mathrm{Log}\,(1-z^2)$, $\mathrm{Log}\,(1+z^2)$, $\mathrm{Log}\,(1-iz^2)$, $\mathrm{Log}\,(1+iz^2)$, analytic? Explain.

3.140. If $f(z)$ is the principal branch of $z^{1/3}$, find $f'(-1-i)$.

3.141. Prove the following formulas for the inverse trigonometric functions:

$$\frac{d}{dz}\left(\mathrm{Arcsin}^{-1} z\right) = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\left(\mathrm{Arcsinh}^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}$$
$$\frac{d}{dz}\left(\mathrm{Arccos}^{-1} z\right) = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\left(\mathrm{Arccosh}^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}$$
$$\frac{d}{dz}\left(\mathrm{Arctan}^{-1} z\right) = \frac{1}{1+z^2}, \qquad \frac{d}{dz}\left(\mathrm{Arctanh}^{-1} z\right) = \frac{1}{1-z^2}.$$

3.142. Find all solutions of

(a) $e^z = -2i$ (b) $e^z = 1+i$ (c) $\sin z = 2$.

Chapter 4

Complex Integration

The purpose of Section 4.1 is to develop the technical machinery that is required to introduce the complex line integral, also called the contour integral. In Section 4.2, we define the complex line integral and discuss some basic properties and examples of complex line integrals. In Section 4.5, we present a powerful concept of "the number of times" a closed contour winds around a point. In Section 4.7, we show that every analytic function in a simply connected domain possesses derivatives of all orders. As a consequence, in Section 4.10, we discuss certain important properties of analytic functions. For example, every analytic function which is analytic at a point a can be expressed as a Taylor series in the 'vicinity of a' and as an application of the integral representation for the Taylor coefficient a_n of f, we obtain an estimate for a_n. We present the Uniqueness/Identity theorem in Section 4.11. Section 4.12 focuses on Laurent series which is actually a natural generalization of the Taylor's series where the center of expansion is not a point of analyticity. Essentially, our effort is to find a Laurent series expansion for a function that is analytic in an annulus region $\{z : r < |z| < R\}$.

4.1 Curves in the Complex Plane

A *continuous curve* (or simply a *curve* or a *path*) in $\mathbb{C}$ is a continuous mapping γ from a closed interval $[a, b]$, $a < b$, $a, b \in \mathbb{R}$, into $\mathbb{C}$. Here the points $\gamma(a)$ and $\gamma(b)$ are called initial and terminal points of the curve, respectively. A parametric representation of a continuous curve γ is given by

$$\gamma(t) = x(t) + iy(t), \quad t \in [a, b],$$

where $x(t)$ and $y(t)$ are continuous real-valued functions on $[a, b]$. Here the interval $[a, b]$ is called a *parametric interval* of γ. A curve γ with parametric interval $[a, b]$ such that $\gamma(a) = \gamma(b)$ is called a *closed curve.* If $\gamma(t)$ is one-to-one, that is, for t_1, t_2 in $[a,b]$ with $t_1 \neq t_2$ we have $\gamma(t_1) \neq \gamma(t_2)$, we call the curve a *Jordan arc*. For example, $\gamma(t) = e^{it}$, $t \in [0,\pi]$, is a Jordan arc. A Jordan arc γ such that $\gamma(a) = \gamma(b)$, is called a *Jordan curve*.[8] A domain D bounded by a Jordan curve is called a *Jordan domain*. With this notation, we refer to the point $\gamma(t)$ as "a point on the curve γ", although strictly speaking, $\gamma(t)$ is on the image of the mapping γ in $\mathbb{C}$:

$$\gamma([a,b]) = \{\gamma(t) : t \in [a,b]\}.$$

So, we regard a curve as the range of a continuous complex function defined on the interval $[a,b]$. If $[a,b]$ and $[c,d]$ are two closed bounded intervals in $\mathbb{R}$, then (by letting $\Gamma(t) = At + B$ with $\Gamma(a) = c$ and $\Gamma(b) = d$ and then solving for A and B) we observe that $\Gamma : [a,b] \to [c,d]$ defined by

$$\Gamma(t) = \frac{c-d}{a-b}t + \frac{ad-bc}{a-b}, \quad t \in [a,b],$$

is a *bijective* and *bicontinuous* map of $[a,b]$ onto $[c,d]$. If $\gamma : [c,d] \to \mathbb{C}$ is a curve with parametric interval $[c,d]$, then $\gamma \circ \Gamma$ is a path with parametric interval $[a,b]$, whose image is the same as that of γ. Thus, we can make a choice of the parametric interval at our convenience.

Note that it is possible for two different curves to have the same image. For instance, γ_1 and γ_2 defined by

$$\gamma_1(t) = 2t(1+i),\ 0 \le t \le 1/2, \quad \text{and} \quad \gamma_2(t) = t^2(1+i),\ 0 \le t \le 1,$$

both represent the same straight line segment $[0, 1+i]$ in $\mathbb{C}$. If we let $\gamma_3(t) = e^{it}$ and $\gamma_4(t) = e^{2it}$, then we see that the images $\gamma_3([0,2\pi])$ and $\gamma_4([0,2\pi])$ are the same in each case. But they represent different curves.

If a curve $\gamma : [a,b] \to \mathbb{C}$ is given by $\gamma(t) = x(t) + iy(t)$, $t \in [a,b]$, then the opposite/reverse curve $-\gamma$ of γ is defined by

$$(-\gamma)(t) = \gamma(a+b-t), \quad a \le t \le b.$$

Thus $-\gamma$ describes the same curve as γ, but in the "reverse direction" with the initial and terminal points interchanged. For example, let $\gamma(t) = e^{it}$, $0 \le t \le \pi$. If t varies from 0 to π, then $\gamma(t)$ describes the upper half of the unit circle $|z| = 1$ with initial point $\gamma(0) = 1$ and terminal point $\gamma(\pi) = -1$. Therefore, the reverse curve $-\gamma$ of γ is defined by

$$(-\gamma)(t) = \gamma(\pi - t), \quad 0 \le t \le \pi,$$

with starting point $(-\gamma)(0) = -1$ and terminal point $(-\gamma)(-\pi) = 1$.

[8] A prominent result used in the theory of complex variables is the celebrated Jordan Curve Theorem which says that every Jordan curve γ divides the complex plane into two parts, the interior and the exterior of γ. For this deep result, for instance we refer to M.H.A. Newman, Elements of topology of the plane sets of points, Cambridge University Press, London, 1964.

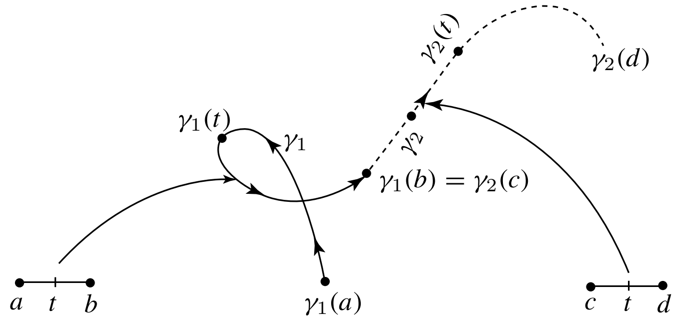


Figure 4.1: Sum $\gamma_1 + \gamma_2$.

If γ_1, γ_2 are two curves with $[a, b]$ as their parametric interval, then γ_1^* and γ_2^* defined by

$$\gamma_1^*(t) = \gamma_1(2t - a),\ t \in \left[a, \frac{a+b}{2}\right], \quad \gamma_2^*(t) = \gamma_2(2t - b),\ t \in \left[\frac{a+b}{2}, b\right]$$

also define same curves, and

$$\gamma_1^*\left(\frac{a+b}{2}\right) = \gamma_1(b);\ \gamma_2^*\left(\frac{a+b}{2}\right) = \gamma_2(a).$$

If γ_1, γ_2 are such that $\gamma_1(b) = \gamma_2(a)$, then we can define a continuous function $\gamma_1 + \gamma_2$ ($\equiv \gamma_1 \cup \gamma_2$) on $[a, b]$ by

$$\gamma_1 + \gamma_2 = \begin{cases} \gamma_1^*(t) & \text{if } t \in [a, (a+b)/2] \\ \gamma_2^*(t) & \text{if } t \in [(a+b)/2, b]. \end{cases}$$

Thus, the formal sum $\gamma_1 + \gamma_2$ is said to represent the *sum* or *union* or *join* of γ_1 and γ_2. Roughly speaking, $\gamma_1 + \gamma_2$ consists of the points of γ_1 followed by those of γ_2. Similarly, $\gamma_1 - \gamma_2$ is defined to be $\gamma_1 + (-\gamma_2)$. For instance, if, for a fixed $r > 0$,

$$\gamma_1 : \gamma_1(t) = rt\ (0 \le t \le 1) \ \text{ and } \ \gamma_2 : \gamma_2(t) = re^{it}\ (0 \le t \le 3\pi/2),$$

then, with the notation $\gamma = \gamma_1 + \gamma_2$ (see Figures 4.2 and 4.1), we write

$$\gamma_1 + \gamma_2 : \gamma(t) = \begin{cases} rt & \text{for } 0 \le t \le 1 \\ re^{i(t-1)} & \text{for } 1 \le t \le 3(\pi/2) + 1. \end{cases}$$

A curve $\gamma : [a, b] \to \mathbb{C}$ can be written, since we can choose the parametric interval conveniently, equivalently in the form

$$\widetilde{\gamma} : [0, 1] \to \mathbb{C}; \quad \text{i.e. } \ \widetilde{\gamma}(t) = \gamma(a + (b - a)t),\ t \in [0, 1].$$

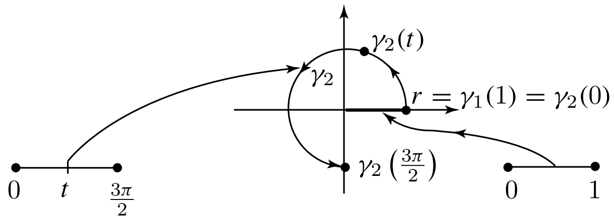


Figure 4.2: Sum of two curves γ_1, γ_2.

A polygon γ with vertices $z_0, z_1, z_3, \ldots, z_n$ is parameterized by

$$\gamma_k(t) = (k+1-t)z_k + (t-k)z_{k+1},\ t \in [k, k+1],$$

where $0 \le k \le n-1$ and $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_{n-1}$.

A curve γ defined on $[a, b]$ is called *simple* if it does not intersect itself, that is, if $\gamma(t_1) \neq \gamma(t_2)$ for $t_1 \neq t_2$, where the possible exception $\gamma(a) = \gamma(b)$ is allowed. In the latter case the curve is said to be a *simple closed curve.* For example, circles, ellipses, rectangles, and triangles are simple closed curves. For instance, curve γ defined by $\gamma(t) = \cos t$ $(-\pi \le t \le \pi)$ is the segment $[-1, 1]$ traversed twice from -1 to 1 and then from 1 to -1, and therefore considered as a closed curve. But each point of $[-1, 1]$ is a self intersection point for the curve so that it is not simple. Note that a simple closed curve is a Jordan curve.

Now we shall deal with continuously differentiable curves.

4.1. Definition. A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *continuously differentiable* on $[a, b]$ or a *curve of class* C^1 on $[a, b]$ or simply a C^1-curve on $[a, b]$ if the function $\gamma(t) = x(t) + iy(t)$ is continuously differentiable on $[a, b]$, i.e. $x'(t)$ and $y'(t)$ exist on $[a, b]$ and are continuous functions on $[a, b]$ (Note that $\gamma(t)$ is differentiable on $[a, b]$ means that $\gamma'(t)$ exists on (a, b), and

$$\lim_{h \to 0^+} \frac{\gamma(a+h) - \gamma(a)}{h}, \quad \lim_{h \to 0^-} \frac{\gamma(b+h) - \gamma(b)}{h}$$

both exist. We denote these limits by $\gamma'(a+)$ (or $\gamma'_+(a)$) and $\gamma'(b-)$ (or $\gamma'_-(b)$), respectively. We call $\gamma'(a+)$ and $\gamma'(b-)$ as the right-hand derivative at a and the left-hand derivative at b, respectively). A continuously differentiable curve is referred to as a *smooth curve.*[9]

For example, $\gamma(t) = t^3$, $t \in [-1, 1]$, is a Jordan arc of class C^1, since

$$-1 \le s < t \le 1 \text{ implies } \gamma(s) < \gamma(t)$$

[9]While defining "smooth curve" some authors insist an additional condition that $\gamma'(t) \neq 0$ on $[a, b]$. So the reader is advised to be aware of this inconsistency while referring other texts in complex variables.

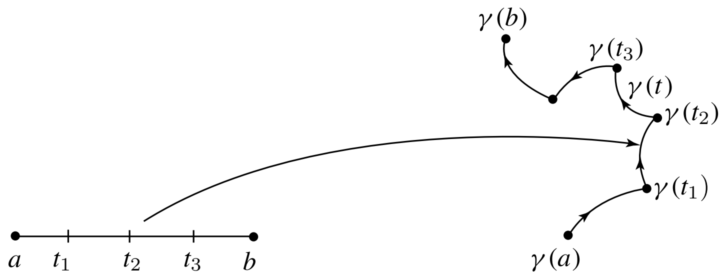


Figure 4.3: Description for piecewise C^1 curve.

and $\gamma(t)$ is a differentiable function of t on $[-1,1]$. Similarly, $\gamma(t) = t^3 + it^2$, $t \in [-1,1]$, is also a Jordan arc of class C^1, since $\gamma(t)$ is a continuously differentiable function of t on $[-1,1]$ and

$$-1 \leq s < t \leq 1 \text{ implies } |\gamma(s)| = \sqrt{s^6 + s^4} < \sqrt{t^6 + t^4} = |\gamma(t)|.$$

Further, the line segment $[z_1, z_2]$ from z_1 to z_2 parameterized by

$$\gamma(t) = (1-t)z_1 + tz_2, \quad t \in [0,1],$$

is continuously differentiable. Similarly, the circular arc parameterized by

$$\gamma(t) = z_0 + re^{it}, \quad t \in [a,b] \subseteq [0, 2\pi],$$

is continuously differentiable. The case $a = 0$ and $b = 2\pi$ yields a circle with center z_0 and radius r.

A curve $\gamma(t)$, $a \leq t \leq b$, is called *piecewise* C^1 (or *piecewise smooth curve*) if there is a subdivision $a = t_0 < t_1 < \cdots < t_j \cdots < t_n = b$ of the interval $[a,b]$ such that the restriction of γ to each subinterval $[t_j, t_{j+1}]$, $0 \leq j \leq n-1$ is a smooth curve (see Figures 4.3). A *contour* is just a continuous curve that is piecewise smooth. Given a domain D in $\mathbb{C}$ and two points z_1 and z_2 in D (need not be distinct), there exists a contour in D with initial point z_1 and terminal point z_2. This fact is clear because any two points in D can be connected by a polygonal path in D.

Consider $\gamma(t) = t + i|t|$, $t \in [-1,1]$. Then (see Figure 4.4)

$$\gamma(t) = \begin{cases} t - it & \text{if } t \in [-1,0] \\ t + it & \text{if } t \in [0,1]. \end{cases}$$

It is easy to see that the restrictions of γ to $[-1,0]$ and to $[0,1]$ are smooth, even though γ is not smooth because $\gamma'(t)$ fails to exist at $t = 0$. Note that $\gamma'(t)$ is discontinuous at 0 but γ is piecewise continuously differentiable, since $\gamma'(t) = 1 - i$ on $[-1,0)$ and $\gamma'(t) = 1 + i$ on $(0,1]$. Accordingly, γ is a piecewise smooth curve.

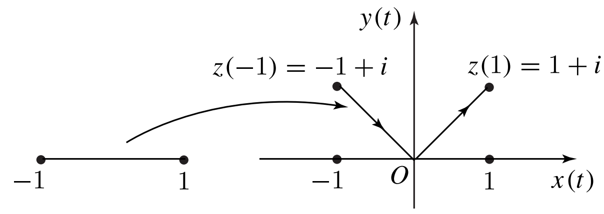


Figure 4.4: The curve $z(t) = t + i|t|$, $t \in [-1, 1]$.

4.2 Properties of Complex Line Integrals

We assume the following facts from real analysis: if a real-valued function F is continuous on $[a, b]$, then the Riemann integral $\int_a^b F(t)\, dt$ exists. It is a trivial exercise to extend this definition for a continuous function $F : [a, b] \to \mathbb{C}$, where $F = U + iV$. Indeed,

$$\int_a^b F(t)\, dt = \int_a^b U(t)\, dt + i \int_a^b V(t)\, dt.$$

A complex-valued function f is said to be *continuous* on a continuously differentiable curve $\gamma : [a, b] \to \mathbb{C}$ (or more generally on a contour) if $\phi(t) = f(z) = f(\gamma(t)) = u(t) + iv(t)$ is continuous for $a \le t \le b$.

Suppose f is a complex-valued function that is continuous on an open set $D \subseteq \mathbb{C}$ and that $\gamma : [a, b] \to \mathbb{C}$ is a contour with $\gamma([a, b]) \subset D$. We define the *complex line integral* or *contour integral* of f along the contour γ, denoted by $\int_\gamma f(z)\, dz$, as follows:

$$(4.2) \qquad \int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\gamma'(t)\, dt,$$

where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, and $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$, being the intervals in which γ is differentiable, and the integrals in the sum are Riemann integrals. The contour γ is called the *path of integration* of the contour integral. Note that the product given by $F(t) = f(\gamma(t))\gamma'(t)$ is piecewise continuous on $[a, b]$. So the second integral in (4.2) is well-defined.

For example, if $\gamma(t) = a + re^{it}$ is a circle, then for an arbitrary continuous function f defined on $\gamma : |z - a| = r$,

$$\int_{|z-a|=r} f(z)\, dz = \int_0^{2\pi} f(a + re^{it}) ire^{it}\, dt.$$

In the most important special case, namely, $f(z) = (z - a)^{-1}$, we easily have

$$\int_{|z-a|=r} f(z)\, dz = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\, dt = 2\pi i.$$

Note that the expression under the integral sign on the right of (4.2) can be obtained by the formal substitution

$$z = \gamma(t), \quad dz = \gamma'(t)\,dt.$$

If $f = u + iv$ and $z = x + iy$, x, y, u, v being real-valued, i.e.

$$f(\gamma(t)) = f(x(t) + iy(t)) = u(x(t), y(t)) + iv(x(t), y(t)),$$

then (4.2) is really

$$\begin{aligned} \int_\gamma f(z)\,dz &= \sum_{j=0}^{n-1} \left\{ \int_{t_j}^{t_{j+1}} [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]\,dt \right. \\ &\qquad \left. + i \int_{t_j}^{t_{j+1}} [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]\,dt \right\}. \end{aligned}$$

Using a change of variable in the definition of the Riemann integral, one has

$$\int_\gamma f(z)\,dz = \int_\gamma [u(x, y)\,dx - v(x, y)\,dy] + i \int_\gamma [u(x, y)\,dy + v(x, y)\,dx],$$

since $\gamma'(t) = x'(t) + iy'(t)$. In the above equality it is tacitly assumed that in the integrands $(x(t), y(t))$ being on the curve γ, that either y is a function of x or x is a function of y. Also note that the limits of integration will depend on the values of $x(a), x(b)$ or $y(a), y(b)$ as the case may be. It should be observed that the expressions under the integral signs can be formally equated as

$$f(z)\,dz = (u + iv)(dx + i\,dy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy).$$

Note that if γ in (4.2) is real-valued, then the path of integration is part of $\mathbb{R}$. On the other hand the path of integration is in the z-plane.

4.3. Definition. A piecewise smooth curve γ with parametric interval $[a, b]$ is said to be a *reparameterization* of $\Gamma(t)$ $(A \leq t \leq B)$ iff there is a C^1-map $\alpha : [A, B] \to [a, b]$ such that $\alpha'(t) > 0$, $\alpha(A) = a$, $\alpha(B) = b$ and $\Gamma(t) = \gamma(\alpha(t))$. Sometimes γ and Γ are said to be equivalent.

The conditions, $\alpha'(t) > 0$, $\alpha(A) = a$ and $\alpha(B) = b$ are to ensure the direction of tracing γ as Γ does. Suppose that f is continuous in an open set D containing all the points of $\gamma(t)$. Then, we have

$$\int_\Gamma f(z)\,dz \ = \ \int_\Gamma f(\Gamma(t))\Gamma'(t)\,dt$$

$$
\begin{aligned}
&= \int_A^B f\{\gamma(\alpha(t)\}\gamma'(\alpha(t))\alpha'(t)\,dt \\
&= \int_{\alpha(A)}^{\alpha(B)} f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_\gamma f(z)\,dz.
\end{aligned}
$$

Therefore, it is immaterial which parameterization is used. Often the computation is simplified if we use some particular equivalent path in evaluating an integral. For certain situations, the choice is by an important property which will be stated in later theorems (see Theorem 4.16). Suppose we want to evaluate

$$
I = \int_\gamma f(z)\,dz, \quad f(z) = z^n, \tag{4.4}
$$

where $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, and n is any integer. It follows that

$$
I = r^{n+1} \int_0^{2\pi} ie^{(n+1)it}\,dt = \begin{cases} r^{n+1} \left[\dfrac{e^{it(n+1)}}{n+1}\right]\Bigg|_0^{2\pi} & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1, \end{cases}
$$

that is,

$$
\int_\gamma f(z)\,dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1. \end{cases} \tag{4.5}
$$

We also note that (4.5) continues to hold if γ and f in (4.4) are replaced by any circle centered at z_0 and $(z - z_0)^n$, respectively. This means that

$$
\int_{|z-z_0|=r} (z - z_0)^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \text{ and integer} \\ 2\pi i & \text{if } n = -1. \end{cases}
$$

We now have the following useful application. For $n \in \mathbb{N}$,

$$
\begin{aligned}
\int_0^{2\pi} \cos^{2n}(t)\,dt &= \int_0^{2\pi} \left[\frac{e^{it} + e^{-it}}{2}\right]^{2n} dt \\
&= \frac{1}{2^{2n}} \int_0^{2\pi} \sum_{k=0}^{2n} \binom{2n}{k} e^{i(2n-2k)t}\,dt \\
&= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \int_0^{2\pi} e^{2i(n-k)t}\,dt \\
&= \frac{\pi}{2^{2n-1}} \binom{2n}{n}.
\end{aligned}
$$

4.6. Example. We evaluate $I_j = \int_{\gamma_j} x\,dz$, $j = 1$ to 7, where

(i) γ_1 is the straight line segment from 0 to $a+ib$ $(a, b \in \mathbb{R})$
(ii) γ_2 is the circle $|z| = R$
(iii) γ_3 is the boundary of the square $[0,1] \times [0,1]$ with $\mathbb{C}$ considered as $\mathbb{R}^2$
(iv) γ_4 is the ellipse $x^2/a^2 + y^2/b^2 = 1$
(v) γ_5 is the line from 0 to $2i$ and then from $2i$ to $4+2i$
(vi) γ_6 is the line segment from 0 to 1, 1 to $1+i$ and then from $1+i$ to 0
(vii) γ_7 is given by $\gamma_7(t) = t + it^2$ on $[0,1]$.

(i) γ_1 may be parameterized by $\gamma_1(t) = (a+ib)t$, $0 \le t \le 1$. Clearly,

$$I_1 = \int_0^1 [\operatorname{Re}\gamma_1(t)]\ \gamma_1'(t)\,dt = \int_0^1 ta(a+ib)\,dt = \frac{a(a+ib)}{2}.$$

(ii) Parameterizing γ_2 by $\gamma_2(t) = Re^{it}$, $0 \le t \le 2\pi$, we have

$$\begin{aligned} I_2 &= \int_0^{2\pi} (R\cos t)(iRe^{it})\,dt \\ &= iR^2 \int_0^{2\pi} [\cos^2 t + i \sin t \cos t]\,dt \\ &= iR^2 \left[\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt + \frac{i}{2}\int_0^{2\pi} \sin 2t\,dt \right] \\ &= iR^2 \left[\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi} + \frac{i}{2}\left(-\frac{\cos 2t}{2}\right)\Big|_0^{2\pi} \right] \\ &= iR^2\pi. \end{aligned}$$

(iii) In this case, we have (see Figure 4.5)

$$I_3 = \left(\int_O^A + \int_A^B + \int_B^C + \int_C^O \right) x\,dz.$$

For the sake of convenience, we parameterize OA, AB, BC and OC, as follows:

$$\begin{aligned} C_1(t) &= (1-t)\cdot 0 + t \cdot 1 = t \\ C_2(t) &= (1-t)\cdot 1 + t\cdot(1+i) = 1 + ti \\ C_3(t) &= (1-t)\cdot(1+i) + t\cdot i = 1 - t + i \\ C_4(t) &= (1-t)\cdot i + t \cdot 0 = (1-t)i, \end{aligned}$$

where $t \in [0,1]$ and $\gamma_3 = C_1 + C_2 + C_3 + C_4$ (see Figure 4.5). Utilizing these, i.e. without reparameterizing, we have

$$I_3 = \sum_{k=1}^{4} \int_0^1 (\operatorname{Re} C_k(t)) C_k'(t)\,dt$$

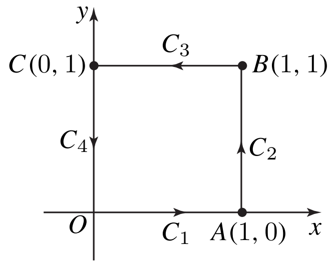


Figure 4.5: Curve $\gamma_3 = C_1 + C_2 + C_3 + C_4$.

$$\begin{aligned} &= \int_0^1 t\,dt + \int_0^1 1 \cdot i\,dt + \int_0^1 (1-t)(-1)\,dt + \int_0^1 0 \cdot (-i)\,dt \\ &= \frac{1}{2} + i - \frac{1}{2} + 0 = i. \end{aligned}$$

(iv) Write γ_4 as $\gamma_4(t) = a\cos t + ib\sin t,\ 0 \leq t \leq 2\pi$. Then,

$$\begin{aligned} I_4 &= \int_0^{2\pi} (a\cos t)(-a\sin t + ib\cos t)\,dt \\ &= -\frac{a^2}{2}\int_0^{2\pi} \sin 2t\,dt + iab\int_0^{2\pi} \cos^2 t\,dt = iab\pi. \end{aligned}$$

The cases (v), (vi) and (vii) may be evaluated similarly and are left as a simple exercise. ●

4.7. Definition. If $\gamma : [a, b] \to \mathbb{C}$ is a smooth and rectifiable curve such that $\gamma(t) = x(t) + iy(t)$, then its (Euclidean) length $L(\gamma)$ is defined by

$$L(\gamma) = \int_\gamma |d\gamma(t)| = \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt. \tag{4.8}$$

If γ is merely piecewise smooth and rectifiable, then its length is the sum of the integrals (4.8) over all the smooth paths of γ.

We consider a few simple examples to demonstrate the use of the formula (4.8). The length of a circle of radius r (use the parametric equation $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, so that $\gamma'(t) = ire^{it}$) is found to be

$$\int_0^{2\pi} |ire^{it}|\,dt = 2\pi r$$

as expected. The line segment $[z_1, z_2]$ parameterized by $\gamma(t) = (1-t)z_1 + tz_2$, $0 \leq t \leq 1$, has its length

$$L(\gamma) = \int_0^1 |\gamma'(t)|\,dt = \int_0^1 |z_2 - z_1|\,dt = |z_1 - z_2|.$$

Similarly, the perimeter of the rectangle may be obtained using this formula with a convenient parameterization. Finally, if γ is an ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then its parametric form is $\gamma(t) = a\cos t + ib\sin t$, $t \in [0, 2\pi]$, so that

$$L(\gamma) = \int_0^{2\pi} \sqrt{a^2\cos^2 t + b^2 \sin^2 t}\, dt = a\int_0^{2\pi} \sqrt{1 + [(b^2/a^2) - 1]\sin^2 t}\, dt.$$

What is the value of this integral? Does there exist a simple formula to compute the arc length of an ellipse with semi-axes of length a and b?

There is a second type of line integral that may be introduced. Let $\gamma : \gamma(t)$, $a \leq t \leq b$, be a smooth curve and $s(t)$ denote the arc-length function for γ. Let f be a continuous function on D with $\gamma([a,b]) \subset D$. Then, $\int_\gamma f(z)\,|dz|$ is defined to be the approximating sums of the form

$$S(\gamma, P) = \sum_{j=1}^{n} f(z_j^*)|s(t_j) - s(t_{j-1})|,$$

where $P : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ ranges over all possible partitions of the interval $[a,b]$. Here z_j^* lies on γ between $\gamma(t_{j-1}) = z_{j-1}$ and $\gamma(t_j) = z_j$. The standard procedure then shows that (with $s'(t) = |\gamma'(t)|$)

$$\int_\gamma f(z)\,|dz| = \int_\gamma f\,ds = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

If f is real-valued, then $\int_\gamma f\,ds$ is a real number. Clearly, $f(z) = 1$ gives that $\int_\gamma f\,ds = L(\gamma)$.

As we have seen earlier, the complex integral may be put in the form

$$\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy).$$

Therefore, the usual rules of integration for real integrals must also apply to contour integrals. The following theorem summarizes some useful properties of complex line integrals. The conclusion of Theorem 4.9 continues to hold if $\gamma, \gamma_1, \gamma_2$ are piecewise smooth (= contour) although we state and prove the theorem for smooth curves for the sake of simplicity .

4.9. Theorem. *Let γ be a smooth curve defined on $[a,b]$ and let f and g be continuous functions on an open set D containing $\gamma([a,b])$ and let α be a complex constant. Then,*

(i) $\displaystyle\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz.$

(ii) $\int_\gamma [\alpha f(z) + g(z)]\, dz = \alpha \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz.$

(iii) *If* $L = L(\gamma)$ *is the length of the curve and* $M = \max_{t\in[a,b]} |f(\gamma(t))|$, *then* $\left|\int_\gamma f(z)\, dz\right| \le ML$. *This property is called the standard estimate for integrals, or M-L inequality.*

(iv) $\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$ *whenever* γ_1 *and* γ_2 *are two smooth curves such that* $\gamma_1(b) = \gamma_2(a)$ *and* $\gamma_j([a,b]) \subset D$ *for* $j = 1, 2$.

Proof. Note that

$$\begin{aligned}
\int_{-\gamma} f(z)\, dz &= \int_a^b f(\gamma(b+a-t))\, d(\gamma(b+a-t)) \\
&= \int_b^a f(\gamma(s))\, d(\gamma(s)), \quad \text{by a change of variable } s = b+a-t, \\
&= -\int_a^b f(\gamma(s))\, d(\gamma(s)) \\
&= -\int_\gamma f(z)\, dz.
\end{aligned}$$

The case (i) now follows. The case (ii) follows from the definition and the linearity property of the Riemann integral.

To prove (iii), we notice that $L = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\, dt$ and for a real-valued Riemann integrable function ϕ on $[a,b]$, we know that

$$\left|\int_a^b \phi(t)\, dt\right| \le \int_a^b |\phi(t)|\, dt. \tag{4.10}$$

If $\int_\gamma f(z)\, dz = 0$, there is nothing to prove. Therefore, we let $\int_\gamma f(z)\, dz \neq 0$ and write

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = Re^{i\theta}, \quad \text{say,}$$

where $R > 0$ and $\theta = \operatorname{Arg}\left(\int_\gamma f(z)\, dz\right)$. We have

$$R = \int_a^b e^{-i\theta} f(\gamma(t))\gamma'(t)\, dt = \int_a^b \operatorname{Re}[e^{-i\theta} f(\gamma t))\gamma'(t)]\, dt.$$

We apply (4.10), with $\phi(t) = \operatorname{Re}[e^{-i\theta} f(\gamma(t))\gamma'(t)]$, to get

$$\phi(t) \le \left|e^{-i\theta} f(\gamma(t))\gamma'(t)]\right|$$

so that $R \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\, dt$. Since $|f(\gamma(t))| \le M$ for all $t \in [a,b]$ and since for positive integrands the Riemann integral is larger for a larger integrand, we have the assertion in (iii).

(iv) As γ_1 and γ_2 are smooth curves with $\gamma_1(b) = \gamma_2(a)$, $\gamma = \gamma_1 + \gamma_2$ is then defined by

$$\gamma(t) = \begin{cases} \gamma_1(2t-a) & \text{if } a \le t \le (b+a)/2 \\ \gamma_2(2t-b) & \text{if } (b+a)/2 \le t \le b. \end{cases}$$

The assertion now follows from the definition after noting that for γ_1 and γ_2 a reparameterization has been made. ■

4.11. Remark. In general, $\mathrm{Re}\left[\int_\gamma f(z)\,dz\right] \neq \int_\gamma \mathrm{Re}\,[f(z)]\,dz$ as can be seen by choosing $\gamma(t) = it$ and $f(z) = 1$. This is to caution the reader to be careful while taking real and imaginary parts of an integral. ●

4.12. Example. From Theorem 4.9(iv) we obtain the following:

(i) If $\gamma(t) = (1+i)t$, $t \in [0,1]$, the line segment from 0 to $1+i$, then for any point ζ on γ we have $|\zeta| \le \sqrt{2}$, $|\zeta^3 + 2| \le 2^{3/2} + 2 = M$, say, and $|\zeta^3 + 2| \ge 2^{3/2} - 2$. As a result of this, we have the estimates

$$\left|\int_\gamma (z^3+2)\,dz\right| \le M \int_\gamma |dz| = (2^{3/2}+2)\sqrt{2} = 4 + 2\sqrt{2}$$

and

$$\left|\int_\gamma (z^3+2)^{-1}\,dz\right| \le \frac{\sqrt{2}}{2^{3/2}-2} = \frac{1}{2-\sqrt{2}},$$

since the line segment $[0, 1+i]$ has length $\sqrt{2}$.

(ii) Consider $\gamma(t) = e^{it}$, $0 \le t \le \pi$. Then, $\left|\int_\gamma z^{-1}e^z\,dz\right| \le ML(\gamma) = e\pi$ because the length $L(\gamma)$ is π and

$$M = \max_{t\in[0,\pi]} \left|e^{\gamma(t)}/\gamma(t)\right| = \max_{t\in[0,\pi]} e^{\cos t} = e.$$

(iii) If $\gamma(t) = (1-t)(1+i) + t(1+3i)$, $0 \le t \le 1$, the directed line segment from $1+i$ to $1+3i$, then $\left|\int_\gamma z^{-2}\,dz\right| \le ML(\gamma) = 1$ because the length $L(\gamma)$ is $|1+3i-(1+i)| = 2$ and

$$M = \max_{t\in[0,1]} \left|\frac{1}{\gamma^2(t)}\right| = \max_{t\in[0,1]} \frac{1}{1+(1+2t)^2} = \frac{1}{2}.$$

(iv) If $|f(z)| \le M$ on γ, then we directly have $\left|\int_\gamma f(z)\,dz\right| \le ML(\gamma)$.

(v) The following inequalities may also be checked in a similar fashion:

(a) $\left|\int_{|z|=1} e^z\,dz\right| \leq 2\pi e$, $\left|\int_{|z|=1} e^{1/z}\,dz\right| \leq 2\pi e$, $\left|\int_{|z|=1} \frac{1}{z}\,dz\right| \leq 2\pi$.
Later, we shall actually see that the value of first integral is 0 while the second and the third each has the value $2\pi i$.

(b) $\left|\int_\gamma (z+1)^2\,dz\right| \leq 9\sqrt{5}$, where γ is the line segment parameterized by $\gamma(t) = 2 - t(2-i)$, $t \in [0,1]$.

(c) $\left|\int_\gamma e^{iz^2}\,dz\right| \leq \dfrac{\pi(e-1)}{4e}$, where $\gamma(t) = e^{it}$, $0 \leq t \leq \pi/4$.

(d) $\left|\int_\gamma [(\mathrm{Re}\,z)^2 + i(\mathrm{Im}\,z)^2]\,dz\right| \leq 2$, where γ is the interval $[-i,i]$ on the imaginary axis. ●

4.13. Definition. Let γ be a curve with parametric interval $[a,b]$ and let $\{f_n\}$ be a sequence of functions on an open set D containing $\gamma([a,b])$. If for a function f defined on D, $|f_n(\gamma(t)) - f(\gamma(t))| \to 0$ uniformly on $[a,b]$, we say that $f_n \to f$ uniformly on γ. If for a function f on D, $|S_n(\gamma(t)) - f(\gamma(t))| \to 0$ uniformly on $[a,b]$, where $S_n = \sum_{k=1}^n f_k$, we say that $\sum_{n\geq 1} f_n \to f$ uniformly on γ.

4.14. Theorem. (Interchange of limit and integration) *Let $\{f_n\}$ be a sequence of continuous functions defined on an open set containing a contour γ. Suppose that $f_n \to f$ uniformly on γ. Then,*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

Proof. Let $\epsilon > 0$ and let f_n converge uniformly on γ with parametric interval $[a,b]$. Then, there is an N such that

$$|f_n(\gamma(t)) - f(\gamma(t))| < \epsilon \quad \text{for } t \in [a,b] \text{ and } n \geq N.$$

First we observe that f is continuous on γ (see Theorem 2.57). Using Theorem 4.9(iv), we have

$$\begin{aligned}
\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| &= \left|\int_a^b [f_n(\gamma(t)) - f(\gamma(t))]\gamma'(t)\,dt\right| \\
&\leq \int_a^b |f_n(\gamma(t)) - f(\gamma(t))|\,|\gamma'(t)|\,dt \\
&< \epsilon \int_a^b |\gamma'(t)|\,dt \quad \text{for } n \geq N.
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, the proof is complete. ■

A straightforward proof gives

4.15. Corollary. (Interchange of summation and integration) *Let $\sum_{n\geq 1} f_n$ be a series of continuous functions defined on an open set containing a contour γ and $\sum_{n\geq 1} f_n \to f$ uniformly on a contour γ. Then,*

$$\sum_{n\geq 1}\int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

There are two versions of the fundamental theorem of calculus for real-valued functions:

(i) $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ for $x\in[a,b]$ where f is continuous on $[a,b]$ and one-sided derivatives are meant at a or b

(ii) $\displaystyle\int_a^b f'(t)\,dt = f(b)-f(a)$ whenever $f'(t)$ is continuous on $[a,b]$.

The following weaker form of Cauchy's Theorem (see Theorem 4.33), which is actually the analogue of the second statement of the fundamental theorem of calculus, is helpful, and integration of familiar functions is facilitated by this result.

4.16. Theorem. (Weak form of Cauchy's theorem) *If $f=u+iv$ is analytic in an open set D containing a contour γ with parametric interval $[a,b]$, i.e. $\gamma([a,b])\subset D$, then*

$$\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a)).$$

That is, the value of the integral is independent of the path. In particular, we have $\int_\gamma f'(z)\,dz = 0$ if γ is closed.

Proof. Let f be analytic on D and γ be, initially, a smooth curve with $\gamma([a,b])\subset D$. Then, we must have (see Corollary 2.21 and (3.3))

$$f(\gamma(t)) = f(\gamma(t_0)) + (\gamma(t)-\gamma(t_0))f'(\gamma(t_0)) + (\gamma(t)-\gamma(t_0))\eta(\gamma(t)),$$

for t near $t_0\in[a,b]$, where $\eta\circ\gamma$ is a continuous function of t on $[a,b]$ such that $\lim_{t\to t_0}\eta(\gamma(t)) = 0$. Therefore, as t_0 is arbitrary, the Chain rule for differentiation gives

$$\frac{d}{dt}[f(\gamma(t))] = f'(\gamma(t))\gamma'(t)$$

on $[a,b]$ and $f(\gamma(t))$ is continuously differentiable for $t\in[a,b]$. The result now follows from the "second statement of the fundamental theorem of calculus for real variable". Indeed, we let

$$f(\gamma(t)) = f(x(t)+iy(t)) = u(x(t),y(t))+iv(x(t),y(t)) = U(t)+iV(t),\ \text{ say,}$$

so that

$$\frac{d}{dt}[f(\gamma(t))] = f'(\gamma(t))\gamma'(t) = \frac{d}{dt}(U(t)) + i\frac{d}{dt}(V(t)).$$

Since γ is smooth, we integrate both sides of the last expression to get

$$\begin{aligned} \int_\gamma f'(z)\,dz &= \int_a^b \frac{d}{dt}[f(\gamma(t))]\,dt \\ &= U(t) + iV(t)\Big|_a^b \\ &= f(\gamma(b)) - f(\gamma(a)). \end{aligned}$$

Suppose that γ is piecewise smooth. Then, we can by definition choose a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction γ_k of γ to (t_k, t_{k+1}), $k = 0, 1, \ldots, n-1$, is smooth. In view of Theorem 4.9(iii) and what has been just proved,

$$\begin{aligned} \int_\gamma f'(z)\,dz &= \sum_{k=0}^{n-1} \int_{\gamma_k} f'(z)\,dz = \sum_{k=0}^{n-1} f(\gamma_k(t_{k+1})) - f(\gamma_k(t_k)) \\ &= f(\gamma(b)) - f(\gamma(a)), \end{aligned}$$

again as asserted. ∎

Theorem 4.16 is also known as the fundamental theorem of line integrals (or contour integration) in the complex plane. Moreover, Theorem 4.16 shows that, if $F(z) = f'(z)$ then one has

$$\int_{z_1}^{z_2} F(z)\,dz = f(z_2) - f(z_1).$$

In particular, for closed curves γ independent of paths, we conclude that $\int_\gamma F(z)\,dz = 0$.

One of the objectives in Section 4.3 is to extend this for those functions $F(z)$ for which no f such that $F(z) = f'(z)$ is at hand. The examples of such functions are $\cos(z^2)$, $\sin(z^2)$, $\exp(z^2)$, etc. On the other hand, it will be shown that

$$\int_\gamma \cos(z^2)\,dz = \int_\gamma \sin(z^2)\,dz = \int_\gamma \exp(z^2)\,dz = 0$$

for each closed contour γ.

4.17. Example. Consider $I_j = \int_{\gamma_j} \overline{z}\,dz$, $j = 1, 2, 3$, where

(i) γ_1 is the directed line segment from 0 to $1+i$;
(ii) γ_2 is the arc of the circle $\gamma_2(t) = 1 + e^{it}$ joining 0 and $1+i$;

(iii) γ_3 is the directed line segment from 0 to 1 and then from 1 to $1+i$.

In this example we observe that $f(z)=\overline{z}$ is nowhere analytic and so $\int_\gamma \overline{z}\,dz$ need not be independent of the choice of the curve γ connecting the points 0 and $1+i$. In fact, it can be checked easily that

$$I_1 = 1,\ I_2 = 1 + i(\pi/2 - 1) \ \text{ and } \ I_3 = 1 + i.$$

Thus $I_1 \neq I_2$, $I_2 \neq I_3$ and $I_1 \neq I_3$, even though γ_1, γ_2 and γ_3 have the same initial and the same terminal points. ●

Let $z_1 = -1$, $z_2 = 1$ and $z_3 = i$. Consider $\gamma_1 = [z_1, z_2] \cup [z_2, z_3]$, and $\gamma_2 = [z_1, z_3]$. Then both γ_1 and γ_2 are curves of class C^1 whose initial and the terminal points are the same. But it is easy to see that

$$\int_{\gamma_1} \overline{z}\,dz = i \quad \text{and} \quad \int_{\gamma_2} \overline{z}\,dz = -i.$$

Recall that $f(z)=\overline{z}$ is nowhere analytic and hence, has no primitive in a domain containing the points $-1, 1$ and i. The above examples show that the integral of a complex function depends on the path of integration.

However, there are a few important Corollaries to Theorem 4.16. Keeping in mind the definition of primitive, we have

4.18. Corollary. *If F is a primitive of f on D and $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve in D, then $\int_\gamma f(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a))$.*

For instance, if $n \geq 0$ is an integer then

$$\int_{z_1}^{z_2} z^n\,dz = \frac{z_2^{n+1} - z_1^{n+1}}{n+1}$$

since '$z^{n+1}/(n+1)$ + constant' is a primitive of z^n on $\mathbb{C}$. If $n < -1$ is an integer, the above equality holds provided the path of integration omits the origin. In particular, $\int_\gamma z^n\,dz = 0$ if γ is any smooth closed path omitting the origin and $n \neq -1$.

4.19. Example. Let $\gamma : [0, 1] \to \mathbb{C}$ be defined by

$$\gamma(t) = 1 - t - \sin \pi t + i(t + \cos \pi t)$$

which is an arc connecting $1+i$ to 0 as in Figure 4.6. To evaluate

$$I = \int_\gamma z^2\,dz = \int_0^1 \gamma^2(t)\gamma'(t)\,dt$$

we first note that, since z^2 is analytic in $\mathbb{C}$, γ lies in the domain of definition of z^2. Since the value of the integral is independent of the choice of the

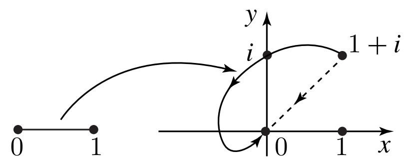


Figure 4.6: Curve $\gamma(t) = 1 - t - \sin \pi t + i(t + \cos \pi t)$.

curve connecting $1 + i$ to 0, we have

$$I = \int_0^1 \gamma_1^2(t)\gamma_1'(t)\, dt,$$

where $\gamma_1(t) = (1 - t)(1 + i) + t \cdot 0,\ t \in [0, 1]$. Thus, we easily find that

$$I = \int_0^1 (1 + i)^2(1 - t)^2[-(1 + i)]\, dt = -\frac{(1 + i)^3}{3}.$$

Indeed, by Corollary 4.18, we directly obtain that $I = F(\gamma_1(1)) - F(\gamma_1(0))$, where $F(z) = z^3/3 + K$, and so $I = F(0) - F(1 + i) = -(1 + i)^3/3$. ●

4.20. Corollary. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ with its radius of convergence $R > 0$. Then, for any closed contour γ in Δ_R, we have $\int_\gamma f'(z)\, dz = 0$.*

Corollary 4.20 follows from the fact that f possesses a primitive F in Δ_R, namely, $F(z) = \sum_{n\geq 0}(n + 1)^{-1}a_n z^{n+1}$.

4.21. Corollary. *Theorem 3.31(i) follows from Corollary 4.18.*

Proof. By the assumption of Theorem 3.31(i), we note that f is a primitive of the zero function $g(z) = 0$ on D. Therefore, for any disk $\Delta(z_0; r)$ entirely within D and each $\zeta \in \Delta(z_0; r)$,

$$0 = \int_\zeta^{z_0} 0 \cdot dz = f(\zeta) - f(z_0)$$

so that $f(z) = f(z_0)$ for $z \in \Delta(z_0; r)$. ■

4.22. Example. Let us evaluate $\int_\gamma (z^2 + z)\, dz$ where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ as in Figure 4.7. First we note that $f(z) = z^2 + z$ is analytic in $\mathbb{C}$ and so it is analytic in any domain containing the points -1 and 1. Therefore, Theorem 4.16 allows us to choose any path connecting -1 and 1. The most convenient path in this case is the line segment γ connecting -1 and 1:

$$\gamma(t) = (1 - t)(-1) + t(1) = 2t - 1,\ t \in [0, 1].$$

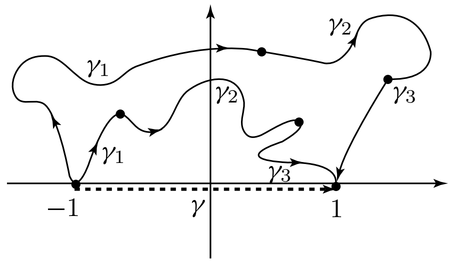


Figure 4.7: Curves connecting -1 and 1.

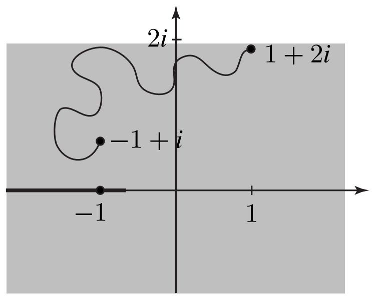


Figure 4.8: Region for the analyticity of $\operatorname{Log}(1+z)$.

Therefore, as $f(\gamma(t))\gamma'(t) = [(2t-1)^2 - (2t-1)](2) = 8t^2 - 4t$, we have

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = 8\int_0^1 t^2\,dt - 4\int_0^1 t\,dt = \frac{8}{3} - \frac{4}{2} = \frac{2}{3}.$$

Moreover, since $f(z) = F'(z)$ with $F(z) = \frac{z^3}{3} + \frac{z^2}{2} + K$, we can directly use Corollary 4.18 to obtain $\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 2/3$. ●

4.23. Example. We next evaluate $I = \int_\gamma \frac{dz}{1+z}$, where γ is any contour in $D = \{z : \operatorname{Im} z > 0\}$, which joins $-1+i$ to $1+2i$ (see Figure 4.8). Suppose that we consider the principal logarithm. We observe that $(1+z)^{-1}$ is the derivative of $F(z) = \operatorname{Log}(1+z)$ and, since F is analytic in $\mathbb{C}\backslash\{(-\infty,-1]\}$, F is analytic in D. Then the value of the integral is independent of the path joining $-1+i$ and $1+2i$ and hence,

$$\begin{aligned} \int_\gamma \frac{dz}{1+z} &= \int_\gamma F'(z)\,dz \\ &= F(1+2i) - F(-1+i) \\ &= \operatorname{Log}[2(1+i)] - \operatorname{Log} i \\ &= [\ln\sqrt{8} + i\frac{\pi}{4}] - i\frac{\pi}{2} = \ln\sqrt{8} - \pi i/4. \end{aligned}$$ ●

4.24. Example. Suppose we wish to evaluate the integral $\int_\gamma \frac{dz}{z}$, where γ is an arc joining $1-i$ to $1+i$. Then, in this case, it is necessary to consider a domain D containing the arc γ. Note that the integrand $1/z$ is analytic in $\mathbb{C}\setminus\{0\}$. As we have observed before if $\alpha \in \mathbb{R}$ is fixed, $D_\alpha = \mathbb{C}\setminus\{Re^{i\alpha} : R > 0\}$ and $\arg_\alpha z$ is the choice of $\arg z$ in $(\alpha - 2\pi, \alpha)$, then

$$f_\alpha(z) = \ln|z| + i \arg_\alpha z$$

is the anti-derivative of $1/z$ in D_α. Suppose we choose $\alpha = \pi$. Then,

$$|\arg_\alpha z| < \pi,\ D_\alpha = D_\pi = \mathbb{C}\setminus\{-R :\ R > 0\}$$

and therefore, $f_\alpha(z)$ becomes the principal logarithm $\operatorname{Log} z$. So if γ is any curve in D_π which joins $1-i$ to $1+i$, then by Corollary 4.18

$$I = f_\pi(z)\Big|_{1-i}^{1+i} = \operatorname{Log} z\Big|_{1-i}^{1+i}$$

which gives

$$I = \operatorname{Log}(1+i) - \operatorname{Log}(1-i) = i[\operatorname{Arg}(1+i) - \operatorname{Arg}(1-i)] = i\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

so that $I = \pi i/2$.

Suppose we choose $\alpha = 2\pi$. Then,

$$0 < \arg_\alpha z < 2\pi,\ D_{2\pi} = \mathbb{C}\setminus\{R :\ R > 0\}.$$

Therefore if γ is any curve in $D_{2\pi}$ which joins $1-i$ to $1+i$, then

$$I = f_{2\pi}(z)\Big|_{1-i}^{1+i} = i[\arg_{2\pi}(1+i) - \arg_{2\pi}(1-i)] = i\left[\frac{\pi}{4} - \left(2\pi - \frac{\pi}{4}\right)\right]$$

so that $I = -3\pi i/2$.

If γ_1 and γ_2 are given by

$$\gamma_1(t) = e^{it},\ t \in [-\pi/4, \pi/4] \ \text{ and } \ \gamma_2(t) = e^{it},\ t \in [\pi/4, 2\pi - \pi/4],$$

then $\gamma_1 \in D_\pi$ and $\gamma_2 \in D_{2\pi}$. Then $\gamma = \gamma_1 + \gamma_2$ is a closed curve and, in this case, we have

$$\int_{|z|=1} \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} - \int_{-\gamma_2} \frac{dz}{z} = \frac{\pi i}{2} - \left(-\frac{3\pi i}{2}\right) = 2\pi i.$$

With this idea, it is also clear that $\int_{|z|=r<1} \operatorname{Log}(1+z)\,dz = 0$, since the function $\operatorname{Log}(1+z)$ is analytic in $\mathbb{C}\setminus\{(-\infty, -1]\}$. ●

4.3 Cauchy-Goursat Theorem

The simplest version of Cauchy's theorem utilizes a theorem from calculus known as 'Green's Theorem' which states that given two real-valued functions $M = M(x, y)$ and $N = N(x, y)$, which are continuous together with their partial derivatives inside and on a simple closed contour γ

$$\int_\gamma M\,dx + N\,dy = \iint_\Omega \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dx\,dy, \tag{4.25}$$

where Ω is the interior of γ.

4.26. Theorem. *If f is analytic with f' continuous inside and on a simple closed contour γ, then $\int_\gamma f(z)\,dz = 0$.*

Proof. Let $f(z) = u(x, y) + iv(x, y)$ and $\Omega = \operatorname{Int}\gamma$, the interior of γ. Then, according to the discussion in Section 4.2, the integral of f over γ can be written as

$$\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (u\,dy + v\,dx). \tag{4.27}$$

Since f is analytic in Ω (and hence f is continuous in Ω), u and v are also continuous therein and $f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z)$. Further, as f' is continuous in Ω, the partial derivatives of u and v are also continuous in Ω. By applying Green's Theorem (see (4.25)) to each of the integrals on the right of (4.27), we obtain

$$\int_\gamma f(z)\,dz = \iint_\Omega (-v_x - u_y)\,dx\,dy + i \iint_\Omega (u_x - v_y)\,dx\,dy.$$

But, in view of the C-R equations, both terms of these double integrals are zero in Ω. ∎

The continuity requirement on f' may be dropped from Theorem 4.26. The result without this condition is called the *Cauchy-Goursat Theorem*. More explicitly, the Cauchy-Goursat theorem asserts that the integral of a function, that is analytic in a simply connected domain D, along any closed contour $\gamma \subset D$ is always zero.

In the following theorem we show that Cauchy's theorem is true for arbitrary analytic functions in D if we restrict the curve to be the triangular curve contained in D. By a (closed) triangle in $\mathbb{C}$ we mean T, the set of points in $\mathbb{C}$ of the form $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3$, where $0 \le \lambda_j \le 1$ ($j = 1, 2, 3$), with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Denote by ∂T the boundary of T, composed of the three line segments $[z_1, z_2]$, $[z_2, z_3]$, $[z_3, z_1]$, described once in the positive direction. Note that T is geometrically the closed triangle and is therefore compact.

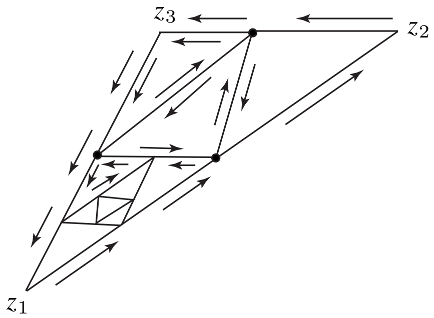


Figure 4.9: Triangular curve.

4.28. Theorem. (Cauchy-Goursat Theorem) *Let f be analytic in an open set $D \subset \mathbb{C}$ and let z_1, z_2, z_3 be in D. Assume that the closed triangle T with vertices z_1, z_2, z_3 is contained in D. Then, $\int_{\partial T} f(z)\,dz = 0$.*

Proof. Let m_1, m_2 and m_3 be the mid points of the segments $[z_1, z_2]$, $[z_2, z_3]$ and $[z_3, z_1]$, respectively. Then we get four smaller triangles T_k, $1 \leq k \leq 4$, as shown in Figure 4.9. We note that

$$|m_1 - m_2| = \left| \frac{z_1 + z_2}{2} - \frac{z_2 + z_3}{2} \right| = \frac{|z_1 - z_3|}{2}.$$

Similarly,

$$|m_2 - m_3| = \frac{|z_2 - z_1|}{2}, \quad |m_3 - m_1| = \frac{|z_3 - z_2|}{2}$$

and hence, $L(\partial T_k) = \frac{1}{2} L(\partial T)$ for each $k = 1, 2, 3, 4$, where ∂T_k denotes the boundary of T_k and $L(\gamma)$, as usual, denotes the length of γ. Define

$$I = \int_{\partial T} f(z)\,dz \quad \text{and} \quad I_k = \int_{\partial T_k} f(z)\,dz \quad \text{for } k = 1, 2, 3, 4.$$

Then, from the properties of complex integrals, we see that

$$\text{(4.29)} \qquad I = \sum_{k=1}^{4} I_k, \quad \text{i.e.} \quad \int_{\partial T} f(z)\,dz = \sum_{k=1}^{4} \int_{\partial T_k} f(z)\,dz,$$

since we are integrating the R.H.S of (4.29) twice in the opposite directions (as we see clearly in Figure 4.9) over each of the boundary ∂T_k which is not part of the sides of the triangle T and consequently corresponding boundary integrals cancel each other (by the reversal rule, see Theorem 4.9(i)). The triangle inequality gives us that

$$|I| \leq \sum_{k=1}^{4} |I_k|.$$

Consequently, there exists at least one k in $\{1,2,3,4\}$ such that the inequality

$$|I_k| \geq |I|/4$$

is satisfied, for otherwise $|I| \leq \sum_{k=1}^{4} |I_k| < 4(I/4) = I$ which is a contradiction. If more than one k satisfies the inequality, we retain the least. We relabel such a triangle as T^1. The diameter of T^1, viz. the length of its largest side, is one-half of the diameter of T, that is

$$\text{diam}\,(T^1) = \frac{1}{2}\text{diam}\,(T),$$

where the diameter of a bounded set S is defined to be $\sup\{|a-b| : a, b \in S\}$. Thus, with $I^{(1)} = \int_{\partial T^1} f(z)\,dz$, we obtain

$$|I| \leq 4|I^{(1)}|, \;\; L(\partial T^1) = \frac{1}{2}L(\partial T), \;\;\text{and}\;\; \text{diam}\,(T^1) = \frac{1}{2}\text{diam}\,(T).$$

Apply the argument with T^1 in place of T to obtain another triangle T^2. Repeating the subdivision process to successively obtained triangles, we have following relations:

(i) for each n, $T^n \subset T^{n-1}$ $(T^0 = T)$,

(ii) if $I^{(j)} = \int_{\partial T^j} f(z)\,dz$, then $|I^{(j)}| \leq 4|I^{(j+1)}|$ $(I^{(0)} = I)$,

(iii) $L(\partial T^{j+1}) = \frac{1}{2}L(\partial T^j)$,

(iv) $\text{diam}\,(T^{j+1}) = \frac{1}{2}\text{diam}\,(T^j)$, where $j = 1, 2, \ldots$.

Note also that $\text{diam}\,(\partial T^j) \leq L(\partial T^j)$. In particular, at the n-th stage

(i)$'$ $T^n \subset T^{n-1} \subset \cdots \subset T^2 \subset T^1 \subset T^0 = T$,

(ii)$'$ $|I| \leq 4^n |I^{(n)}|$,

(iii)$'$ $L(\partial(T^n) = \left(\frac{1}{2}\right)^n L(\partial T)$,

(iv)$'$ $\text{diam}\,(T^n) = \left(\frac{1}{2}\right)^n \text{diam}\,(T)$.

Since $\lim_{n\to\infty} \text{diam}\,(T^n) = 0$ and, by (i)$'$, $T^1 \supset T^2 \supset \cdots \supset T^n \supset \cdots$ is a nested sequence of non-empty compact subsets of $\mathbb{C}$, the intersection $\cap_{n=0}^{\infty} T^n$ is non-empty and so it follows that this intersection contains at least one point, say z_0, common to all triangles T^n (a use has been made of Cantor's theorem). In particular, $z_0 \in T$, so f is analytic at z_0. Since f is analytic at z_0, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)\eta(z),$$

with η continuous and $|\eta(z)| < \epsilon$ for $|z - z_0| < \delta$. Therefore, we have

$$\begin{aligned}\int_{\partial T^n} f(z)\,dz &= [f(z_0) - z_0 f'(z_0)] \int_{\partial T^n} dz + f'(z_0) \int_{\partial T^n} z\,dz \\ &\quad + \int_{\partial T^n} (z - z_0)\eta(z)\,dz.\end{aligned}$$

We know that if γ is any closed contour, then $\int_\gamma dz = \int_\gamma (z - z_0)\, dz = 0$. As the first two integrals vanish,

$$\int_{\partial T^n} f(z)\, dz = \int_{\partial T^n} (z - z_0)\eta(z)\, dz.$$

By (i)′, we also note that, for sufficiently large n, the triangle T^n is contained in $\{z : |z - z_0| < \delta\}$. Using the standard estimate (see Theorem 4.9(iv)) together with the above observations, we get

$$\begin{aligned} |I^{(n)}| &= \left| \int_{\partial T^n} f(z)\, dz \right| \\ &= \left| \int_{\partial T^n} (z - z_0)\eta(z)\, dz \right| \\ &\leq \epsilon \max_{z \in \partial T^n} |z - z_0| \cdot L(\partial T^n) \\ &\leq \epsilon \cdot \text{diam}\,(T^n) L(\partial T^n) \\ &\leq \epsilon \cdot \left(\frac{1}{2}\right)^n \text{diam}\,(T) \left(\frac{1}{2}\right)^n L(\partial T), \quad \text{by (iii)}' \text{ and (iv)}', \\ &= \epsilon 4^{-n} \text{diam}\,(T) L(\partial T). \end{aligned}$$

Taking (ii)′ into account we have the inequalities

$$4^{-n}|I| \leq |I^{(n)}| \leq \epsilon 4^{-n} \text{diam}\,(T) L(\partial T).$$

Consequently, we have $|I| \leq \epsilon \text{diam}\,(T) L(\partial T)$. Since $\epsilon > 0$ in this inequality is an arbitrary positive number, we must have $I = 0$ and so the proof is complete. ■

If γ is a quadrilateral, it can be divided into two triangles T and T' (see Figure 4.10) so that

$$\int_\gamma f(z)\, dz = \int_{\partial T} f(z)\, dz + \int_{\partial T'} f(z)\, dz = 0,$$

since the integrals along AC and CA cancel each other. In general, if γ is any simple polygon then we can decompose such a polygon into triangles so that $\displaystyle\int_\gamma f(z)\, dz = 0$.

An important point in the above proof is that it is not necessary to assume the continuity of the derivative $f'(z)$ of $f(z)$. Next we present a simple extension of Theorem 4.28 with a relaxed condition on the differentiability of f.

4.30. Theorem. *Let D be an open set and let f be analytic on D except possibly at $a \in D$. Assume that f is continuous on D. Then, we have $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle T in D.*

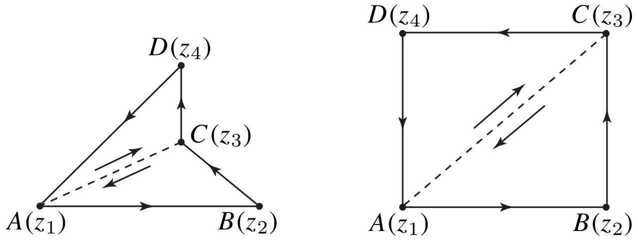


Figure 4.10: Triangles $T = [z_1, z_2, z_3]$ and $T' = [z_1, z_3, z_4]$.

Proof. For a closed triangle T in D, we may simply assume that a lies in T, for the result is a consequence of Theorem 4.28 otherwise. Given a positive integer, we can subdivide T into n^2 congruent triangles T_{jk} by adjoining the midpoints of opposite sides. Then, we have

$$\int_{\partial T} f(z)\,dz = \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\partial T_{jk}} f(z)\,dz$$

since the dividing segments cancel in pairs. If a is not a point of T_{jk}, then, by Theorem 4.28, $\int_{\partial T_{jk}} f(z)\,dz = 0$. If a belongs to the triangle T_{jk}, then the M-L inequality shows that

$$\left|\int_{\partial T_{jk}} f(z)\,dz\right| \le \int_{\partial T_{jk}} |f(z)|\,|dz| \le M\,L(\partial T_{jk}) = \frac{ML(\partial T)}{n},$$

where $M = \max_{z\in\partial T}|f(z)|$. Note that $|f(z)|$ is a continuous function on the compact set C. Note that the point a at the worst can belong to one of the four triangles ∂T_{jk}. It follows that

$$\left|\int_{\partial T} f(z)\,dz\right| = \left|\sum_{a\in\partial T_{jk}}\int_{\partial T_{jk}} f(z)\,dz\right| \le \sum_{a\in\partial T_{jk}}\left|\int_{\partial T_{jk}} f(z)\,dz\right| \le \frac{4ML(C)}{n}.$$

Since n was arbitrary, $\left|\int_{\partial T} f(z)\,dz\right| = 0$ and the proof is complete. ■

4.31. Theorem. *Let D be a domain that is starlike with respect to a and f be analytic on D. Then, there exists an analytic function F on D such that $F'(z) = f(z)$ in D. In particular, $\int_C f(z)\,dz = 0$ for every closed contour C in D.*

Proof. Since D is starlike (see Figure 4.11) with respect to a, $[a, z] \subset D$ for every $z \in D$. Define F on D by

$$F(z) = \int_{[a,z]} f(\zeta)\,d\zeta.$$

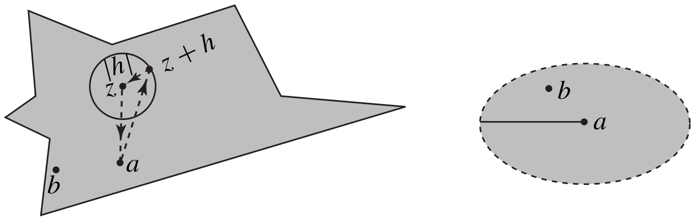


Figure 4.11: Starlike domain with respect to a.

Note that $F(a) = 0$. Fix $z \in D\backslash\{a\}$. Then there exists $h \in \mathbb{C}$ with $|h|$ sufficiently small such that $\overline{\Delta}(z;|h|) \subset D$ and $[z, z+h] \subset \Delta(z;|h|) \subset D$. As D is starlike, the triangle $T = [z, a, z+h]$ lies in D. By Theorem 4.28,

$$0 = \int_{\partial T} f(\zeta)\, d\zeta = \left(\int_z^a + \int_a^{z+h} + \int_{z+h}^z\right) f(\zeta)\, d\zeta,$$

or

$$\int_a^{z+h} f(\zeta)\, d\zeta - \int_a^z f(\zeta)\, d\zeta = \int_z^{z+h} f(\zeta)\, d\zeta.$$

Therefore, we have

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(\zeta)\, d\zeta - \frac{f(z)}{h}\int_z^{z+h} d\zeta$$

and so

$$\begin{aligned}\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_z^{z+h} [f(\zeta) - f(z)]\, d\zeta\right| \\ &\leq \frac{1}{|h|}\left[\sup_{\zeta\in[z,z+h]} |f(\zeta) - f(z)|\right] |h|,\end{aligned}$$

using the standard integral estimate (see Theorem 4.9(iv)). Continuity of f immediately yields that

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \to 0 \quad \text{as} \quad h \to 0,$$

and so $F'(z)$ exists and is equal to $f(z)$. Since z is arbitrary, $F'(z) = f(z)$ in D. That is, F is a primitive of f on D, as required. The second part follows from Theorem 4.16. ■

Since every disk is a starlike domain, Theorem 4.31 yields the "local form" of Cauchy's theorem.

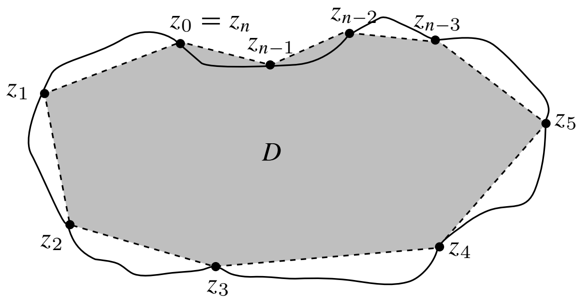


Figure 4.12: Closed polygon with vertices on C.

4.32. Corollary. (Cauchy's Theorem for a Disk) *Let f be analytic in a disk $\Delta(z_0; R)$ (or more generally, f is continuous in $\Delta(z_0; R)$ and analytic in $\Delta(z_0; R)\backslash\{a\}$ for some $a \in \Delta(z_0; R)$). Then, $\int_\gamma f(z)\, dz = 0$ for every closed contour γ in $\Delta(z_0; R)$.*

The function $f(z) = 1/z$ defined in the annulus $A = \{z : 1 < |z| < 2\}$ shows that the conclusion of Theorem 4.31 fails if D is not a starlike domain with respect to a point in D. Further, the contour in Theorem 4.31 can have self-intersection.

Now we are in position to prove the long waited "Cauchy integral theorem".

4.33. Theorem. (Cauchy's Integral Theorem) *If f is analytic in a simply connected domain D, then there exists a function F in D such that $F'(z) = f(z)$. In particular, $\int_\gamma f(z)\, dz = 0$ for each simple closed contour γ in D.*

Proof. Let C be a simple closed contour in D, and $D_0 = C \cup \operatorname{int} C$. So, $D_0 \subseteq D$ is compact and hence, for every $z \in D_0$ there exists $\delta_z > 0$ such that $\overline{\Delta}(z; \delta_z) \subset D$. Now $D_1 = \cup_{z \in D_0} \overline{\Delta}(z; \delta)$ is compact and so f is uniformly continuous on D_1. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|f(z) - f(\zeta)| < \frac{\epsilon}{2L(C)}$$

for all $z, \zeta \in D_1$ and $|z - \zeta| < \delta$. Here $L(C)$ denotes the length of C. Now without loss of generality we may assume that $\delta_z \leq \delta/2$ for every $z \in D_0$. Since C is compact, there exist points $z_0, z_1, \dots, z_n = z_0$ in C such that $C \subseteq \cup_{k=0}^{n-1} \overline{\Delta}(z_k; \delta_{z_k})$. Let Λ be a closed polygon in D_1 with vertices $z_0, z_1, \dots, z_n$, see Figure 4.12. Therefore,

$$|f(z) - f(z_k)| < \frac{\epsilon}{2L(C)} \quad \text{for } k = 1, 2, \dots, n, \tag{4.34}$$

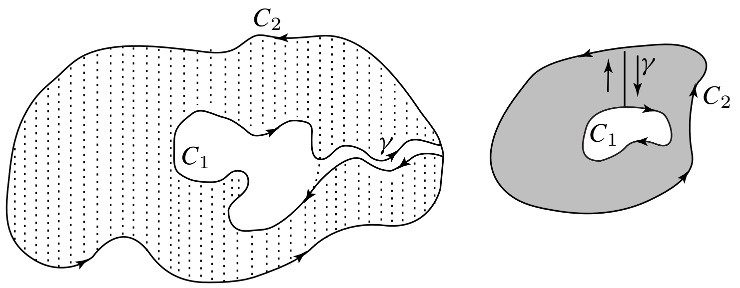


Figure 4.13: Region bounded by two curves C_1, C_2.

whenever $z \in [z_{k-1}, z_k]$, $k = 0, 1, 2, \ldots, n-1$. Further,

$$\begin{aligned}
0 = \int_\Lambda f(z)\,dz &= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} f(z)\,dz \\
&= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z) - f(z_k) + f(z_k)]\,dz \\
&= -\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z_k) - f(z)]\,dz + \sum_{k=1}^{n} f(z_k) \int_{z_{k-1}}^{z_k} dz \\
&= -\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z_k) - f(z)]\,dz + \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}),
\end{aligned}$$

that is,

$$\sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [f(z_k) - f(z)]\,dz. \tag{4.35}$$

By (4.34) and (4.35), we have

$$\left| \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) \right| \le \frac{\epsilon}{2L(C)} \sum_{k=1}^{n} |z_k - z_{k-1}| < \frac{\epsilon}{2}.$$

Since

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) = \int_C f(z)\,dz$$

and since ϵ is arbitrary, the theorem is proved. ∎

Suppose f is analytic in a ring shaped space bounded by two simple closed contours C_1 and C_2 as shown in Figure 4.13.

Let γ be any contour or a line from C_1 to C_2, also shown in Figure 4.13. Then the region bounded by $C = C_1 + \gamma - C_2 - \gamma$ is simply connected.

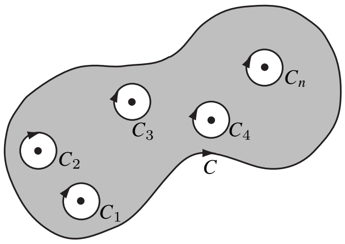


Figure 4.14: Illustration for Cauchy's deformation of contour

From Theorem 4.33,

$$\begin{aligned} 0 = \int_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_\gamma f(z)\,dz - \int_{C_2} f(z)\,dz - \int_\gamma f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz, \text{ by Theorem 4.9(i),} \end{aligned}$$

which gives $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$. This result is referred to as the *Cauchy deformation theorem.* In Figure 4.14, we illustrate Cauchy's theorem for domain with $(n-1)$ holes. In a manner similar to that used above, we get

$$\int_{C_1+C_2+\cdots+c_n} f(z)\,dz = 0. \tag{4.36}$$

Equation (4.36) can be written in the form

$$\begin{aligned} \oint_{C_1} f(z)\,dz &= -\left[\oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz\right] \\ &= \oint_{-C_2} f(z)\,dz + \cdots + \oint_{-C_n} f(z)\,dz. \end{aligned}$$

In other words, by integrating along each inner contour in the counterclockwise direction, so that the $(n-1)$ inner contours have negative orientation, it follows that the value of the integral along the outer contour is equal to the sum of the values along the inner contours.

Annulus regions are classified as follows: Let $a \in \mathbb{C}$ and $0 \le r < R \le \infty$. Then the open subset

$$D = D(a; r, R) = \{z \in \mathbb{C} : r < |z-a| < R\}$$

of $\mathbb{C}$ is called the *annulus* or *circular ring* around a with inner radius r and outer radius R. If $r = 0$ and $R < \infty$, then D is a disk with center removed, i.e. the punctured disk: $D = D(a; 0, R) = \Delta(a; R)\backslash\{a\}$. If $r = 0$ and $R = \infty$, then $D(a; 0, \infty) = \mathbb{C}\backslash\{a\}$. Finally, D defined above is the exterior circle excluding the point ∞ if $R = \infty$ and $r > 0$. In particular, the theorem of Cauchy's deformation of contour gives the following:

4.37. Theorem. *Let $0 \le R_1 < r < R_2 \le \infty$, and $g(z)$ be analytic in the annulus domain $D = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$. If $C_r = \{z : |z-a| = r\}$, then $\int_{C_r} g(z)\,dz$ is independent of r.*

4.4 Consequence of Simply Connectivity

How do we produce a non-vanishing analytic function f in a simply connected domain Ω? Take an arbitrary analytic function h in Ω. Then, the desired function f is given by $f(z) = \exp(h(z))$. In the following theorem we actually show that every non-vanishing analytic function f arises in this way.

4.38. Theorem. *Let Ω be a simply connected domain and $f \in \mathcal{H}(\Omega)$ with $f(z) \neq 0$ on Ω. Then, there exists a $h \in \mathcal{H}(\Omega)$ such that $e^{h(z)} = f(z)$.*

Proof. As $f(z) \neq 0$ on Ω, $f'(z)/f(z)$ is analytic on Ω. By Theorem 4.33 (see also Corollary 4.61) there exists an $h \in \mathcal{H}(\Omega)$ such that

$$h'(z) = \frac{f'(z)}{f(z)} \quad \text{for } z \in \Omega.$$

We claim that $f(z)e^{-h(z)} = 1$ for $z \in \Omega$. To do this, we define $g(z) = f(z)e^{-h(z)}$. Clearly, $g \in \mathcal{H}(\Omega)$ and

$$g'(z) = (f'(z) - f(z)h'(z))e^{-h(z)} = 0 \quad \text{for } z \in \Omega.$$

Fix $c \in \Omega$. Then,

$$g(z) - g(c) = \int_c^z g'(\zeta)\,d\zeta = 0$$

so that $f(z)e^{-h(z)} = g(c)$ or $f(z) = e^{h(z)}g(c)$. As $g(c) \neq 0$, we can set $g(c) = e^k$ for some k. Then

$$f(z) = \exp(h(z) + k) = \exp(H(z)),$$

where $H(z) = h(z) + k$. ∎

Another consequence of this result is the following square root property.

4.39. Theorem. *Assume the hypotheses of Theorem 4.38. Then, f has an analytic square root- that is there exists a $g \in \mathcal{H}(\Omega)$ with $g^2(z) = f(z)$ for $z \in \Omega$.*

Proof. The desired conclusion follows if we choose h as in the previous theorem and set $g(z) = \exp(h(z)/2)$. ∎

We have a direct proof of Theorem 4.38 at least when $\Omega = \mathbb{C}$.

4.40. Theorem. *Let $f \in \mathcal{H}(\mathbb{C})$ with $f(z) \neq 0$ on $\mathbb{C}$. Then, there exists a $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = e^{h(z)}$.*

Proof. By hypothesis, $f'(z)/f(z)$ belongs to $\mathcal{H}(\mathbb{C})$ and therefore, it admits a Taylor series (about the origin) converging in the whole of $\mathbb{C}$:

$$\frac{f'(z)}{f(z)} = \sum_{k=0}^{\infty} a_k z^k \quad \text{for } z \in \mathbb{C}.$$

Define

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1} = g(z), \qquad \text{i.e. } \frac{f'(z)}{f(z)} = g'(z). \tag{4.41}$$

Since

$$\limsup_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{1/n} = \limsup_{n\to\infty} (n+1)^{-1/n}|a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} = 0,$$

the radius of convergence of the series (4.41) that represents the function g is infinity and therefore, $g \in \mathcal{H}(\mathbb{C})$. Set $H(z) = e^{g(z)}$. Then,

$$\frac{H'(z)}{H(z)} = g'(z) = \frac{f'(z)}{f(z)}, \qquad \text{i.e. } f(z)H'(z) - f'(z)H(z) = 0 \quad \text{for } z \in \mathbb{C},$$

which gives

$$\frac{d}{dz}\left(\frac{H(z)}{f(z)}\right) = 0 \quad \text{for } z \in \mathbb{C},$$

so that $H(z) = kf(z)$ for some constant k. Note that $H(0)/f(0) = k$, $k \neq 0$ and so there exists $a \in \mathbb{C}$ such that $e^a = k$. Thus, $H(z) = e^a f(z)$ so that $e^{g(z)-a} = f(z)$. Hence, $h(z) = g(z) - a$ is the desired function. ∎

4.5 Winding Number or Index of a Curve

Suppose that γ is a closed contour in $\mathbb{C}$. Let a be a given point in $\mathbb{C}\backslash\{\gamma\}$. Then, there is a useful formula that measures how often γ winds around a. For example if $\gamma : \gamma(t) = \{z : z - a = re^{it},\ 0 \leq t \leq 2k\pi\}$, then γ encircles the point a k times (counterclockwise). Further,

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2k\pi} \frac{ire^{it}}{re^{it}} dt = 2k\pi i, \quad \text{i.e. } \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = k.$$

From this we also observe that if γ encircles the point a k-times in the clockwise direction, then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = -k.$$

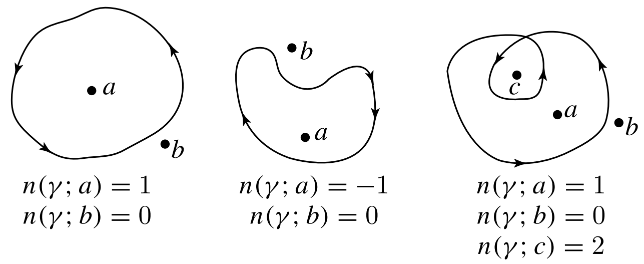


Figure 4.15: Description for winding number.

In either case, $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer. Here is the analytic definition of the winding number of a, which captures the intuitive notion of "the number of times γ wraps around a in the counterclockwise direction" (see Figure 4.15):

4.42. Definition. Let γ be a closed contour in $\mathbb{C}$ that avoids a point $a \in \mathbb{C}$. The index (or winding number) of γ about a, denoted by $n(\gamma; a)$ or $\mathrm{Ind}\,(\gamma; a)$, is given by the integral

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

Actually, from our later discussion, Cauchy's theorem will imply that $n(\gamma; a) = n(\gamma_0; a)$ for all closed curves γ_0 that are homotopic to γ as closed curves in $\mathbb{C}\setminus\{a\}$.

In the following we will collect some properties of the index $n(\gamma; a)$.

4.43. Theorem. *For every closed contour γ in $\mathbb{C}$ and $a \in \mathbb{C}\setminus\gamma$, $n(\gamma; a)$ is an integer.*

Proof. If the parametric interval of γ is $[0,1]$, then $\gamma(0) = \gamma(1)$. Consider the functions $g : [0,1] \to \mathbb{C}$ and $h : [0,1] \to \mathbb{C}$, defined by

$$g(t) = \int_0^t \frac{\gamma'(s)\,ds}{\gamma(s) - a} \quad \text{and} \quad h(t) = (\gamma(t) - a)e^{-g(t)}, \tag{4.44}$$

respectively. Then, $g(0) = 0$ and g is continuous on $[0,1]$. Likewise h is continuous on $[0,1]$. Moreover, for $t \in [0,1]$, g is piecewise smooth and has the derivative

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

at every point t where $\gamma'(t)$ is continuous. Consequently, h has the derivative

$$h'(t) = [\gamma'(t) - g'(t)(\gamma(t) - a)]e^{-g(t)} = 0$$

at every point t where $\gamma'(t)$ is continuous. Because γ is piecewise smooth, $h'(t) = 0$ fails to hold only at a finite number points in the interval $[0, 1]$. Thus, by the continuity of h, it follows that h must reduce to a constant k on $[0, 1]$. In particular,

$$h(0) = h(1), \quad \text{or} \quad \gamma(0) - a = e^{-g(1)}(\gamma(1) - a).$$

Since $\gamma(1) = \gamma(0)$, the last equation means that $e^{-g(1)} = 1$. We conclude that $g(1) = 2\pi ik$ for some integer k. Hence, by (4.44), we have $n(\gamma; a) = k \in \mathbb{Z}$. ■

To avoid the little technicality, one could simply supply the proof of Theorem 4.43 just for smooth curves as it is easy. For convenience, we shall replace a in Theorem 4.43 by ζ and obtain

4.45. Theorem. *If γ is a closed contour in $\mathbb{C}$, then the mapping $\zeta \longmapsto n(\gamma; \zeta)$ is a continuous function of ζ at any point $\zeta \notin \gamma$.*

Proof. Let D be an open set containing no point of γ, $a \in D$, and $\delta = \text{dist}\,(a, \gamma)$. Then $\delta > 0$ and for $z \in \gamma$, $|z - a| \geq \delta$. Now, for all h with $|h| < \delta/2$, we have

$$|z - a - h| \geq |z - a| - |h| \geq \delta - \delta/2$$

and so, using the standard estimate for integrals (see Theorem 4.9(iv)), we have

$$\begin{aligned} |n(\gamma; a + h) - n(\gamma; a)| &= \left| \frac{1}{2\pi} \int_\gamma \left(\frac{1}{z - a - h} - \frac{1}{z - a} \right) dz \right| \\ &\leq \frac{|h|}{2\pi} \int_\gamma \left| \frac{1}{(z - a - h)(z - a)} \right| |dz| \\ &\leq \frac{|h|}{2\pi(\delta/2)\delta} L(\gamma) \end{aligned}$$

which tends to zero as h approaches 0, where $L(\gamma)$ denotes the length of γ. Hence, as a is arbitrary, $n(\gamma; \zeta)$ is a continuous function of ζ in $\mathbb{C}\backslash\{\gamma\}$. ■

Integer-valuedness and the continuity of $n(\gamma; \zeta)$ yield

4.46. Corollary. *The function $n(\gamma; \zeta)$, $\zeta \in \mathbb{C}\backslash\{\gamma\}$, is constant in the components of $\mathbb{C}\backslash\{\gamma\}$.*

4.47. Theorem. *We have $n(\gamma; \zeta) = 0$ in the unbounded component of the closed contour γ.*

Proof. Suppose that $\gamma \subset \Delta_R$. Then, for any ζ in the unbounded component of $\mathbb{C}\backslash\overline{\Delta}_R$, we have

$$|n(\gamma;\zeta)| = \left|\frac{1}{2\pi}\int_\gamma \frac{dz}{z-\zeta}\right| < \frac{1}{2\pi}\frac{L(\gamma)}{|\zeta|-R},$$

since $|z-\zeta| \geq |\zeta|-|z| > |\zeta|-R$. So $|n(\gamma;\zeta)| < 1$ for $|\zeta|$ sufficiently large (for instance for $|\zeta| > \frac{L(\gamma)}{2\pi}+R$). Since $|n(\gamma;\zeta)|$ is a non-negative integer and is constant for every ζ, it follows that $n(\gamma;\zeta) = 0$ for every ζ in the unbounded component of γ. ∎

Returning to the initial example where we had $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = k \in \mathbb{Z}$ whenever $\gamma(t) = a+re^{2\pi ikt}$, we can use this result to conclude that $n(\gamma;a) = k$ for $|z-a| < r$ and $n(\gamma;a) = 0$ for $|z-a| > r$.

The concept of the winding number is useful to characterize what is meant by the inside (interior) and the outside (exterior) of a closed curve γ, respectively, in the following way:

$$\text{Int}\,(\gamma) = \{z \notin \gamma : n(\gamma;z) \neq 0\} \text{ and } \text{Ext}\,(\gamma) = \{z \notin \gamma : n(\gamma;z) = 0\}.$$

Moreover, a closed curve $\gamma : [a,b] \to \mathbb{C}$ is said to be *positively oriented* if $n(\gamma;z) > 0$ for every z inside γ and is *negatively oriented* if $n(\gamma;z) < 0$ for every z outside γ. As we have seen, for the circle $\partial\Delta(z_0;r)$ about z_0, the positive orientation is counterclockwise. More precisely, we have following results which we state without proof.

4.48. Theorem. *If γ consists of finitely many closed contours $\gamma_1, \gamma_2, \dots, \gamma_k$ in $\mathbb{C}$, then for every $a \notin \gamma_k$ (i.e. not on any one of the γ_j),*

(i) $n(\gamma;a) = n(\gamma_1;a) + n(\gamma_2;a) + \cdots + n(\gamma_k;a)$

(ii) $n(-\gamma_1;a) = -n(\gamma_1;a)$.

4.6 Homotopy Version of Cauchy's Theorem

In this section we discuss more general conditions under which we may vary the curve γ continuously and so that $\int_\gamma f(z)\,dz$ unchanged when $f \in \mathcal{H}(D)$ and γ is contained in D. For this, we introduce the important notion of homotopy of continuous maps. We consider two curves which are parameterized by the same interval $[0,1]$ and formulate the following definition.

Let γ_0, γ_1 be two curves in a domain $D \subseteq \mathbb{C}$ with parametric interval $[0,1]$ having common initial and terminal points:

$$a = \gamma_0(0) = \gamma_1(0),\ b = \gamma_0(1) = \gamma_1(1).$$

The set of all curves in D which connect a and b is denoted by $\Gamma(D;a,b)$. If $a = b$, then we say that the curve is closed and a is called the base point.

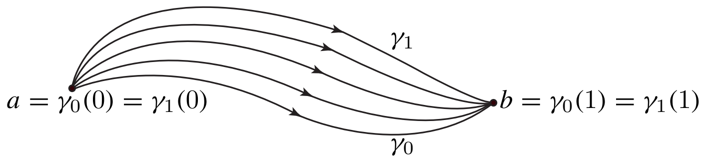


Figure 4.16: Description for homotopic curves.

$\Gamma(D; a, a)$ is then the family of all closed curves (loops in D with base point a).

4.49. Definition. Let γ_0 and $\gamma_1 \in \Gamma(D; a, b)$. We say that γ_0 and γ_1 are homotopic (or that γ_0 is homotopic to γ_1) with fixed end points if there exists a continuous map $F : [0,1] \times [0,1] \to \mathbb{C}$ such that

(i) $F(t, u) \in D$ for all $0 \leq t, u \leq 1$

(ii) $F(t, 0) = \gamma_0(t)$ and $F(t, 1) = \gamma_1(t)$ for all $0 \leq t \leq 1$

(iii) $F(0, u) = a$ and $F(1, u) = b$, for all $0 \leq u \leq 1$.

We write $\gamma_0 \simeq \gamma_1$ (or $F : \gamma_0 \simeq \gamma_1$) to indicate that γ_0 is homotopic to γ_1.

To have a better understanding of the definition, we write

$$\gamma_u(t) = F(t, u)$$

and say that F is a homotopy between γ_0 and γ_1. Note that, for each $u \in [0,1]$, $\gamma_u : [0,1] \to D$ is a continuous map with $\gamma_u(0) = a$ and $\gamma_u(1) = b$. At the start, $u = 0$ and $\gamma_u = \gamma_0$; as u varies, the map γ_u varies continuously so that at the end $u = 1$ we have $\gamma_u = \gamma_1$, i.e. γ_0 can be transformed continuously into γ_1 in D (see Figure 4.16). We observe the following:

(i) Definition 4.49(iii) means that the homotopy F fixes both initial and terminal points.

(ii) If $\gamma_1 \in \Gamma(D; a, b)$ is a closed curve (i.e. $a = \gamma_1(0) = \gamma_1(1) = b$), $\gamma_0(t) \equiv a$ for all $t \in [0,1]$ (i.e. γ_0 is a constant curve) and $\gamma_0 \simeq \gamma_1$, then we say that the curve γ_1 can be continuously deformed into the point a. The point a is called the *base point* of γ_1. In other words, if γ is a closed curve with base point a then γ is said to be *homotopic to* 0 in D, or simply *null-homotopic* (written as $\gamma \simeq 0$) if $\gamma \simeq \gamma_0$. The family of closed curves in D with base point a is denoted by $\Gamma(D; a)$.

(iii) Homotopy is an equivalence relation in $\Gamma(D; a, b)$. Clearly $\gamma_0 \simeq \gamma_0$. For $\gamma_0, \gamma_1 \in \Gamma(D; a, b)$, we have

$$\gamma_0 \simeq \gamma_1 \Longrightarrow \gamma_1 \simeq \gamma_0.$$

To check this, let $F : \gamma_0 \simeq \gamma_1$ be a homotopy carrying γ_0 into γ_1. Then, H defined by $H(t,u) = F(t, 1-u)$ is a homotopy carrying γ_1 into γ_0, i.e. $H : \gamma_1 \simeq \gamma_0$.

For $\gamma_0, \gamma_1, \gamma_2 \in \Gamma(D; a, b)$, we have

$$\gamma_0 \simeq \gamma_1 \text{ and } \gamma_1 \simeq \gamma_2 \Longrightarrow \gamma_0 \simeq \gamma_2.$$

To verify this, let $F : \gamma_0 \simeq \gamma_1$ and $G : \gamma_1 \simeq \gamma_2$. Define H by

$$H(t,u) = \begin{cases} F(t, 2u) & \text{if } 0 \leq u \leq 1/2 \\ G(t, 2u-1) & \text{if } 1/2 \leq u \leq 1. \end{cases}$$

Then, $H : \gamma_0 \simeq \gamma_2$. This proves our claim.

4.50. Definition. A domain D is said to be *simply connected* if every closed curve in D is homotopic to a point in D.

If D is a simply connected domain, then for any pair $\gamma_0, \gamma_1 \in \Gamma(D; a, b)$ we have $\gamma_0 \simeq \gamma_1$ by (iii) above, whereas any closed curve in D is homotopic to zero.

Suppose D is starlike with respect to a, and γ_1 be a closed curve in D. Putting

$$F(t,u) = u\gamma_1(t) + (1-u)a, \ u \in [0,1],$$

we see that $F(t,u)$ is a continuous function defined on the rectangle $\{(t,u) : t, u \in [0,1]\}$ such that $F(t,u) \in D$. Also

$$F(t,0) = \gamma_0(t), \quad F(t,1) = \gamma_1(t), \quad \text{for all } 0 \leq t \leq 1$$

$$F(0,u) = u\gamma_1(0) + (1-u)a = ua + (1-u)a = a$$

and

$$F(1,u) = u\gamma_1(1) + (1-u)a = ua + (1-u)a = a,$$

since $\gamma_1 \in \Gamma(D; a)$. So, $\gamma_1 \simeq \gamma_0$ in D and thus D is simply connected.

If D is a convex domain then it is starlike and so it is simply connected. Therefore, if γ_0 and γ_1 are any two closed curves in $\Gamma(D; a)$ then $\gamma_0 \simeq \gamma_1$ in D.

4.51. Definition. Let D be an open set and γ_0 and γ_1 be two curves defined on $[0,1]$. We say that γ_0 and γ_1 are *close together* if there exists a partition $\mathcal{P}$ of $[0,1]$, $\mathcal{P} : \ 0 = t_0 < t_1 < \cdots < t_n = 1$, and a sequence of disks D_j, $j = 0, 1, \dots, n-1$, contained in D such that for each $j = 0, 1, \dots, n-1$, D_j contains the images $\gamma_0([t_j, t_j + 1])$ and $\gamma_1([t_j, t_j + 1])$.

Let γ_0 and γ_1 be closed curves in D that are close together. Let $\gamma_0(t_j) = z_j$, $\gamma_1(t_j) = \zeta_j$. Consider $f \in \mathcal{H}(D)$ and let g_j be a primitive of f on D_j,

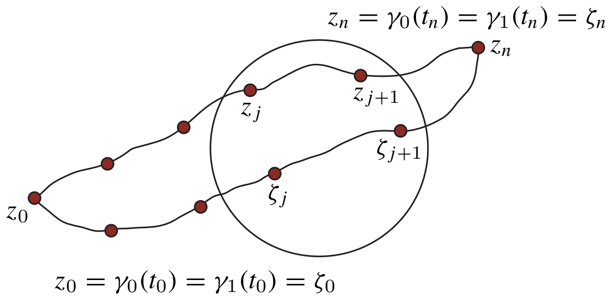


Figure 4.17: Description for close together curves.

which exists by Theorem 4.31. Further, if γ_0 and γ_1 are closed curves of class C^1 and are close together, then

$$I_0 = \int_{\gamma_0} f(z)\,dz = \sum_{j=0}^{n-1} [g_j(z_{j+1}) - g_j(z_j)]$$

and

$$I_1 = \int_{\gamma_1} f(z)\,dz = \sum_{j=0}^{n-1} [g_j(\zeta_{j+1}) - g_j(\zeta_j)]$$

so that $I_0 - I_1 = [g_{n-1}(z_n) - g_{n-1}(\zeta_n)] - [g_0(z_0) - g_0(\zeta_0)]$. Since γ_0 and γ_1 are closed and since the primitives g_{n-1} and g_0 differ by a constant, $I_0 - I_1 = 0$ and hence $I_0 = I_1$.

The following theorem summarizes the above discussion.

4.52. Theorem. *Let D be an open set in $\mathbb{C}$ and γ_0 and γ_1 be two closed contours in D. Suppose that γ_0 and γ_1 are close together. Then, for each $f \in \mathcal{H}(D)$, we have $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.*

Let $\gamma : [0,1] \to D$ be a curve in an open set D. Then, $\gamma([0,1])$ is compact. Define

$$\phi(t) = \min_{\zeta \in \mathbb{C} \setminus D} |\gamma(t) - \zeta|.$$

Then, $\phi(t) > 0$ on $[0,1]$. Clearly ϕ is continuous and ϕ has a minimum on $[0,1]$. Further, γ is uniformly continuous on $[0,1]$. Let $R = \min_{0 \le t \le 1} \phi(t)$. Therefore, for $\epsilon = R/2 > 0$ there exists a $\delta > 0$ such that

$$|\gamma(t) - \gamma(u)| < R/2 \quad \text{whenever} \quad |t-u| < \delta \quad (t, u \in [0,1]).$$

Choose points $t_0 = 0 < t_1 < \cdots < t_n = 1$ in $[0,1]$ such that $|t_j - t_{j+1}| < \delta$. Then (see Figure 4.17),

$$\gamma([t_j, t_{j+1}]) \subset \Delta(\gamma(t_j); R/2) = D_j \subset D.$$

For all such j, let $\gamma_j : [t_j, t_{j+1}] \to D$ be the restriction of γ to $[t_j, t_{j+1}]$. Since the disk $\Delta(\gamma(t_j); R/2)$ is convex and lies in D, by Theorem 4.31, we have $\left(\int_\gamma - \sum_{j=1}^{n-1} \int_{\gamma_j}\right) f(z)\,dz = 0$ for each analytic function f on D. That is,

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n-1} \int_{\gamma_j} f(z)\,dz. \tag{4.53}$$

This also follows from the properties of the Riemann integral. Note also that, by Theorem 4.31, there exists an F_j on D_j such that $F_j'(z) = f(z)$ on D_j. Thus, if γ_j is of class C^1, (4.53) and Corollary 4.18 give

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \sum_{j=1}^{n-1} \int_{\gamma_j} f(\gamma_j(t))\gamma_j'(t)\,dt \\
&= \sum_{j=1}^{n-1} \int_{\gamma_j} F_j'(\gamma_j(t))\gamma_j'(t)\,dt \\
&= \sum_{j=1}^{n-1} \int_{\gamma_j} d[F_j(\gamma_j(t))] \\
&= \sum_{j=1}^{n-1} [F_j(z_{j+1}) - F_j(z_j)]
\end{aligned}$$

where $\gamma(t_j) = z_j$.

Our next result is the famous Cauchy's theorem.

4.54. Theorem. (Homotopy Version of Cauchy's Theorem) *Let D be domain in $\mathbb{C}$ and γ_0 and γ_1 be two closed contours in D such that $\gamma_0 \simeq \gamma_1$ in D. Then, for each $f \in \mathcal{H}(D)$, we have $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.*

Proof. Let $F : \gamma_0 \simeq \gamma_1$ be a homotopy in D. Since F is continuous on the square $R = [0,1] \times [0,1]$ which is compact, the image $F(R)$ is compact and F is uniformly continuous on R. Hence, $F(R)$ has a positive distance from $\mathbb{C} \backslash D$. Choose partitions

$$\begin{aligned}
u_0 = 0 < u_1 < \cdots < u_m = 1 \\
t_0 = 0 < t_1 < \cdots < t_n = 1
\end{aligned}$$

and let $R_{jk} = [t_j, t_{j+1}] \times [u_k, u_{k+1}]$ ($j = 0, 1, \ldots, n-1$; $k = 0, 1, \ldots, m-1$), a rectangle. Then, $F(R_{jk}) = D_{jk} \subset D$. Define Γ_k by

$$\Gamma_k(t) = F(t, u_k),\ k = 0, 1, \ldots, m.$$

Then, Γ_k's are continuous and the curves Γ_k, Γ_{k+1} are close together. By Theorem 4.52,

$$\int_{\Gamma_k} f(z)\,dz = \int_{\Gamma_{k+1}} f(z)\,dz,\ k = 0, 1, \ldots, m-1.$$

As $\Gamma_0 = \gamma_0$ and $\Gamma_m = \gamma_1$, the desired equality follows. ■

4.55. Corollary. *If D is a simply connected domain in $\mathbb{C}$, then, for any $f \in \mathcal{H}(D)$ and any closed contour γ in D, we have $\int_\gamma f(z)\,dz = 0$.*

Proof. Let γ_0 be a constant curve. Then, $\gamma_0'(t) = 0$ and so $\int_{\gamma_0} f(z)\,dz = 0$. The result follows from Theorem 4.54 upon taking $\gamma_1 = \gamma$. ■

Another version (see also Corollary 4.61) of Corollary 4.55 is the following.

4.56. Corollary. *If D is a simply connected domain in $\mathbb{C}$, then, for any $f \in \mathcal{H}(D)$, and $\gamma_0, \gamma_1 \in \Gamma(D; a, b)$ in D, we have $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.*

Proof. Using Corollary 4.55, we obtain $\int_{\gamma_0 - \gamma_1} f(z)\,dz = 0$ and the desired conclusion now follows. ■

4.57. Corollary. *If γ_0 and γ_1 are two closed contours in a domain D of $\mathbb{C}$ such that $\gamma_0 \simeq \gamma_1$ then, for each $a \in \mathbb{C}\backslash D$, we have $n(\gamma_0; a) = n(\gamma_1; a)$.*

4.58. Example. Let γ_0 and γ_1 be defined by

$$\gamma_0(t) = e^{2\pi i t} \text{ and } \gamma_1(t) = e^{-2\pi i t},\ t \in [0, 1].$$

Then, γ_0 and γ_1 are the circle $|w| = 1$. Define $F(t, u) = e^{2\pi i t} - 2iu \sin(2\pi t)$. Then, $F : \gamma_0 \simeq \gamma_1$ in $D = \mathbb{C}$.

Further, if $D = \mathbb{C}\backslash\{0\}$ and $f(z) = 1/z$ then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_0} \frac{1}{z}\,dz = \int_0^1 \frac{1}{e^{2\pi i t}} 2\pi i e^{2\pi i t}\,dt = 2\pi i$$

so that $n(\gamma_0; 0) = 1$. Similarly we see that $n(\gamma_1; 0) = -1$. Thus, γ_0 and γ_1 are not homotopic in $D = \mathbb{C}\backslash\{0\}$. ●

4.59. Example. Consider two circles

$$\gamma_0(t) = Re^{2\pi i t} \text{ and } \gamma_1(t) = (R + 1)e^{2\pi i t} \ (t \in [0, 1]),$$

where $R > 0$ is fixed. Define $F(t, u) = (R + u)e^{2\pi i t}$. Then, for $t, u \in [0, 1]$,

$$F(t, 0) = \gamma_0(t),\ F(t, 1) = \gamma_1(t) \text{ and } F(0, u) = F(1, u) = R + u.$$

Further, we also have $|F(t, u)| \le R + 1$ for all $u \in [0, 1]$. Therefore, F is a homotopy of γ_0 to γ_1 in any region D containing $\overline{\Delta}_{R+1}$. ●

4.60. Example. Let $\gamma_0(t) = 2t$ and $\gamma_1(t) = 1 + e^{\pi i(1-t)}$, $t \in [0,1]$. Define $F(t,u) = (1-u)\gamma_0(t) + u\gamma_1(t)$. Then, $|F(t,u)| \le 2$ for all $u \in [0,1]$. Clearly, $F : \gamma_0 \simeq \gamma_1$ in any region D containing $\overline{\Delta}_2$. ●

Next we give an alternate proof of Theorem 4.33 in the following form.

4.61. Corollary. *If f is analytic in a simply connected domain D, then there is a function F in D such that $F'(z) = f(z)$, and F is unique up to an additive constant.*

Proof. Let $z_0 \in D$. For each $z \in D$, let γ_z be a curve in D from z_0 to z. Define F by

$$F(z) = \int_{\gamma_z} f(\zeta)\, d\zeta.$$

By Corollary 4.57, the value of $F(z)$ is independent of the choice of γ_z. Therefore, for any contour γ in D from z_1 to z_2 we have

$$F(z_2) - F(z_1) = \int_{\gamma} f(\zeta)\, d\zeta. \tag{4.62}$$

In particular, (4.62) holds for any line segment in D connecting z_1 and z_2. That is

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(\zeta)\, d\zeta.$$

Here z_2 is sufficiently close to z_1 so that $[z_1, z_2] \subset D$. It follows, from the method of proof of Theorem 4.31 (take $z = z_1$, $z + h = z_2$ in Theorem 4.31), that $F'(z) = f(z)$ in D.

Now let G be any other function in D such that $G'(z) = f(z)$. Applying Corollary 4.18, it follows that for any points z_1, z_2 in D and any contour γ in D from z_1 to z_2, we have

$$F(z_2) - F(z_1) = \int_{\gamma} F'(z)\, dz = \int_{\gamma} f(z)\, dz = \int_{\gamma} G'(z)\, dz = G(z_2) - G(z_1)$$

and therefore, $F - G$ is constant in D. ∎

4.7 Cauchy Integral Formula

The Cauchy integral formula expresses a remarkable fact about an analytic function. Its values everywhere inside a simple closed contour are completely determined by its values on the boundary. The integral representation allows us to show that analytic functions are infinitely differentiable. In fact the values of each derivative of an analytic function are determined just by the values of the function on the boundary. Later in Section 4.10, we use the integral representation to obtain power series expansions for analytic functions.

4.63. Theorem. (Cauchy Integral Formula) *If D is a simply connected domain and γ is a closed contour in D, then for $f \in \mathcal{H}(D)$ and $a \in D \setminus \{\gamma\}$,*

$$f(a)n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a}\, d\zeta. \tag{4.64}$$

Proof. Let $f \in \mathcal{H}(D)$ and consider

$$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{for } z \neq a \\ f'(a) & \text{for } z = a. \end{cases}$$

Then, F is analytic in $D \setminus \{a\}$ and continuous on D. Hence, by Corollary 4.32, $\int_C F(\zeta)\, d\zeta = 0$ for every closed contour C in some disk $\Delta(z_0; \delta) \subseteq D$ with $a \in \Delta(z_0; \delta)$. If we rearrange this, then we obtain

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta) - f(a)}{\zeta - a}\, d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - a}\, d\zeta - f(a)n(C; a) = 0.$$

Thus, we have proved a local version of the Cauchy integral formula. The general version follows once we prove that $F \in \mathcal{H}(D)$. This can be done with the same idea used in Theorem 4.30. ∎

Most of the applications of the above theorem are in the case when f is analytic inside and on γ where γ is a simple closed contour and a is inside γ. We have then $n(\gamma; a) = 1$, so that (4.64) becomes

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a}\, d\zeta.$$

From the formula given by (4.64), in the case of a point a for which $n(\gamma; a) \neq 0$, it follows that "the value of f at an interior point" is determined by "its boundary values." This fact has many valuable consequences in complex analysis, which we shall soon see in a number of examples.

4.65. Example. Using the Cauchy integral formula for simple closed contour (i.e. when $n(\gamma; a) = 1$) we have

(a) $\displaystyle\int_{|z|=1} \frac{\sin z}{z}\, dz = 2\pi i (\sin z)\Big|_{z=0} = 0$

(b) $\displaystyle\int_{|z|=1} \frac{\cos z}{z(z-4)}\, dz = 2\pi i \frac{\cos z}{z-4}\Big|_{z=0} = -\frac{\pi}{2} i$

(c) $\displaystyle\int_{|z-a|=1} \frac{e^{2\pi z}}{z - a}\, dz = 2\pi i e^{2\pi a}$

(d) $\displaystyle\int_{|z|=2} \frac{e^z + z^2}{z - 1}\, dz = 2\pi i [e + 1]$

(e) If $f(z) = \sin(\pi z^2) + \cos(\pi z^2)$, then by Cauchy's deformation of contour

$$\begin{aligned}\int_{|z|=4} \frac{f(z)}{(z-2)(z-3)}\,dz &= \int_{|z-2|=1/2} \frac{f(z)/(z-3)}{z-2}\,dz \\ &\quad + \int_{|z-3|=1/2} \frac{f(z)/(z-2)}{z-3}\,dz \\ &= 2\pi i(-f(2)+f(3)) = 0.\end{aligned}$$

(f) By Cauchy's deformation of contour

$$\begin{aligned}\int_{|z|=4} \frac{e^z}{z(z-1)}\,dz &= \int_{|z|=1/2} \frac{e^z/(z-1)}{z}\,dz + \int_{|z-1|=1/2} \frac{e^z/z}{z-1}\,dz \\ &= 2\pi i(-1+e).\end{aligned}$$

(g) $\displaystyle\int_{|z|=1} \frac{\cos(e^z)}{z}\,dz = 2\pi i \cos 1 = \pi i(e^i + e^{-i}).$ •

Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[a,b]$ and $F'(x) = f(x)$ on $[a,b]$. Then, the "mean/average value" of $f(x)$ on $[a,b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{F(b)-F(a)}{b-a}.$$

In view of the mean-value theorem, the right hand side expression is equal to $F'(c)$ (which is now $f(c)$ because of our assumption) for some $c \in (a,b)$.

As an immediate special case of Theorem 4.63, we have an important result which shows that for functions analytic inside and on a circle, the average of the values on the circumference is equal to the value of the function at the center of the circle.

4.66. Theorem. (Gauss's Mean-Value Theorem) *If D is a domain and $\overline{\Delta}(z_0;r) \subset D$, then for $f \in \mathcal{H}(D)$ and $a \in \Delta(z_0;r)$,*

$$f(a) = \frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} \frac{f(\zeta)}{\zeta - a}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

where $\rho > 0$ is such that $\overline{\Delta}(a;\rho) \subseteq \overline{\Delta}(z_0;r)$.

Proof. By hypothesis, $n(\gamma;a) = 1$ with $\gamma = \partial\Delta(z_0;r)$. Let $\zeta - a = \rho e^{i\theta}$. Then, $d\zeta = i\rho e^{i\theta}\,d\theta$ and the result follows. ∎

If $f \in \mathcal{H}(\Delta)$ and L is the length of the image curve Γ_r of $\partial\Delta_r$ $(0 < r < 1)$ under f, then it is easy to see that $L \geq 2\pi r|f'(0)|$. Indeed,

$$\begin{aligned} L = \int_{\Gamma_r} |dw| &= \int_{\partial\Delta_r} |f'(z)|\,|dz| \\ &= r\int_0^{2\pi} |f'(re^{i\theta})|\,d\theta \\ &\geq r\left|\int_0^{2\pi} f'(re^{i\theta})\,d\theta\right| \\ &= 2\pi r|f'(0)|, \quad \text{by the mean value property.} \end{aligned}$$

4.67. Example. Let us evaluate the integral $I = \int_{|z|=r} \mathrm{Re}\, z\, dz$. On the circle $|z| = r$, we have $z = re^{i\theta}$ and $z\overline{z} = r^2$ so that $|dz| = rd\theta = rdz/iz$ and

$$\mathrm{Re}\, z = \frac{z + \overline{z}}{2} = \frac{1}{2}\left(z + \frac{r^2}{z}\right).$$

Using this simple observation, the given integral may be rewritten as

$$I = \frac{1}{2}\int_{|z|=r} z\, dz + \frac{r^2}{2}\int_{|z|=r} \frac{dz}{z}.$$

Observe that the first integral vanishes by Cauchy's theorem while the value of second integral is $2\pi i$, by Cauchy's integral formula. Thus, the value of the given integral I is $i\pi r^2$. Similarly, we obtain the following:

$$\int_{|z|=r} \mathrm{Im}\, z\, dz = -\pi r^2 \quad \text{and} \quad \int_{|z|=r} \mathrm{Im}\, z\, |dz| = \frac{r}{2i}\int_{|z|=r}\left(z - \frac{r^2}{z}\right)\frac{dz}{iz} = 0.$$

Finally, on the circle $|z - a| = r$, we have $(z - a)(\overline{z} - \overline{a}) = r^2$ so that $\overline{z} = \overline{a} + r^2/(z - a)$ and therefore,

$$\int_{|z-a|=r} \overline{z}\, dz = \overline{a}\int_{|z-a|=r} dz + r^2\int_{|z-a|=r} \frac{dz}{z - a} = 2\pi i r^2. \qquad \bullet$$

4.68. Example. Let f be analytic inside and on the ellipse C given by

$$\left\{z = x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \quad a \geq b > 0.$$

Suppose that

(i) $|f(z)| \leq M_1$ for $z \in C_1 = \{z \in C : \mathrm{Re}\, z \geq 0\}$ and,

(ii) $|f(z)| \leq M_2$ for $z \in C_2 = \{z \in C : \mathrm{Re}\, z < 0\}$.

Using the Cauchy integral formula, let us find an upper bound for $|f(0)|$. To do this, we consider an auxiliary function $\phi(z)$ defined by

$$\phi(z) = f(z)f(-z) \quad \text{for} \quad z \in C = C_1 \cup C_2.$$

Clearly, $\phi(z)$ is analytic inside and on the ellipse C. First we observe the following:

(i) $z \in C \iff \{z : |z - \sqrt{a^2-b^2}| + |z + \sqrt{a^2-b^2}| = 2a\}$, see Example 1.33.

(ii) $z \in C_1$ iff $-z \in C_2$, whenever $\operatorname{Re} z > 0$ and $z \in C$. This observation shows that $|\phi(z)| = |f(z)f(-z)| \leq M_1M_2$ for $z \in C_j$ $(j = 1, 2)$. Note that $\operatorname{Re} z = 0$ iff $\operatorname{Re}(-z) = 0$, and so $|\phi(z)| < M_1^2$ when $\operatorname{Re} z = 0$ but $z \neq 0$.

(iii) For $z \in C$, we have $b \leq |z| \leq a$.

By the Cauchy integral formula,

$$\phi(0) = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{z}\,dz = \frac{1}{2\pi i}\left(\int_{C_1} \frac{\phi(z)}{z}\,dz + \int_{C_2} \frac{\phi(z)}{z}\,dz\right).$$

Therefore, if $L(\gamma)$ denotes the length of the curve γ then $L(C) = L(C_1) + L(C_2)$ and we obtain the estimate

$$\begin{aligned} |f(0)|^2 = |\phi(0)| &\leq \frac{1}{2\pi}\left(\left|\int_{C_1} \frac{\phi(z)}{z}\,dz\right| + \left|\int_{C_2} \frac{\phi(z)}{z}\,dz\right|\right) \\ &\leq \frac{1}{2\pi}\left(\frac{M_1M_2}{b}L(C_1) + \frac{M_1M_2}{b}L(C_2)\right) \\ &= \frac{M_1M_2L(C)}{2\pi b} \end{aligned}$$

which gives that

$$|f(0)| \leq \sqrt{\frac{M_1M_2L(C)}{2\pi b}}.$$

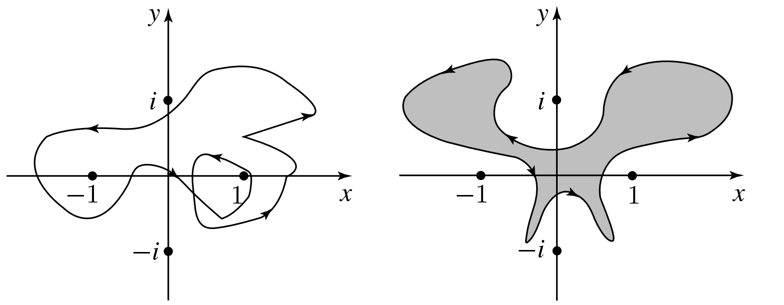
What happens when $a = b$? In this special case, the ellipse becomes the circle $|z| = a$. So, $L(C) = 2\pi a$ and $|f(0)| \leq \sqrt{M_1M_2}$. ●

4.69. Example. Let us discuss the process of evaluating the integral

$$I = \int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz,$$

where closed contour γ is yet to be specified. Using the partial fraction decomposition, we can rewrite

$$I = \frac{1}{2}\int_\gamma \frac{\operatorname{Arctan} z}{z-1}\,dz - \frac{1}{2}\int_\gamma \frac{\operatorname{Arctan} z}{z+1}\,dz.$$



(i) γ encloses both -1 and 1 (ii) γ does not enclose both -1 and 1

Figure 4.18: Illustration for winding number.

Here $\text{Arctan}\, z$ is the principal branch of the inverse tangent function defined by

$$\text{Arctan}\, z = -\frac{i}{2}\,\text{Log}\left(\frac{1+iz}{1-iz}\right).$$

What is the region of analyticity of $\text{Arctan}\, z$? To find this we need to remove those points $z \in \mathbb{C}$ for which

$$\frac{1+iz}{1-iz} = u, \quad u \in \mathbb{R} \text{ with } u \leq 0.$$

Note that

$$\frac{1+iz}{1-iz} = \frac{(1+iz)(1+i\overline{z})}{|1-iz|^2} = \frac{1-|z|^2+2i\text{Re}\, z}{|1-iz|^2}$$

and therefore, the points to be removed from $\mathbb{C}$ are those points z for which $\text{Re}\, z = 0$ and $|z| \geq 1$. This gives $x = 0$ and $|y| \geq 1$ and thus, $\text{Arctan}\, z$ is analytic on $\mathbb{C} \setminus \{iy : y \in \mathbb{R},\ |y| \geq 1\}$. Now, γ could be specified as any closed contour which does not touch the two slits. For example,

(i) if γ is as in Figure 4.18(i), then

$$I = 2^{-1}[2\pi i\, n(\gamma;1)f(1) - 2\pi i\, n(\gamma;-1)f(-1)]$$

where $f(z) = \text{Arctan}\, z$. We compute

$$f(1) = -\frac{i}{2}\,\text{Log}\left(\frac{1+i}{1-i}\right) = -\frac{i}{2}\,\text{Log}\, i \quad \text{and} \quad f(-1) = -\frac{i}{2}\,\text{Log}\,(-i).$$

Therefore,

$$I = \pi i\left[2\left(\frac{\pi}{4}\right) - 1\left(-\frac{\pi}{4}\right)\right] = i\frac{3\pi^2}{4}.$$

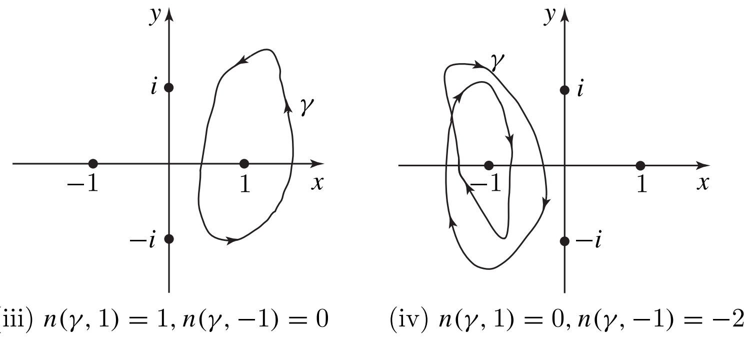


Figure 4.19: Curve encloses either 1 or -1.

(ii) if γ is as in Figure 4.18(ii), i.e. γ does not enclose both -1 and 1, then

$$\int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz = 0.$$

(iii) if γ encloses only one of the two points 1 and -1, namely, the point 1 (see Figure 4.19(i)) then

$$\int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz = \int_\gamma \frac{g(z)}{z-1}\,dz, \quad g(z)=\frac{\operatorname{Arctan} z}{z+1}.$$

We compute $g(1)=(1/2)\operatorname{Arctan} 1 = \pi/8$ and therefore, in this case,

$$\int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz = 2\pi i\, n(\gamma;1)g(1) = 2\pi i\frac{\pi}{8} = \frac{i\pi^2}{4}.$$

(iv) Similarly, if γ encloses only the point $z=-1$, then we write

$$\int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz = 2\pi i\, n(\gamma,1)\,\phi(-1), \quad \phi(z)=\frac{\operatorname{Arctan} z}{z-1}.$$

Since $n(\gamma,-1)=-2$ and $\phi(-1)=-(1/2)\operatorname{Arctan}(-1)=\pi/8$, it follows that (see Figure 4.19(ii))

$$\int_\gamma \frac{\operatorname{Arctan} z}{z^2-1}\,dz = 2\pi i(-2)\left(\frac{\pi}{8}\right) = -\frac{i\pi^2}{2}.$$ ●

For the proof of our next theorem, we need the following:

4.70. Theorem. *Let ϕ be a complex-valued function which is continuous in an open set D containing a contour γ. Then, for all $z\notin\gamma$, the function F_n defined by*

(4.71) $$F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^n}\,d\zeta, \quad n=1,2,\ldots,$$

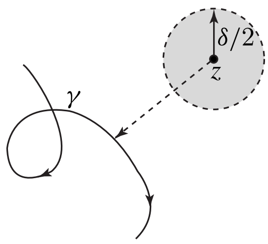


Figure 4.20: A disk $\Delta(z;\delta)$ disjoint from $\gamma([a,b])$.

is analytic and satisfies the equation $F_n'(z) = nF_{n+1}(z)$*; or equivalently*

$$F^{(k)}(z) = k!\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta, \quad k=1,2,\dots, \tag{4.72}$$

with $F_1(z) = F(z)$.

Proof. Let $\gamma:\ \gamma(t)$, $0\le t\le 1$, be a given contour in D. Then, γ being continuous on $[0,1]$, $\gamma([0,1])$ is compact. The function ϕ being continuous on $\gamma([0,1])$, there exists $M>0$ such that

$$|\phi(\zeta)| \le M \quad \text{on} \quad \gamma([0,1]).$$

For $z\notin\gamma$, let $\Delta(z;\delta)$ (e.g. choose $\delta = \operatorname{dist}(z,\gamma)$) be a disk about z disjoint from $\gamma([0,1])$ so that $|\zeta - z|\ge\delta$ for ζ on the curve γ. Suppose further that $a\in\Delta(z;\delta/2)$ and

$$|h| \le \frac{1}{2}\left[\frac{\delta}{2} - |a-z|\right] = \delta_0, \quad \text{say} .$$

Then, as

$$|a+h-z| \le |a-z| + |h| \le \frac{\delta}{4} + \frac{|a-z|}{2} < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2},$$

$a+h\in\Delta(z;\delta/2)$ (see Figure 4.20) and so (with $F_1(z)=F(z)$), we have

$$\frac{F(a+h)-F(a)}{h} = \int_\gamma \frac{\phi(\zeta)}{(\zeta-a)(\zeta-a-h)}\,d\zeta.$$

Hence,

$$\frac{F(a+h)-F(a)}{h} - \int_\gamma \frac{\phi(\zeta)}{(\zeta-a)^2}\,d\zeta = h\int_\gamma \frac{\phi(\zeta)}{(\zeta-a)^2(\zeta-a-h)}\,d\zeta. \tag{4.73}$$

For ζ on γ, we note that $|\zeta-a|\ge|\zeta-z|-|a-z|\ge\delta-\delta/2=\delta/2$ and

$$|\zeta-a-h| \ge |\zeta-a| - |h| \ge \frac{\delta}{2} - \frac{1}{2}\left[\frac{\delta}{2} - |a-z|\right] = \frac{\delta}{4} + \frac{|a-z|}{2} \ge \frac{\delta}{4}.$$

Therefore, using the standard estimate for integrals (see Theorem 4.9(iv)), it follows that the modulus of the right hand side of (4.73) is less than or equal to

$$\frac{|h|ML(\gamma)}{(\delta/2)^2(\delta/4)}$$

which clearly approaches zero as $h \to 0$. Since a was an arbitrary point of D, this implies that F_1 $(= F)$, defined by (4.71), is analytic for all $z \in D \setminus \{\gamma\}$ and that

$$F'(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - a)^2}\, d\zeta, \quad \text{i.e.} \quad F_1'(z) = 1 \cdot F_2(z).$$

Now the case when $n = 1$, i.e. $k = 1$ in (4.72), follows. Therefore, to prove the general case we proceed by induction on n. Assume that the result holds for some value of $k \geq 1$. The integral formula (4.72) for $F^{(k)}(z)$ then yields the expression

$$F^{(k)}(a+h) - F^{(k)}(a) = k! \int_\gamma \left[\frac{1}{(\zeta - a - h)^{k+1}} - \frac{1}{(\zeta - a)^{k+1}}\right] \phi(\zeta)\, d\zeta.$$

Now, we must show that (4.72) holds for $F^{(k+1)}(a)$. To do this, since

$$(k+1) \int_{[a,a+h]} \frac{du}{(\zeta - u)^{k+2}} = \frac{1}{(\zeta - a - h)^{k+1}} - \frac{1}{(\zeta - a)^{k+1}},$$

we write

$$\begin{aligned}
&\frac{F^{(k)}(a+h) - F^{(k)}(a)}{h} - (k+1)! \int_\gamma \frac{\phi(\zeta)}{(\zeta - a)^{k+2}}\, d\zeta \\
&\quad = (k+1)! \int_\gamma \left[\frac{1}{h} \int_{[a,a+h]} \frac{du}{(\zeta - u)^{k+2}} - \frac{1}{(\zeta - a)^{k+2}}\right] \phi(\zeta)\, d\zeta \\
&\quad = \frac{(k+1)!}{h} \int_\gamma \left[\int_{[a,a+h]} \left(\frac{1}{(\zeta - u)^{k+2}} - \frac{1}{(\zeta - a)^{k+2}}\right) du\right] \phi(\zeta)\, d\zeta \\
&\quad = \frac{(k+2)!}{h} \int_\gamma \left[\int_{[a,a+h]} \left(\int_{[a,u]} \frac{dv}{(\zeta - v)^{k+3}}\right) du\right] \phi(\zeta)\, d\zeta \\
&\quad = M(h), \quad \text{say.}
\end{aligned}$$

Now $v \in [a, u] \subset [a, a+h]$ and so for all ζ on γ,

$$|\zeta - v| \geq |\zeta - a| - |v - a| \geq |\zeta - a| - |h| \geq \frac{\delta}{2} - \frac{1}{2}\left[\frac{\delta}{2} - |a - z|\right] = \frac{\delta}{4} + \frac{|a - z|}{2}$$

and therefore, $|\zeta - v| \geq \delta/4$. Hence

$$|M(h)| \leq \frac{(k+2)!}{|h|} M \cdot L(\gamma). \frac{|h| \cdot |h|}{(\delta/4)^{k+3}},$$

which approaches zero as $h \to 0$. Hence, $F^{(k+1)}$ exists at a and is given by

$$F^{(k+1)}(a) = (k+1)! \int_\gamma \frac{\phi(\zeta)}{(\zeta - a)^{k+2}}\, d\zeta, \quad k = 1, 2, \ldots .$$

Since a was an arbitrary point of D, the conclusion of the theorem follows by induction. ■

We now apply Theorem 4.70 to the Cauchy integral formula to obtain

4.74. Theorem. (Cauchy Integral Formula for Derivatives) *Under the assumptions of Theorem* 4.63, *we have*

$$\frac{f^{(k)}(a)}{k!} n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{k+1}}\, d\zeta, \quad k = 0, 1, 2, \ldots .$$

In particular, for $n(\gamma; a) = 1$, *we have*

$$\frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{k+1}}\, d\zeta, \quad k = 0, 1, 2, \ldots .$$

Proof. Let $\phi(z) = f(z)$ in Theorem 4.70. Then

$$F_1(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z) n(\gamma; z),$$

by the Cauchy integral formula for all points z inside γ. Note that $n(\gamma; z)$ is an analytic function of z on $D \setminus \{\gamma\}$ and is continuous on $D \setminus \{\gamma\}$. Since $n(\gamma; z)$ is an integer, it is constant in the neighborhood of z. Thus, $n(\gamma; z)$ is locally constant. Therefore,

$$F_1^{(k)}(z) = 2\pi i f^{(k)}(z) n(\gamma; z).$$

But, by Theorem 4.70, we have

$$F_{k+1}(z) = \frac{F_k'(z)}{k} = \frac{F_{k-1}''(z)}{k(k-1)} = \cdots = \frac{F_1^{(k)}(z)}{k!} = \frac{2\pi i}{k!} f^{(k)}(z) n(\gamma; z);$$

that is,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta = \frac{f^{(k)}(z)}{k!} n(\gamma; z)$$

and the result follows, by replacing z by a. ■

4.75. Corollary. *If f is analytic in a domain D then the derivatives of all orders exist and each of its derivatives $f^{(k)}$ is analytic in D and may be obtained by differentiating under the integral sign in the Cauchy integral formula.*

Proof. In Theorem 4.74 take $a \in D$ and $\gamma = \{\zeta : |\zeta - a| = r\} \subset D$. Then $f^{(k)}(a)$ exists for all $k = 1, 2, 3, \dots$, so $f^{(k)}(z)$ is differentiable in a neighborhood of a and is therefore analytic at a. Since $a \in D$ was arbitrary, the result follows. ∎

Let us now demonstrate how different this corollary is from the situation of a function of a real variable. For example, consider $f(x) = (x+1)^{5/3}$ for $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$ we have

$$f'(x) = \frac{5}{3}(x+1)^{2/3} \quad \text{and} \quad f'(-1) = 0.$$

On the other hand,

$$f''(x) = \frac{10}{9}(x+1)^{-1/3} \quad \text{for } x \neq -1$$

but $f''(-1)$ does not exist. For the complex analog of the same function, namely $f(z) = (z+1)^{5/3}$, the point $z = -1$ is a branch point and the analytic branch of $f(z)$ exists in $\mathbb{C} \backslash \{x + i0 : x \leq -1\}$.

Thus, it is possible for a function of real variable $g(x)$ in $(a, b) \subseteq \mathbb{R}$ to have a derivative $g'(x)$ without $g'(x)$ being continuous therein. Here is another example. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then, we see that g is continuously differentiable on $\mathbb{R}$ but $g'(x)$ is not continuous at the origin. How about $h(x) = |x|^3$ on $\mathbb{R}$?

4.76. Example. Let $D = \{(x, y) : |x| \leq a,\ |y| \leq b,\ a \geq b\}$, a rectangular region. Suppose that $f \in \mathcal{H}(D)$ satisfies the inequality $|f(z)| \leq M$ on ∂D. Then,

$$|f'(0)| \leq \frac{2M(a+b)}{\pi b^2}.$$

Indeed, by Theorem 4.74 and the fact that $|\zeta| \geq b$ on ∂D, we have the inequality

$$|f'(0)| = \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta^2}\, d\zeta \right| \leq \frac{1}{2\pi} \frac{M}{b^2} L(\partial D) = \frac{M(4a+4b)}{2\pi b^2}$$

which gives the desired result. ●

4.77. Corollary. *If $f = u + iv$ is analytic in a domain D, then all the partial derivatives of u and v exist and are continuous in D.*

Proof. As $f' = u_x + iv_x = v_y - iu_y$ in D and since f' is analytic, the conclusion for the first partial derivatives follows. As

$$f'' = u_{xx} + iv_{xx} = v_{yx} - iu_{yx} = v_{xy} - iu_{xy} = -u_{yy} - iv_{yy}$$

and since f'' is analytic, the result for the second partial derivatives follows. Finally, the proof follows by induction. ■

4.78. Example. Let us use the Cauchy integral formula to evaluate $\int_{|z-1|=1} (\overline{z})^n \, dz$, $n \in \mathbb{Z}$. To do this, we use the parameterization $z - 1 = e^{it}$ so that

$$\overline{z} = 1 + e^{-it} = 1 + \frac{1}{z-1} = \frac{z}{z-1}.$$

Therefore,

$$\int_{|z-1|=1} (\overline{z})^n \, dz = \int_{|z-1|=1} \frac{z^n}{(z-1)^n} \, dz$$

which is clearly 0, by Cauchy's theorem, whenever $n = 0, -1, -2, \dots$. If $f(z) = z^n$ ($n \in \mathbb{N}$), then, by Cauchy's integral formula, the value of the integral is seen to be $2\pi i n$. ●

4.79. Theorem. (Cauchy's Inequality) *Let f be analytic in the open disk $\Delta(a; R)$ and $|f(\zeta)| \le M$ for $\zeta \in \partial\Delta(a; r)$, $0 < r < R$. Then, for each $k \in \mathbb{N}$ one has $|f^{(k)}(a)| \le M k! r^{-k}$.*

Proof. Apply the standard estimate (see Theorem 4.9) to the Cauchy integral formula for derivatives with $\gamma = \partial\Delta(a; r)$ and find that

$$|f^{(k)}(a)| \le \frac{k!}{2\pi} \cdot \frac{M}{r^{k+1}} \cdot 2\pi r = \frac{Mk!}{r^k}.$$ ■

4.80. Remark. The number M in Theorem 4.79 depends on the circle $|\zeta - a| = r$. But notice that for $k = 0$, one has $|f(a)| \le M$ for any circle centered at a as long as this circle lies entirely within the disk $\Delta(a; R)$. This observation shows that an upper bound M of $|f(z)|$ on any circle about a cannot be smaller than $|f(a)|$.

It is also important to know how different this result is from the case of a real-valued function of a real variable. So we raise the following question: *Does the Cauchy inequality hold in the real variable case?* Consider

$$u(x) = \sin(1/x) \quad \text{for} \quad x \in \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}.$$

Then, u is differentiable on $\mathbb{R}^+$, $|u(x)| \le 1$ for $x \in \mathbb{R}^+$ and

$$u'(x) = -x^{-2} \cos(1/x)$$

which is unbounded, since $u'(1/(2n\pi)) = -4n^2\pi^2$ for every $n \in \mathbb{N}$.

Another simple example may be given by the function $u_n(x) = \sin nx$. Then, for each $n \in \mathbb{Z}$, $|u_n(x)| \le 1$ on $\mathbb{R}$. Further, $u_n'(x)$ exists for all $x \in \mathbb{R}$ and $u_n'(0) = n$ so that $u_n'(x)$ is unbounded on $\mathbb{R}$ when $n \to \infty$. ●

The above theorem combined with Theorem 3.71 gives the following result known as *Cauchy's Inequality for Taylor's coefficients.*

4.81. Theorem. *Let $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ be a series with positive radius of convergence R. If $0 < r < R$ and $M(r) = \max_{|z-z_0|=r} |f(z)|$, then, for each $k \in \mathbb{N}$, $|a_k| \leq r^{-k}M(r)$.*

4.82. Example. Consider the geometric series $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$, $|z| < 1$. Then, for this function (with $z_0 = 0$, $R = 1$, and $f(z) = (1-z)^{-1}$ in Theorem 4.81), we have $M(r) = (1-r)^{-1}$ for $0 < r < 1$, since the function $w = (1-z)^{-1}$ maps $|z| = r$ onto

$$\left| w - \frac{1}{1-r^2} \right| = \frac{r}{1-r^2}.$$

So, $|a_k| \leq (1-r)^{-1}r^{-k}$ for $k \in \mathbb{N}$. But we note that $a_k = 1$ for each $k \in \mathbb{N}$ and hence, the estimate is not good! ●

4.83. Example. Suppose that $f \in \mathcal{H}(\Delta)$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $|f(z)| \leq |z|/(1-|z|)$. Then, by Cauchy's inequality, it can be easily seen that $|a_n| < ne$ for $n \in \mathbb{N}$. Indeed, for each $r \in (0,1)$,

$$|a_n| \leq \frac{1}{(1-r)r^{n-1}}.$$

In particular, for $r = 1 - 1/(n+1)$, this inequality gives

$$|a_n| \leq (n+1)\left(\tfrac{n+1}{n}\right)^{n-1} = n\left(1+\tfrac{1}{n}\right)^n < ne. \qquad \bullet$$

Next we give a straightforward application of the Cauchy integral formula.

4.84. Theorem. (Weierstrass' Theorem for Sequences) *Let D be open and $f_n : D \to \mathbb{C}$ be analytic for each $n \in \mathbb{N}$. If $f_n \to f$ uniformly on each compact subset of D, then f is analytic in D. Moreover, for each k, $f_n^{(k)}(z) \to f^{(k)}(z)$ uniformly as $n \to \infty$ for each compact subset of D.*

Proof. Since D is open, there exists an open disk, $D_0 = \Delta(z_0;\epsilon)$, centered at z_0 with $\overline{D}_0 \subset D$. By hypothesis, $f_n \to f$ uniformly on $\overline{D}_0$. In $\overline{D}_0$, f is the uniform limit of continuous functions; hence f is continuous on $\overline{D}_0$. It follows from Cauchy's theorem that $\int_C f_n(z)\,dz = 0$ for any closed contour C such that C and its interior lie in D_0. But, since $f_n \to f$ uniformly on the compact set C, we have

$$\begin{aligned}
\left| \int_C f_n(z)\,dz - \int_C f(z)\,dz \right| &= \left| \int_C [f_n(z) - f(z)]\,dz \right| \\
&\leq \left\{ \sup_{z\in C} |f_n(z) - f(z)| \right\} \times \text{ length of } C \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\int_C f(z)\,dz = \lim_{n\to\infty}\int_C f_n(z)\,dz = 0$$

and, since C is arbitrary, it follows from Morera's Theorem (see Theorem 4.86) that f is analytic in D.

Next let $\gamma = \{z : z - z_0 = re^{it},\ 0 \le t < 2\pi\}$. Then, the length of γ is $L(\gamma) = 2\pi r$. The Cauchy integral formula yields

$$\begin{aligned} |f_n^{(k)}(z_0) - f^{(k)}(z_0)| &= \left|\frac{k!}{2\pi i}\int_\gamma \frac{f_n(z)}{(z-z_0)^{k+1}}\,dz - \frac{k!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}\right| \\ &\le \left\{\sup_{z\in\gamma}\left|\frac{f_n(z)-f(z)}{(z-z_0)^{k+1}}\right|\right\}\cdot\frac{k!}{2\pi}(2\pi r) \\ &= \frac{k!}{r^k}\cdot\left[\sup_{z\in\gamma}|f_n(z)-f(z)|\right] \\ &\to 0 \ \text{ as } \ n\to\infty, \end{aligned}$$

which proves the second part. Since $z_0 \in D$ is arbitrary, for each k, $f_n^{(k)}(z) \to f^{(k)}(z)$ locally uniformly on D. ∎

We observe that, in the real case, a sequence of infinitely differentiable functions can converge uniformly to a nowhere differentiable functions. Theorem 4.84 applied to partial sums of a series gives the following:

4.85. Theorem. **(Weierstrass' Theorem for Series)** *Let D be open and $f_n : D \to \mathbb{C}$ be analytic for each $n \in \mathbb{N}$. If a series $\sum_{n\ge1} f_n$ converges uniformly on each compact subset of D, then the sum $f = \sum_{n\ge1} f_n$ is analytic in D. Moreover, the series can be differentiated term-by-term for each $z_0 \in D$.*

Note that Weierstrass' theorem can be applied to the series $\sum_{n\ge1} a_n z^n$, since each term in this series is analytic in $\mathbb{C}$. Further, we note that the uniform convergence of $\sum_{n\ge1} f_n$ on D does not necessarily imply the uniform convergence of $\sum_{n\ge1} f_n^{(k)}$ on D. For example, let

$$f_n(z) = \frac{z^{n+1}}{n(n+1)}.$$

Then, $|f_n(z)| < n^{-2}$ for $z \in \Delta$ and all $n \in \mathbb{N}$. By Weierstrass' M-test (see Theorem 2.59), the series $\sum_{n\ge1} f_n$ is uniformly convergent on Δ. On the other hand, $f_n''(z) = z^{n-1}$ and so, by Example 2.55, $\sum_{n\ge1} f_n''(z)$ is not uniformly convergent on Δ.

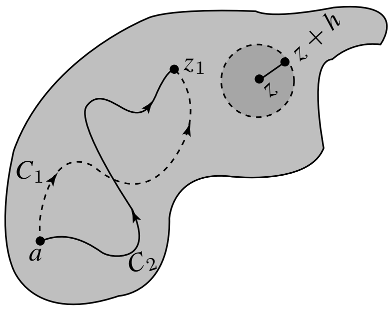


Figure 4.21: Two curves connecting a and z_1.

4.8 Morera's Theorem

If $n \in \mathbb{N}$ and $|a| > 1$, then Cauchy's theorem immediately yields that

$$I = \int_{|z|=1} \frac{dz}{(z-a)^n} = 0.$$

This is due to the fact that the integrand in I is analytic for all z in Δ, since $|a| > 1$. On the other hand, we know that

$$J = \int_{|z-a|=r} \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \in \mathbb{Z} \setminus \{1\}, \\ 2\pi i & \text{if } n = 1. \end{cases}$$

Here the non-zero result arises because of the fact that the integrand in J is not analytic at $z = a$ and this point lies inside the circle $|z-a| = r$ which violates the condition of Cauchy's theorem. A partial converse of Cauchy's theorem is the following. The power of this theorem resides in the fact that only the continuity of f is assumed.

4.86. Theorem. (Morera's Theorem) *Suppose that f is continuous in an open set D with the property that $\int_C f(z)\,dz = 0$ for each closed contour C in D. Then, f is analytic in D.*

Proof. Let a be an arbitrary fixed point in D. Since the integral $\int_C f(z)\,dz = 0$ holds for each closed contour C, for any two contours C_1 and C_2 in D (see Figure 4.21) which connect a with z_1, we have

$$0 = \int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz.$$

This means that the value of the integral $\int_a^{z_1} f(z)\,dz$ does not depend upon the path which connects a with z_1. Define

$$F(z) = \int_a^z f(\zeta)\,d\zeta \quad \text{for} \quad z \in D.$$

Since D is open, there exists an $r > 0$ such that $\Delta(z;r) \subseteq D$. Let $|h|$ be sufficiently small so that $z + h \in \Delta(z;r)$. Since $\int_a^z f(\zeta)\,d\zeta$ is independent of the path, as usual, we may write

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\,d\zeta$$

so that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} [f(\zeta) - f(z)]\,d\zeta.$$

Since f is continuous, as in the proof of Theorem 4.31, it follows that

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \to 0 \quad \text{as} \quad |h| \to 0$$

and so F is differentiable with $F'(z) = f(z)$ in $\Delta(z;r)$. Since F is analytic in $\Delta(z;r)$, it follows from the Cauchy integral formula for derivatives that f is analytic in $\Delta(z;r)$. Since z was an arbitrary point of D, we conclude that f is also analytic in D. ∎

4.87. Remark. Suppose that

$$f(z) = \begin{cases} \dfrac{1}{(z-1)^2} & \text{if } z \in \Delta(1;r)\backslash\{1\} \\ 1 & \text{if } z = 1 \end{cases}, \quad g(z) = \begin{cases} \dfrac{\cos z}{z^2} & \text{if } z \in \Delta\backslash\{0\} \\ 0 & \text{if } z = 0. \end{cases}$$

Then (see Example 4.4), for any simple closed contour C in $\Delta(1;r)$, we have $\int_C f(z)\,dz = 0$. However f is not analytic for $z \in \Delta(1;r)$, since f is not even continuous at $z = 1$. Note that Morera's Theorem is not applicable since the continuity requirement is not satisfied.

Similarly for the $g(z)$ defined above, we know that $\int_{|z|=1} g(z)\,dz = 0$. Again g is not analytic in Δ, since g is not continuous at $z = 0$. •

Most often we use Morera's Theorem but for a different situation other than that is stated.

4.88. Corollary. (Riemann's Removability Theorem) *Suppose that f is continuous on a domain D and analytic on $D\backslash\{z_0\}$ for some $z_0 \in D$. Then, f is analytic on D.*

Proof. It is enough to prove the result for a disk $|z - z_0| < \epsilon$. By Cauchy's theorem for a disk, we have $\int_C f(z)\,dz = 0$ for all closed contours C inside the disk $|z - z_0| < \epsilon$. Hence, f is analytic on $|z - z_0| < \epsilon$ by Morera's Theorem. ∎

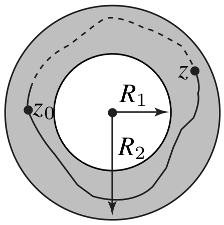


Figure 4.22: Curve in a ring shaped region.

4.9 Existence of Harmonic Conjugate

Corollary 4.75 helps us in finding a harmonic conjugate of a harmonic function. We shall now give the idea behind the formal solution to the statement of Theorem 3.39. Suppose $\phi = \phi(x,y)$ is the real part of an analytic function f in a simply connected domain D and suppose we can find $\psi = \psi(x,y)$ such that $f = \phi + i\psi$. The C-R equations would then imply $f' = \phi_x - i\phi_y$ and so if γ is a path which connects z to z_0 in D,

$$f(z) - f(z_0) = \int_\gamma f'(\zeta)\,d\zeta = \int_\gamma [\phi_x - i\phi_y]\,d\zeta$$

which is the only possible solution if we are given ϕ. The following example shows that we cannot choose D to be just a domain.

4.89. Example. For $u(x,y) = \ln\sqrt{x^2+y^2}$ in $D = \{z : R_1 < |z| < R_2\}$ where $R_2 > R_1 \geq 0$, there does not exist a $v(x,y)$ such that $f = u + iv$ is analytic in D. To prove this assertion, we suppose that $f = u + iv$ is analytic in D. Then, f' is also analytic in D. By hypothesis,

$$u_x(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad u_y(x,y) = \frac{y}{x^2+y^2}$$

so that

$$f'(z) = u_x(x,y) + iv_x(x,y) = u_x(x,y) - iu_y(x,y) = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$

So, by Theorem 4.16, the value of the integral $\int_{z_0}^{z} f'(z)\,dz$ is independent of the path joining z_0 and z in D (see Figure 4.22). Thus, we have

$$f(z) - f(z_0) = \int_{z_0}^{z} f'(\zeta)\,d\zeta = \int_{z_0}^{z} \frac{1}{\zeta}\,d\zeta.$$

In particular, if we choose the closed contour $|\zeta| = R$ with initial point $z_0 = R$ and the terminal point $z = Re^{2\pi i}$ ($R \in (R_1, R_2)$), then the above gives that

$$0 = f(Re^{2\pi i}) - f(R) = \int_{|\zeta|=R} \frac{1}{\zeta}\,d\zeta = \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}}\,d\theta = 2\pi i$$

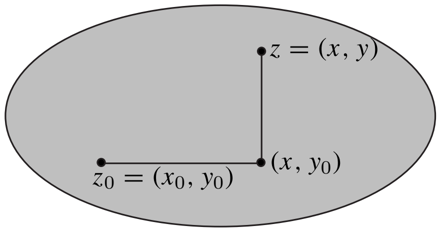


Figure 4.23: Determination of conjugate harmonic.

and this contradiction shows that no such $v(x, y)$ can exist in D such that $f = u + iv$ is analytic in D. ●

Proof of Theorem 3.39. Since ϕ is harmonic in D, $M_x - N_y = 0$, where $M = \phi_x$ and $N = -\phi_y$. Now M and N are differentiable functions of (x, y) in D and $N dx + M dy$ is an exact differential for $(x, y) \in D$, i.e. there exists a function ψ such that

$$d\psi = N dx + M dy, \quad \text{for } (x, y) \in D.$$

With respect to a reference point $(x_0, y_0) \in D$, let γ be a path (see Figure 4.23) consisting of the line segment connecting (x_0, y_0) to (x, y_0) and (x, y_0) to (x, y). Define

$$\begin{aligned} \psi(x, y) &= \int_\gamma N\, dx + M\, dy + k \\ &= \int_{x_0}^{x} -\phi_y(t, y_0)\, dt + \int_{y_0}^{y} \phi_x(x, s)\, ds + k \end{aligned} \tag{4.90}$$

where k is some real constant. The partial derivative of (4.90) with respect to y is given by

$$\psi_y(x, y) = \frac{\partial}{\partial y}\left(\int_{y_0}^{y} \phi_x(x, s)\, ds\right) = \phi_x(x, y),$$

since the first integral in (4.90) is independent of y. Similarly, taking the partial derivative of (4.90) with respect to x yields

$$\psi_x(x, y) = -\phi_y(x, y_0) + \frac{\partial}{\partial x}\left(\int_{y_0}^{y} \phi_x(x, s)\, ds\right). \tag{4.91}$$

Using the differentiation formula under the integral sign for the second term in (4.91), we have (as in the proof of Theorem 3.53)

$$\psi_x(x, y) = -\phi_y(x, y), \quad \psi_y(x, y) = \phi_x(x, y).x$$

Thus, ϕ and ψ satisfy the C-R equations for $F = \phi + i\psi$ and ψ is harmonic in D. By Theorem 3.26, $F = \phi + i\psi$ is analytic in D. Thus, a harmonic conjugate always exists in D and the proof is completed. ■

4.92. Remark. We observe the following:

(i) If the domain is multiply connected and ϕ is harmonic there, then the conjugate function becomes multiple-valued. For instance if

$$\phi(x, y) = \ln\sqrt{x^2 + y^2},$$

then the corresponding multiple-valued conjugate function is

$$\psi(x, y) = \text{Arctan}\left(\frac{y}{x}\right) + 2k\pi + \text{ constant}, \quad k \in \{\pm 1, \pm 2, \dots\}.$$

(ii) From Theorem 4.33, we note that an arbitrary harmonic function ϕ in a simply connected domain can always be considered as the real (or imaginary) part of an analytic function in D. ●

4.10 Taylor's Theorem

In Theorem 3.71, we have shown that every power series $\sum_{n\geq 0} a_n(z-a)^n$ is infinitely differentiable in its disk of convergence $\Delta(a; R)$. Moreover, $a_n = f^{(n)}(a)/n!$ and it does not depend on R. Now, we use complex integration to show that every analytic function in a domain can be expressed locally as a convergent power series. This fact opens the door to a systematic discussion of the local structural properties of analytic functions.

4.93. Theorem. *If f is analytic in Δ_R, then f has a Maclaurin series expansion $f(z) = \sum_{n\geq 0} a_n z^n$ for all $z \in \Delta_R$, where*

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta, \quad n = 0, 1, 2, \dots$$

with $\gamma = \{\zeta : |\zeta| = r\}$ and $0 < r < R$.

Proof. For a given $z \in \Delta_R$, choose r such that $r < R$ and let γ be $\gamma = \{\zeta : |\zeta| = r\}$. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

We know that, for all $|z/\zeta| < 1$,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\cdot\frac{1}{1 - z/\zeta} = \sum_{n\geq 0}\frac{z^n}{\zeta^{n+1}}$$

and the convergence is uniform on γ, and for a fixed z. Now $f(\zeta)$ is bounded on γ, since it is a continuous function on the compact set γ. We form

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n\geq 0} f_n(\zeta) \quad \text{with} \quad f_n(\zeta) = \frac{f(\zeta)}{\zeta^{n+1}} z^n. \tag{4.94}$$

Viewed as a series of functions of ζ, the series (4.94) converges uniformly on γ. Indeed, in $\overline{\Delta}_\rho$, $f(\zeta)$ is bounded for each $\rho < r$, with bound M, say, and

$$|f_n(\zeta)| \leq \left[\sup_{|\zeta|=\rho} |f_n(\zeta)|\right] \leq \frac{M}{r^{n+1}}\rho^n = \frac{M}{r}\left(\frac{\rho}{r}\right)^n$$

and so $\sum_{n\geq 0} f_n(\zeta)$ is uniformly convergent on $\overline{\Delta}_\rho$, by the Weierstrass M-test. Therefore, the series (4.94) may be integrated term-by-term so that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n\geq 0}\left\{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta\right\} z^n = \sum_{n\geq 0} a_n z^n.$$

Since $z \in \Delta_R$ is arbitrary, we have $f(z) = \sum_{n\geq 0} a_n z^n$ for all $|z| < R$ and so the assertion now follows from Cauchy's integral formula for derivatives. Uniqueness of the coefficients a_n follows from the Uniqueness theorem for Taylor series. By Theorem 4.16, the value of the integral is independent of the choice of the curve in $|z| < R$. ∎

Using the simple transformation $z - a = w$, we obtain the following result which shows that every analytic function can be expressed locally as a convergent power series. However, we observe that several power series (sometimes perhaps infinitely many) may be required to represent f throughout the domain.

4.95. Corollary. **(Taylor's Theorem)** *If f is analytic in a domain D then for $z \in \Delta(a;R) \subseteq D$, f has the Taylor series expansion*

$$f(z) = \sum_{n\geq 0} a_n (z-a)^n, \quad a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta,$$

where $C = \{\zeta : |\zeta - a| = r\}$ and $0 < r < R$.

If f is analytic in a domain D with $a \in D$, then f admits a Taylor series expansion about a: $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. What is the radius of convergence of the series we have obtained? It may happen that the circle of convergence encloses points outside D. By Corollary 4.95 what we know is that if f is analytic in $\Delta(a;r)$, then the series about a converges to $f(z)$ in that disk. The series converges in at least the largest disk centered at a that is contained in D. Clearly, r may be increased until the circle

$|z-a| = r$ encounters a singularity of $f(z)$. Thus, the radius of convergence R is the largest number R such that $f(z)$ *extends* to be analytic on the disk $|z-a| < R$ and the extended function is called an analytic continuation of f. We shall have a preliminary discussion on this issue in Chapter 10. However, the example given below will give a little flavor of this idea. When $f(z)$ is a single-valued analytic branch of a multi-valued function, then branch points give obstacles just as much as a singularity. Thus, the radius of convergence is the distance from the center of the expansion to the nearest singularity or branch point.

4.96. Examples. We know that $f(z) = \operatorname{Log}(1-z)$ is analytic in the cut plane $\mathbb{C} \setminus \{x + i0 : x \geq 1\}$. In particular, f is analytic in $|z| < 1$ and has a Taylor series expansion in $|z| < 1$ about 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Note that 1 is the largest number R such that f extends to be analytic in the disk $|z| < R$. It follows that $f(0) = \operatorname{Log} 1 = 0$ and for $n \geq 1$,

$$f^{(n)}(0) = -\left.\frac{1 \cdot 2 \cdots (n-1)}{(1-z)^n}\right|_{z=0} = -(n-1)!.$$

So, for $|z| < 1$,

$$\operatorname{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Note that an application of the Ratio/Root test quickly confirms that the series converges absolutely for $|z| \leq \rho < 1$. Next, we consider the function g defined by

$$g(z) = \frac{\operatorname{Log}(1+z)}{z}.$$

Then $g \in \mathcal{H}(D)$, $D = \mathbb{C} \setminus (\{x + i0 : x \leq -1\} \cup \{0\})$. Although $z = 0$ is a singularity, g can be extended to be analytic at $z = 0$ (it is a removable singularity of g but we shall discuss various singularities in Chapter 7) because

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}, \quad |z| < 1.$$

Moreover, if we let

$$\frac{\operatorname{Log}(1+z)}{z} = \sum_{n=0}^{\infty} a_n (z-6)^n$$

then the radius of convergence of this series is $R = 7$ (the distance from the branch point -1 to 6) not the distance from 0 to 6. •

4.97. Example. Let $f(z) = \operatorname{Log} z$ for $z \in \mathbb{C} \setminus \{x + i0 : x \leq 0\}$ and $a = -1 + i = \sqrt{2}e^{3\pi i/4}$. Then $f(a) = \ln \sqrt{2} + 3\pi i/4$ and for $n \geq 1$,

$$f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n}$$

and

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{(-1)^{n-1}}{n2^{n/2}} e^{-3\pi i n/4}$$

so that the power series expansion of f about a is

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} e^{-3\pi i n/4}}{n2^{n/2}} (z-a)^n.$$

It is easy to see (for example, by the Ratio test) that the radius of convergence of the series on the right is $R = \sqrt{2}$. Again, this does not contradict the discontinuity of $\operatorname{Log} z$ at the point $z = -1$, because $\operatorname{Log} z$ extends to be analytic for $|z - (-1 + i)| < \sqrt{2}$ although the extension does not coincide with $\operatorname{Log} z$ in the part of the disk that lies in the lower half-plane, see Chapter 10 for a discussion on analytic continuation. ●

4.98. Example. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1+x^4}.$$

This function admits a real power series about any point $a \in \mathbb{R}$. Yet the power series about $a = 0$, given by

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

has $(-1, 1)$ as its interval of convergence. On the other hand, its complex analog

$$f(z) = \frac{1}{1+z^4}$$

has singularities at $z_k = e^{i\pi(1+2k)/4}$, $k = 0, 1, 2, 3$. Clearly, the distance from 0 to the nearest singularity is 1, which necessarily is the radius of convergence for the corresponding series about 0. ●

Consider the function $f(z) = z^{-1}$. Then, f is analytic on $\mathbb{C} \setminus \{0\}$. Therefore, for all $z \neq 0$,

$$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

and so by Taylor's Theorem, we have the Taylor series expansion of f about $z_0 \neq 0$,

$$\frac{1}{z} = \sum_{n\geq 0} \frac{(-1)^n}{z_0^{n+1}}(z-z_0)^n \quad \text{for} \quad z \in \Delta(z_0;|z_0|).$$

After k-fold differentiation this gives

$$\frac{1}{z^{k+1}} = \sum_{n\geq k} (-1)^{n-k} \binom{n}{k} \frac{1}{z_0^{n+1}}(z-z_0)^{n-k}$$

for $z \in \Delta(z_0;|z_0|)$. The substitution $z_0 = -1$ with the transformation $z = w-1$ yields the Maclaurin series of $(1-w)^{-k-1}$:

$$\text{(4.99)} \qquad \frac{1}{(1-w)^{k+1}} = \sum_{n\geq k} \binom{n}{k} w^{n-k} \quad \text{for} \quad w \in \Delta,$$

which is the k-fold differentiation of the geometric series $\sum_{n\geq 0} w^n$.

We remark that the coefficient formulae (see Corollary 4.95) may not be directly useful in writing the Taylor expansion for analytic functions. The formula (4.99), the corresponding formulae for e^z, $\sin z$, $\cos z$ etc., and formulae for the Cauchy product are more often useful in such problems.

4.100. Example. Let us develop the function f defined by

$$f(z) = \frac{1}{1-z-z^2}$$

into a Taylor series about 0. By partial fraction decomposition we find that

$$\begin{aligned} f(z) &= \frac{1}{\beta-\alpha}\left[\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right] \\ &= \frac{1}{\beta-\alpha}\left[\frac{1}{\beta}\left(\frac{1}{1-z/\beta}\right) - \frac{1}{\alpha}\left(\frac{1}{1-z/\alpha}\right)\right], \end{aligned}$$

where

$$\alpha = \frac{\sqrt{5}-1}{2}, \quad \beta = -\frac{\sqrt{5}+1}{2}.$$

Note that $\alpha < |\beta|$, $\alpha\beta = -1$ and $\beta - \alpha = -\sqrt{5}$. Therefore, the Taylor series of f about 0 is

$$\begin{aligned} f(z) &= \frac{1}{\beta-\alpha}\left[\sum_{n\geq 0}\frac{z^n}{\beta^{n+1}} - \sum_{n\geq 0}\frac{z^n}{\alpha^{n+1}}\right] \quad (|z|<\alpha), \\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(\alpha\beta)^{n+1}}\right] z^n \quad (|z|<\alpha), \end{aligned}$$

since the first series converges for $|z| < |\beta|$ while the second series converges for $|z| < \alpha$ so that the combined series converges for $|z| < \min\{\alpha, |\beta|\} = \alpha$. As $\alpha\beta = -1$, we rewrite the above equation as

$$\frac{1}{1-z-z^2} = \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left[\left(\frac{\sqrt{5}+1}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right] z^n, \quad |z| < \alpha.$$

For $|z| < \alpha = (\sqrt{5}-1)/2$, we may write

$$\frac{1}{1-z-z^2} = \sum_{n\geq 0} a_n z^n, \quad \text{i.e.} \quad 1 = (1-z-z^2) \left[\sum_{n\geq 0} a_n z^n \right].$$

By uniqueness of Taylor's coefficients of f about 0, on equating the coefficients of z^n on both sides, we see that

$$a_0 = a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + a_{n-1}.$$

Note that the sequence $\{a_n\}_{n\geq 0}$ obtained here gives the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$. •

4.11 Zeros of Analytic Functions

We now discuss the zeros of analytic functions using the Taylor series expansion as a tool. Suppose that f is analytic, $f(z) \not\equiv 0$ in an open set D and f vanishes at some point $a \in D$. Then f admits a Taylor series about a:

$$f(z) = a_1(z-a) + a_2(z-a)^2 + \cdots, \quad |z-a| < R,$$

where $\{z : |z-a| < R\} \subseteq D$. Since $f(z) \not\equiv 0$, not all the coefficients a_k can vanish. This shows that there is a positive integer $m \geq 1$ such that

$$a_1 = \cdots = a_{m-1} = 0, \quad \text{but } a_m \neq 0.$$

Then we say that f has a *zero* (finite) of order m at a. Also, the integer m is referred to as the multiplicity of the zero of f at a. Zeros of order 1 are often called *simple zeros*. In case f has a zero of order m at a, we may rewrite $f(z)$ as

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \cdots = (z-a)^m g(z),$$

where $g(z)$ is analytic at a and $g(a) = a_m \neq 0$ (Note that the radius of convergence of $a_m + a_{m+1}(z-a) + \cdots$ is exactly same as that of

$$a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \cdots).$$

The above discussion leads to

4.101. Proposition. *A function f analytic at a has a zero of order m at a iff $f(z) = (z-a)^m g(z)$, where g is analytic at a and $g(a) \neq 0$.*

Further, one can use this Proposition to establish the L'Hôspital rule for complex functions, see Exercise 4.161.

A zero of an analytic function f is said to be *isolated* if it has a neighborhood in which there is no other zero of f. An important consequence of the following result is the Identity theorem for analytic functions.

4.102. Theorem. *Every zero of an analytic function f ($\not\equiv 0$) is isolated.*

Proof. Suppose that f has a zero of order m at a. Then there exist an $R > 0$ such that

$$f(z) = (z-a)^m g(z), \quad |z-a| < R,$$

where g is analytic at a and $g(a) \neq 0$. Let $|g(a)| = 2\epsilon > 0$. Then for this ϵ, since g is continuous at a, there exists a $\delta > 0$ such that

$$|g(z) - g(a)| < \epsilon \text{ whenever } |z-a| < \delta.$$

Therefore when $|z-a| < \delta$ we have

$$|g(z)| = |g(a) - [g(a) - g(z)]| \geq |g(a)| - |g(z) - g(a)| > 2\epsilon - \epsilon = \epsilon.$$

Thus, $g(z) \neq 0$ in $\Delta(a;\delta)$ (We remind the reader that we have already noticed this point while discussing the limit of a function, see Theorem 2.10). But $|z-a|^m \neq 0$ in $0 < |z-a| < \delta$. Hence, $f(z) = g(z)(z-a)^m \neq 0$ in this neighborhood except at a. This completes the proof. ■

The following theorem plays an important role in complex function theory. In simplest terms this theorem, which is an extension of Theorem 3.75, completely characterizes an analytic function in a domain D just by its behavior in a small subset of D.

4.103. Theorem. (Identity/Uniqueness Theorem) *Suppose that f is analytic in a domain D. If S, the set of zeros of f in D, has a limit point z^* in D. Then $f(z) \equiv 0$ in D.*

The hypothesis that D is connected in Theorem 4.103 is necessary. For example, if $D = \mathbb{C}\backslash\{z : 1 \leq |z| \leq 3\}$ and if $f : D \to \mathbb{C}$ is defined by

$$f(z) = \begin{cases} 0 & \text{for } |z| < 1 \\ 2 & \text{for } |z| > 3, \end{cases}$$

then $f \in \mathcal{H}(D)$ and its zero set ($|z| < 1$) has a limit point in D, yet f is not identically zero in D.

4.104. Remarks. The following observations are important:

(i) Consider a real-valued function f of a real variable defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

For $x \neq 0$, it is clear that f has derivatives of all orders. In fact, it is a simple exercise to see that f is infinitely differentiable at all points of $\mathbb{R}$ and, in particular,

$$f^{(n)}(x) = 0 \text{ for } x \leq 0 \text{ and all } n = 0, 1, 2, \dots .$$

However, $f(x)$ does not vanish in $\mathbb{R}$. Thus, in the case of real-valued functions the behavior of infinitely differentiable functions in one region of its domain of definition has no effect on its behavior on some other region. However, the Uniqueness theorem shows that this is not the case with functions of a complex variable as we shall soon see its remarkable role in a number of elementary connections as well.

(ii) The hypothesis that the limit point a lies in D is not superfluous. For example, consider

$$f(z) = \sin\left(\frac{1}{1-z}\right).$$

Then, $f \in \mathcal{H}(\Delta)$ and the zeros of f are given by $z = 1 - 1/(n\pi)$ $(n \in \mathbb{Z})$. But the zeros of f that lie inside Δ are

$$z_n = 1 - \frac{1}{n\pi} \quad (n \in \mathbb{N})$$

and $z_n \to 1$ as $n \to \infty$ yet, $f(z) \not\equiv 0$. Obviously, f is not analytic at 1 and $1 \notin \Delta$.

(iii) Consider $f(z) = \exp(z/(1-z)) - 1$ for $z \in \Delta$. Then, $f \in \mathcal{H}(\Delta)$. The zeros of f are obtained from solving $z/(1-z) = 2n\pi i$. This gives

$$z_n = \frac{2n\pi i}{1 + 2n\pi i} \quad (n \in \mathbb{Z})$$

so that f has infinitely many zeros and, since $|z_n| < 1$ for each $n \in \mathbb{Z}$, it follows that each of them lies inside Δ. In fact,

$$\left|z_n - \frac{1}{2}\right| = \frac{1}{2}\left|\frac{-1 + 2n\pi i}{1 + 2n\pi i}\right| = \frac{1}{2}$$

which implies that the zeros of f actually lie in $\Delta \cap \partial\Delta(1/2; 1/2)$. Thus, $f(z) \not\equiv 0$ and yet f has infinitely many zeros in Δ on which f is analytic. Obviously, f is not analytic at $z = 1$ and the point 1 is a limit of point of the above sequence of zeros of f. In fact, we may rewrite $f(z)$ as

$$f(z) = e^{-1} \exp\left(\frac{1}{1-z}\right) - 1.$$

The reader with knowledge of isolated singular points can quickly recognize the point $z = 1$ as an essential singularity of f and also of the function f considered in (ii).

(iv) From the last two examples in (ii) and (iii), we also observe that there are non-constant analytic functions in Δ having infinitely many zeros in Δ. ●

A direct consequence of Theorem 4.103 is that *if a set of zeros of an analytic function in a domain D contains an infinite sequence of distinct points that has its limit point in D, then the function is identically zero in D.* Sets such as an interval (a, b) of $\mathbb{R}$ and an open disk in $\mathbb{C}$ always an contain an infinite sequence of points that converges in the set itself. Consequently, the Uniqueness theorem is often helpful in checking the validity in the complex plane of certain functional identities known to be true in $\mathbb{R}$ or in an open subset of $\mathbb{R}$. For instance, consider

$$f_1(z) = \sin^2 z + \cos^2 z - 1 \quad \text{and} \quad f_2(z) = \cosh^2 z - \sinh^2 z - 1.$$

We know from elementary mathematics that the identities

$$f_1(z) = 0 \quad \text{and} \quad f_2(z) = 0 \tag{4.105}$$

hold when z is real. Since trigonometric functions $\sin z, \cos z, \cosh z$ and $\sinh z$ are all entire, both f_1 and f_2 are analytic in $\mathbb{C}$. Since the x-axis contains a sequence of distinct points (for example $1/n$, $-1/n$) converging to an element in it, the Uniqueness theorem immediately shows that the identities in (4.105) hold for all $z \in \mathbb{C}$ (Note that every point of $\mathbb{R}$ is a limit point).

Similarly, we can easily derive the following identities as easy consequences of the Uniqueness theorem:

$$\begin{aligned}
1 + \tan^2 z &= \sec^2 z, \quad \text{for all } z \in \mathbb{C} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}; \\
1 + \cot^2 z &= \csc^2 z, \quad \text{for all } z \in \mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}; \\
1 - \tanh^2 z &= \operatorname{sech}^2 z, \quad \text{for all } z \in \mathbb{C} \setminus \{(2k+1)\pi i/2 : k \in \mathbb{Z}\}; \\
1 + \operatorname{csch}^2 z &= \coth^2 z, \quad \text{for all } z \in \mathbb{C} \setminus \{k\pi i : k \in \mathbb{Z}\},
\end{aligned}$$

(see also Section 3.4). Another interesting identity we derived earlier in (3.78) is

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}.$$

Assume that this holds when z_1 and z_2 are real. Then to check the validity of this for all z_1 and z_2 in $\mathbb{C}$, using the Uniqueness theorem, we proceed as follows: Let $z_2 = x_2$, a fixed real and $z_1 = z$, a complex variable. Then

$$e^{z + x_2} = e^z \cdot e^{x_2}$$

holds when z is real and so, by the Uniqueness theorem, this identity is true for all $z \in \mathbb{C}$. Fixing z as z_1, in particular, $e^{z_1+x_2} = e^{z_1} \cdot e^{x_2}$. This being true for all real x_2, it follows, by the same argument that

$$e^{z_1+z} = e^{z_1} \cdot e^z \text{ for all } z \in \mathbb{C}.$$

Taking $z = z_2$ we get $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$. Similarly, it can be seen that

$$\begin{aligned} \sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\ \cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \end{aligned}$$

which hold for all real values of z_1 and z_2, continues to hold for complex values of z_1 and z_2.

At this point we remind the reader that not all the familiar properties of the functions sin, cos, tan etc. as functions of a real variable remain true when these are viewed as functions of a complex variable. We know that $\sin x$ and $\cos x$ are bounded by 1 for all real x. On the other hand if $y > 0$ we have

$$\cos(iy) = \frac{e^y + e^{-y}}{2} > 1 + \frac{y^2}{2} \quad \text{and} \quad |-i\sin(iy)| = \frac{e^y - e^{-y}}{2} > y$$

which shows that there exists no constant K for which $|\cos z| < K$ and $|\sin z| < K$ in $\mathbb{C}$.

Our next example deals with the binomial expansion. Let

$$f(z) = \frac{1}{(1-z)^{k+1}}, \quad \text{and} \quad g(z) = \sum_{n \geq k} \binom{n}{k} z^{n-k}.$$

Then, from real variable calculus, we know that $f(z) - g(z) = 0$ when $z = x$ is real and $|x| < 1$. Clearly f is analytic for $z \in \mathbb{C} \backslash \{1\}$. The radius of convergence of the series with sum g is 1. Therefore, in particular, f and g are analytic for $z \in \Delta$. The Uniqueness theorem immediately yields $f(z) = g(z)$ for all $z \in \Delta$, i.e.

$$\frac{1}{(1-z)^{k+1}} = \sum_{n \geq k} \binom{n}{k} z^{n-k} \quad \text{for all } z \in \Delta.$$

Using the same method, we can obtain a generalized version of the binomial expansion

$$(1-z)^a = 1 - az + \frac{a(a-1)}{2!} z^2 + \frac{a(a-1)(a-2)}{3!} z^3 - \cdots, \quad |z| < 1,$$

where a is an arbitrary complex number (see also Theorem 3.112).

Proof of Theorem 4.103. Let $\{z_n\}$ be a sequence of zeros of f in D such that $z_n \to z^*$, where z^* is also a point in D. Then we note that, since $z_n \to z^*$, $f(z_n) = 0$ for all n, and f is continuous at z^*,

$$f(z^*) = \lim_{n \to \infty} f(z_n) = 0.$$

Then, by Theorem 4.102, either $f(z) \equiv 0$ in a neighborhood of z^* or $f(z) \neq 0$ in some punctured neighborhood $\Delta(z^*;\delta)\backslash\{z^*\} \subset D$. The second part of the statement contradicts $f(z_n) = 0$ since, for sufficiently large n, z_n lies in this punctured neighborhood of the limit point z^*. So we must have $f(z) \equiv 0$ in any disk $\Delta(z^*;R) \subseteq D$.

To complete the proof we have to show that $f(z) \equiv 0$ in the whole of D, and for this we split D into two sets:

$$\begin{aligned} A &= \{\zeta \in D : \zeta \text{ is a limit point of } S\} \\ B &= \{\zeta \in D : \zeta \notin A\}, \end{aligned}$$

where S is the set of zeros of f in D. Then

$$D = A \cup B \text{ and } A \cap B = \emptyset.$$

Let $\zeta \in A$. Then, since $\zeta \in A$ is a limit point of S, $f(z) \equiv 0$ in a neighborhood of ζ. Thus each point in A is an interior point of A. Indeed, if $z' \in \Delta(\zeta;\delta)$ for some $\delta > 0$, then $|z' - \zeta| < \delta$. Set

$$\epsilon = \delta - |z' - \zeta| \text{ and } z \in \Delta(z';\epsilon), \text{ i.e. } |z - z'| < \epsilon.$$

Then

$$|z - \zeta| \leq |z - z'| + |z' - \zeta| < \epsilon + |z' - \zeta| = \delta.$$

It follows that $\Delta(z';\epsilon) \subset \Delta(\zeta;\delta)$; so A is open and is non-empty, by hypothesis.

To show B is open, let $\zeta' \in B$. Since ζ' is not a limit point of S, by continuity of f at ζ', there exists $\delta > 0$ such that $f(z) \neq 0$ throughout $\Delta(\zeta';\delta) \subset D$. Since D is connected and nonempty, it cannot be written as the union of two non-empty disjoint open sets. Hence we must have either $A = \emptyset$ or $B = \emptyset$. But by hypothesis $z^* \in A$. It now follows that $A \neq \emptyset$ and therefore $B = \emptyset$.

Thus $A = D$ and every $\zeta \in D$ is a limit point of the zeros of f so that $f \equiv 0$ in D. ∎

The following result is also referred to as the Uniqueness theorem which is considered to be a fundamental result for an introduction to the concept of analytic continuation (see Chapter 10).

4.106. Theorem. *Suppose that f and g are analytic in a domain D. If S, the set of zeros of $f - g$ in D, has a limit point z^* in D, then $f(z) \equiv g(z)$ in D.*

Proof. Apply Theorem 4.103 to $h = f - g$. ∎

4.107. Example. The Uniqueness theorem provides another example of fundamental differences between complex-valued functions and real-valued functions. For example, for $k \in \mathbb{N}$, define $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} x^{2k} \sin^k(2\pi/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then, for each k, f_k is continuously differentiable on $\mathbb{R}$ and $f_k(1/n) = 0$ for all $n \in \mathbb{N}$, yet f_k and f_m are distinct functions for $k \neq m$.

Next, we consider a new function $g_k : \mathbb{C} \to \mathbb{R}$ defined by $g_k(z) = f_k(|z|)$, where f_k is as above. Then, for each k, g_k is continuous in $\mathbb{C}$ and $g_k(1/n) = 0$ for all $n \in \mathbb{N}$, yet g_k and g_m are distinct functions in $\mathbb{C}$ for $k \neq m$. This example demonstrates that two different continuous functions in a domain D can assume the same values on an infinite set which has a limit point in D. It follows that the Uniqueness theorem does not hold for continuous functions. ●

4.108. Corollary. *Suppose that f and g are analytic in a domain D such that $f(z)g(z) = 0$ for each $z \in D$. Then either $f(z) = 0$ for $z \in D$ or $g(z) = 0$ for each $z \in D$.*

Proof. Suppose that $f(a) \neq 0$ for some $a \in D$. Then, by the continuity of f, there exists a disk $\Delta(a;\delta) \subseteq D$ such that $f(z) \neq 0$ in $\Delta(a;\delta)$. But then $g(z) = 0$ for all $z \in \Delta(a;\delta)$. By the Uniqueness theorem, $g(z) \equiv 0$ in D. ■

4.109. Corollary. *Suppose that f is analytic in a domain D and $f'(z) = 0$ in some disk contained in D. Then $f(z)$ is constant in D.*

Proof. By Theorem 3.31, f is a constant in a disk $D_1 \subset D$, say c. Therefore, every point of D_1 is a limit point of $\{z : f(z) - c = 0\}$. By the Uniqueness theorem, $f(z) - c = 0$ throughout D. ■

4.110. Example. Let $S = \{1/n : n = 1, 2, \dots\}$. Then $S \subset [0,1]$ and S has a limit point 0.

(i) Suppose that $f \in \mathcal{H}(\mathbb{C})$ and $f(z) = \cos z$ for $z \in S$. Then by the Uniqueness theorem, since $0 \in \mathbb{C}$, $f(z) = \cos z$ in $\mathbb{C}$. Note that f is determined throughout $\mathbb{C}$ just by its values at the points $1/n$.

(ii) Let $f(z) = e^{2\pi i/z} - 1$, $z \neq 0$. Then f is analytic for all $z \in \mathbb{C}\backslash\{0\}$ and $f(z_n) = 0$ for $z_n = 1/n$, $n \in \mathbb{Z}$. Even though $z_n \to 0$ as $n \to \infty$, we cannot claim $f(z) \equiv 0$, since the condition that the limit point 0 must lie in $\mathbb{C}\backslash\{0\}$ is not satisfied. Notice that z_n is an isolated zero of f for each $n = 1, 2, \dots$ whereas 0 is not in $\mathbb{C}\backslash\{0\}$. In this example, f cannot even be defined at $z = 0$ so as to make f to be continuous at 0. Similarly, the function F defined by $F(z) = \sin(\pi/z)$ for $z \neq 0$, is analytic in $\mathbb{C}\backslash\{0\}$ and $F(1/n) = 0$ for $n \in \mathbb{Z}$. Is F identically zero? Is it possible to extend F to make it continuous at 0?

(iii) Let $f(z) = \exp\{\operatorname{Log} z\}$. Since $\operatorname{Log} z$ is seen to be analytic in D_π, where $D_\pi = \mathbb{C}\backslash\{z = x : x \leq 0\}$, f is analytic in D_π. We know that $\exp\{\operatorname{Log} z\} = z$ for $z = x > 0$. Therefore, we have

$$\exp\{\operatorname{Log} z\} = z \quad \text{for } z \in D_\pi,$$

as we have obtained this fact earlier but by a different approach. ●

In general, using the Uniqueness theorem, we conclude the following:

4.111. Corollary. *If $f_1(z)$, $f_2(z)$, ... are analytic functions defined in a domain D and satisfy a certain algebraic identity (or functional identity such as in (4.105)) on a set with a limit point in its domain of analyticity, then these functions satisfy the same identity throughout the domain of analyticity.*

4.112. Example. If $f \in \mathcal{H}(\Delta)$ and $|f(z)| \leq 1 - |z|$ in Δ, then it is easy to see that $f(z) = 0$ in Δ. Indeed, if $a \in \Delta_r$ $(0 < r < 1)$ is fixed, then, by the Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - a}\, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta} - a}\, d\theta$$

so that the standard estimate (see Theorem 4.9(iv)) gives

$$|f(a)| \leq \frac{(1-r)r}{r - |a|} \to 0 \quad \text{as } r \to 1.$$

Thus, $f(z) = 0$ on Δ_r as a is arbitrary. By the Uniqueness theorem, $f(z) = 0$ on Δ. ●

By the method of proof of Theorem 4.103, we next show that an analytic function in a domain cannot vanish together with all its derivatives at some point inside the domain unless it is identically zero.

4.113. Theorem. (Uniqueness Theorem for Power Series) *Let f be analytic in a domain D. Suppose that at some point $a \in D$, $f^{(n)}(a) = 0$ for all $n = 0, 1, 2, \ldots$. Then, $f(z) \equiv 0$.*

Proof. Let $a \in D$. There exists an $r > 0$ such that $\Delta(a; r) \subseteq D$. By Corollary 4.95,

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(z - a)^n$$

holds for $z \in \Delta(a; r) \subseteq D$ so that, by hypothesis, $f(z) = 0$ on $\Delta(a; r)$, which in turn implies that $f(z) \equiv 0$ on D by Theorem 4.103. ■

We now discuss analytic functions in relation to the simple operation of multiplication on their Taylor series.

4.114. Theorem. *Suppose that the power series $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ are convergent for $|z| < R_1$ and $|z| < R_2$ with sums $f(z)$ and $g(z)$, respectively. Then, we have*

$$f(z)g(z) = \sum_{n\geq 0} c_n z^n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \text{for } |z| < \min\{R_1, R_2\} = R.$$

Proof. By Theorem 3.71, f and g are analytic for $|z| < R$ with

$$a_n = f^{(n)}(0)/n! \quad \text{and} \quad b_n = g^{(n)}(0)n!$$

and fg is analytic for $|z| < R$. Therefore, by the Cauchy product

$$f(z)g(z) = \sum_{n\geq 0} c_n z^n$$

converges for $|z| < R$ and hence is analytic. Thus, by Taylor's theorem

$$c_n = \frac{(fg)^{(n)}(0)}{n!} = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{1}{n!} \frac{n!}{k!(n-k)!} f^{(k)}(0) g^{(n-k)}(0). \quad \blacksquare$$

4.12 Laurent Series

Suppose that f is not defined, or is not analytic, at a point a. Then, we cannot express it in a neighborhood of a as a convergent power series expansion of the form $f(z) = \sum_{n\geq 0} a_n (z-a)^n$, for, if we could do so then, by Theorem 3.71, f would be analytic at a.

For instance, consider $f(z) = \sin(1/z)$. The series representation for f is obtained by considering the series for $\sin z$ and replacing z by $1/z$, which gives a series involving negative powers of z:

$$f(z) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \cdots, \quad z \neq 0.$$

It converges for all z with $z \neq 0$. More generally, a series of the form

$$\sum_{n\geq 0} b_n (z-a)^{-n} \tag{4.115}$$

can be thought of as a power series in the variable $1/(z-a)$. Letting $\zeta = 1/(z-a)$, the above series becomes an ordinary power series in ζ:

$$\sum_{n\geq 0} b_n \zeta^n. \tag{4.116}$$

The next theorem shows how the properties of power series in negative powers such as (4.115) can be deduced from the corresponding properties of ordinary power series. In view of (4.116) and Theorem 3.64, we have

4.117. Theorem. *Let $r = r_*^{-1} = \limsup_{n\to\infty} |b_n|^{1/n}$.*

(i) *If* $r = 0$, *then the series* (4.115) *converges absolutely for every* $z \in \mathbb{C}_\infty \setminus \{a\}$.

(ii) *If* $0 < r < \infty$, *then the series* (4.115) *converges absolutely for all* z *with* $|z - a| > r$, *the convergence being uniform on* $|z - a| \geq \rho > r$ *and diverges for* $|z - a| < r$.

(iii) *If* $r = \infty$, *then the series* (4.115) *diverges for all finite* z.

We say that an "infinite series" of the form $\sum_{n=-\infty}^{\infty} A_n$ converges to a limit L iff both $\sum_{n\geq 0} A_n$ and $\sum_{n\geq 1} A_{-n}$ converge and the sum of their limit is L. Equivalently, we say that the double series $\sum_{n=-\infty}^{\infty} A_n$ converges to L iff given $\epsilon > 0$ there exists an N such that

$$\left| \sum_{k=-n}^{m} A_k \right| < \epsilon \quad \text{whenever } m, n \geq N.$$

It is important to observe that m and n are independent here. In fact, the existence of the limit $\lim_{n\to\infty} \sum_{k=-n}^{n} A_k$ does not in general imply that the corresponding double series $\sum_{n=-\infty}^{\infty} A_n$ converges.

Let $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$ and $r = r_*^{-1} = \limsup_{n\to\infty} |b_n|^{1/n}$ and suppose that $0 < R, r < \infty$. Then, by Theorems 3.64 and 4.117, the series

$$f_1(z) = \sum_{n\geq 0} a_n z^n = a_0 + \sum_{n\geq 1} a_n z^n$$

converges absolutely for $|z| < R$ and diverges if $|z| > R$, while

$$f_2(z) = \sum_{n\geq 0} b_n z^{-n} = b_0 + \sum_{n\leq -1} b_{-n} z^n$$

converges absolutely for $|z| > r$ and diverges if $|z| < r$. So there is a non-empty region of convergence for the series of the form

$$\text{(4.118)} \qquad f(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \quad A_n = \begin{cases} a_n & \text{if } n \geq 1 \\ a_0 + b_0 & \text{if } n = 0 \\ b_{-n} & \text{if } n \leq -1 \end{cases},$$

iff $r < R$. That is, the combined series (4.118) converges for all z in the common region $\Omega = \{z : r < |z| < R\}$. At each point lying outside region $\overline{\Omega}$ the combined series is divergent, since the series defined by the sum $f_1(z)$ or $f_2(z)$ is divergent. Also, by Theorems 3.71 and 4.117, we note that f_1 and f_2 are both analytic in the respective domain of convergence. So the combined series defined above represents an analytic function in the annular region Ω . Conversely if f is analytic in Ω for some $0 \leq r < R \leq \infty$, then f has such a series expansion of the form (4.118) valid in Ω (see Theorems 3.71 and 4.123).

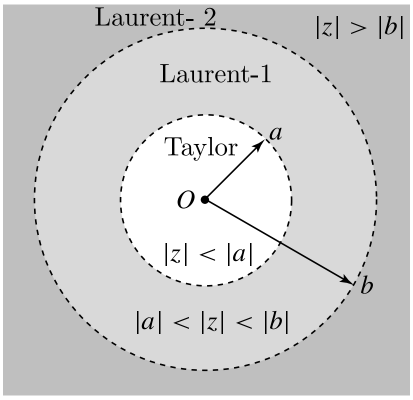


Figure 4.24: Description for Laurent's series.

4.119. Remark. By Theorem 3.64, f_1 converges uniformly for $|z| \le \rho < R$ and by Theorem 4.117, f_2 converges uniformly for $|z| \ge \rho' > r$. Since both series converge uniformly for all z satisfying $\rho' \le |z| \le \rho$, the series defined by (4.118) converges uniformly to f for $\rho' \le |z| \le \rho$. ●

A *Laurent series* about a is a series of the form

$$\sum_{n=-\infty}^{\infty} A_n(z-a)^n := \sum_{n\ge 0} A_n(z-a)^n + \sum_{n\ge 1} A_{-n}(z-a)^{-n}$$

which represents an analytic function in the annulus $r < |z-a| < R$. The numbers A_n are the corresponding coefficients about a. The series of the form (4.118) is then a Laurent series about $z = 0$.

As a motivation for a Laurent series we consider the function

$$f(z) = \frac{1}{(z-a)(z-b)}, \ a \neq b. \tag{4.120}$$

Then, f is analytic everywhere except at $z = a, b$ and therefore, we are unable to express it in the neighborhood of a as a convergent series of positive powers of $z - a$. To obtain a Laurent series for f about $z = 0$, we rewrite (4.120) as (see Figure 4.24)

$$f(z) = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right].$$

If $0 < |a| < |z| < |b|$ (so that $|z/b| < 1, \ |a/z| < 1$),

$$f(z) = \frac{1}{a-b}\left[\frac{1}{z}\sum_{n\ge 0}\left(\frac{a}{z}\right)^n - \frac{1}{b}\sum_{n\ge 0}\left(\frac{z}{b}\right)^n\right]$$

$$= \frac{1}{a-b}\left[\sum_{n\geq 1}\frac{a^{n-1}}{z^n} - \sum_{n\geq 0}\frac{z^n}{b^{n+1}}\right]$$

$$= \frac{1}{a-b}\left[\sum_{n\leq -1} a^{-n-1}z^n - \sum_{n\geq 0}\frac{z^n}{b^{n+1}}\right].$$

This may be written as

$$(4.121) \qquad f(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \quad A_n = \begin{cases} -\dfrac{1}{(a-b)b^{n+1}} & \text{if } n \geq 0 \\ \dfrac{1}{(a-b)a^{n+1}} & \text{if } n \leq -1. \end{cases}$$

Note that the expression in (4.121) involves both positive and negative powers of z and (4.121) is the Laurent series of (4.120) valid for $0 < |a| < |z| < |b|$.

If $|z| > |b| > |a|$ (so that $|b/z| < 1$, $|a/z| < 1$), then we have

$$f(z) = \frac{1}{a-b}\left[\sum_{n\geq 0}\frac{a^n}{z^{n+1}} - \sum_{n\geq 0}\frac{b^n}{z^{n+1}}\right] = \frac{1}{a-b}\left[\sum_{n\geq 0}\frac{a^n-b^n}{z^{n+1}}\right].$$

For example, we also note that

$$\frac{1}{z}, \quad \frac{1}{(z-1)^2}, \quad \frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + (z-1)^2,$$

are themselves the Laurent expansion around 0, 1 or 2 as the case may be.

4.122. Example. Consider the series

$$\sum_{n=-2004}^{\infty}\frac{z^n}{n^2} = \sum_{n=-2004}^{-1}\frac{z^n}{n^2} + \sum_{n=1}^{\infty}\frac{z^n}{n^2}.$$

We note that the first series on the right (which contains only a finite number of terms) converges for all $z \neq 0$ whereas the second series converges for $|z| < 1$, diverges for $|z| > 1$ and converges for all z with $|z| = 1$. Thus, the given double series converges for all $0 < |z| \leq 1$ and diverges for $|z| > 1$. On the other hand if we consider

$$\sum_{n=-\infty,\ n\neq 0}^{\infty}\frac{z^n}{n^2} = \sum_{n=1}^{\infty}\frac{z^{-n}}{n^2} + \sum_{n=1}^{\infty}\frac{z^n}{n^2},$$

then it follows that the first series on the right converges for $|z| \geq 1$ whereas the second series converges for $|z| \leq 1$ so that (the combined series) the double series converges only for $|z| = 1$. ●

We next direct our attention to obtain an analogue of the Cauchy-Taylor representation theorem which shows that a function analytic in an annulus, say $D = \{z : R_1 < |z - a| < R_2\}$, can be expanded into a Laurent series which converges to the function for every z in the annulus D.

4.123. Theorem. (Laurent's Theorem) *If f is analytic in the annulus: $R_1 < |z| < R_2$ (where $0 \le R_1 < R_2 \le \infty$), then f has a unique representation $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ for any z in the annulus, where*

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta, \quad n \in \mathbb{Z}, \tag{4.124}$$

with $C = \{\zeta : |\zeta| = r\}$ and $R_1 < r < R_2$.

4.125. Corollary. *If f is analytic in the annulus: $R_1 < |z-a| < R_2$ (where $R_1 \ge 0$), then, for any z in the annulus, f has a unique representation*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \quad a_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \;\; (n \in \mathbb{Z}),$$

with $R_1 < r < R_2$.

Before proving Theorem 4.123, we note that the above corollary is a consequence of applying this result to $g(z) = f(z + a)$, which is analytic in the annulus $R_1 < |z| < R_2$.

Laurent's theorem continues to hold if $R_1 = 0$ or $R_2 = \infty$ or both. In case $R_1 = 0$, the Laurent series represents an analytic function in a deleted neighborhood: $\Delta(a; R_2)\backslash\{a\} = \{z : 0 < |z - a| < R_2\}$. When $R_2 = \infty$ and $R_1 > 0$, the series represents the function in a deleted neighborhood of infinity: $[\overline{\Delta}(a; R_1)]^c = \{z : |z - a| > R_1\}$. When $R_1 = 0$ and $R_2 = \infty$, we say that the series represents the function in the punctured plane namely, $\{z : |z - a| > 0\}$.

Proof of Theorem 4.123. For a given z in the annulus $R_1 < |z| < R_2$, choose r_1 and r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$ and

$$C_m = \{\zeta : |\zeta| = r_m\} \;\text{ for } m = 1, 2,$$

the positively oriented circles. Note that z lies between C_1 and C_2. By making two cross-cuts from C_1 to C_2 avoiding the point z, we have (see Figure 4.25)

$$\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta \tag{4.126}$$

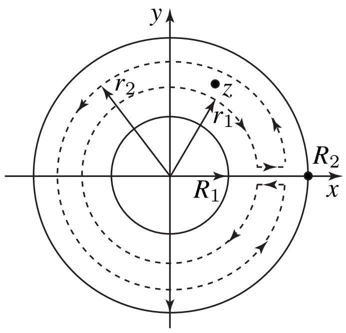


Figure 4.25: Annulus region.

where γ is a small circle containing the point z inside γ. Here the contributions to the integral of the curve along the cross-cuts cancel in pairs. But, by the Cauchy integral formula,

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z).$$

Hence, (4.126) becomes

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{4.127}$$

Proceeding exactly as was done in proving Taylor's theorem, on C_2, we have (see Weierstrass' Theorem, namely Theorem 4.85)

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n\geq 0}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta\right\} z^n = \sum_{n\geq 0} a_n z^n \tag{4.128}$$

with

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \quad \text{for } n = 0, 1, 2, \ldots. \tag{4.129}$$

Now,

$$-\frac{1}{\zeta - z} = \frac{1}{z}\cdot\frac{1}{1 - \zeta/z} = \sum_{n\geq 0}\frac{\zeta^n}{z^{n+1}} = \sum_{n\leq -1}\frac{z^n}{\zeta^{n+1}}, \tag{4.130}$$

since $|\zeta| < |z|$ on C_1, and the convergence is uniform in ζ on C_1 for a fixed z. Using (4.130), we easily have

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n\leq -1} a_n z^n, \tag{4.131}$$

where a_n is the same as defined in (4.129) for $n = -1, -2, -3, \ldots$ but over the contour C_1. Since $g(\zeta) = f(\zeta)/\zeta^{n+1}$ is analytic on the annulus domain

$R_1 < |\zeta| < R_2$, by Cauchy's deformation theorem (see Theorem 4.37), we may replace C_1 and C_2 in the expression for a_n by C (as specified in the statement of the theorem) in the calculation of the coefficients a_n. This observation together with (4.128) and (4.131) shows that (4.127) becomes $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where a_n is defined by (4.124).

The integral formula for the coefficients, namely (4.124) allows us to show that the Laurent series representation for a given function f is unique. That is to say that if we have another Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{for} \quad R_1 < |z| < R_2, \tag{4.132}$$

then $a_n = c_n$ for all n, where a_n is given by (4.124).

Now for each n, by (4.132), (4.124) may be rewritten as

$$a_n = \frac{1}{2\pi i} \int_C \frac{1}{\zeta^{n+1}} \left\{ \sum_{k=-\infty}^{\infty} c_k \zeta^k \right\} d\zeta = \frac{1}{2\pi i} \int_C \sum_{k=-\infty}^{\infty} c_k \zeta^{k-n-1}\, d\zeta.$$

We again make use of the fact that the interchange of summation and integral signs is permissible (see Theorem 4.85). This is because the convergence of $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ in the annulus $R_1 < |z| < R_2$ implies that it converges uniformly along C. We also know that

$$\int_C \zeta^m \, d\zeta = \begin{cases} 0 & \text{for any integer } m \neq -1 \\ 2\pi i & \text{for } m = -1. \end{cases}$$

Thus, for each integer n, we have

$$a_n = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} c_k \int_C \zeta^{k-n-1}\, \zeta = \frac{1}{2\pi i} \left[c_n \int_C \zeta^{-1}\, d\zeta \right] = c_n. \qquad \blacksquare$$

4.133. Remark. Equation (4.127) expresses actually the following: *"Every analytic function f in the annulus $R_1 < |z| < R_2$ can be uniquely decomposed into a sum $f(z) = f_-(z) + f_+(z)$, where $f_+(z)$ is analytic for $|z| < r_2 \ (< R_2)$, and $f_-(z)$ is analytic for $|z| > r_1 \ (> R_1)$"*. The uniqueness assertion in the proof implies that the decomposition is independent of r_1 and r_2, so that both $f_+(z)$ and $f_-(z)$ are defined and analytic in the annulus $R_1 < |z| < R_2$. ●

4.134. Remark. The formula, for $R_1 < 1 < R_2$, gives the interesting relation between Laurent and Fourier series expansions. Let f be analytic in some neighborhood, say $D = \{z : 1 - \epsilon < |z| = 1 < 1 + \epsilon\}$, $\epsilon > 0$, of the unit circle $|z| = 1$. Then, for z in this neighborhood, we get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta.$$

In particular if we let $f(e^{it}) = F(t)$ and $z = e^{it}$, we have

$$(4.135) \qquad F(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{with} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}\,dt.$$

The series in (4.135) is the Fourier series of F in the complex form. ●

4.136. Remark. Again, we note that the part $f_+(z) = \sum_{n\geq 0} a_n z^n$ defines an analytic function for $|z| < R_2$, while $f_-(z) = \sum_{n\leq -1} a_n z^n$ defines an analytic function for $|z| > R_1$. The Laurent series representation for f in two different domains will, in general, be different. If f is analytic at $z = 0$ then the corresponding $f_-(z) = 0$ and the Laurent series becomes the Taylor series about 0. In this case, we have $a_n = f^{(n)}(0)/n!$. On the other hand, we cannot set $a_n = f^{(n)}(0)/n!$ in Theorem 4.117 as we did with Taylor's series representation where we had assumed that f is analytic for $|z| < R$ for some R. In fact, $f^{(n)}(0)$ is not even defined in Theorem 4.123 since 0 is not in the annulus. Similar comments apply for the Laurent series about $z = a$.

The coefficients a_n are not often obtained by using the integral formula defined by (4.124). Because of the uniqueness property of the Laurent series expansion, it is enough to find a valid expansion for the analytic function in the same annulus by some other easy techniques by computation or otherwise. Therefore, whenever possible, we can make use of a simple known expansion to obtain the Laurent expansion of quite complicated functions. ●

4.137. Example. The function f defined by $f(z) = 1/z$ is itself a Laurent series at $z = 0$ in the annulus $0 < |z| < \infty$. To determine the Laurent expansion of f at $z = a$ $(\neq 0)$ we proceed as follows:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{z-a}\cdot\left[\frac{1}{1+a/(z-a)}\right] \\ &= \frac{1}{z-a}\cdot\sum_{n\geq 0}(-1)^n\frac{a^n}{(z-a)^n}, \quad |z-a| > |a|,\end{aligned}$$

valid for $|z-a| > |a|$. Similarly, for $|z-a| < |a|$, we have

$$\frac{1}{z} = \frac{1}{a}\left[\frac{1}{1+(z-a)/a}\right] = \sum_{n\geq 0}(-1)^n\frac{(z-a)^n}{a^{n+1}}$$

which is the Taylor series expansion of f at a valid in $|z-a| < |a|$. ●

4.138. Example. Let $c \in \mathbb{C}$ be fixed such that $|c| > 1$. Let us discuss the convergence of

$$\sum_{n=-\infty}^{\infty} f_n(z), \quad f_n(z) = \frac{z^n}{c^{|n|}}.$$

Note that

$$\sum_{n\geq 0} f_n(z) = \sum_{n\geq 0} \left(\frac{z}{c}\right)^n = \frac{1}{1-z/c} = \frac{c}{c-z} \quad \text{for } |z| < |c|.$$

Similarly,

$$\sum_{n\leq -1} f_n(z) = \sum_{m\geq 1} \frac{1}{(cz)^m} = \frac{1}{cz-1} \quad \text{for } \ |z| > |c|^{-1}.$$

Therefore, the combined series $\sum_{n=-\infty}^{\infty} f_n(z)$ converges for $|c|^{-1} < |z| < |c|$ and diverges at various other values of z. •

4.139. Theorem. *If f is analytic in a neighborhood of infinity and f is bounded there, then the Laurent coefficients $a_n = 0$ for $n = 1, 2, 3, \ldots$, where each a_n is given by* (4.129).

Proof. By hypothesis there exists $R > 0$ such that f is analytic for $|z| > R$, and $|f(z)| \leq M$ for some $M > 0$. Then, for $r > R$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{with } \ a_n = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$$

and therefore, we have $|a_n| \leq Mr^{-n}$. But r can be chosen as large as we please; so $r^{-n} \to 0$ as $r \to \infty$ for $n > 0$, which means $a_n = 0$ for $n > 0$. ■

4.140. Example. Let $\alpha \in \mathbb{R}$ and $f(z) = \exp(\frac{1}{2}\alpha(z - z^{-1}))$. Then, $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$. Moreover,

$$\exp\left(\frac{\alpha z}{2}\right) = \sum_{n\geq 0} \frac{1}{n!} \left(\frac{\alpha z}{2}\right)^n \quad \text{for all } z$$

and

$$\exp\left(-\frac{\alpha}{2z}\right) = \sum_{n\geq 0} \frac{(-1)^n}{n!} \left(\frac{\alpha z^{-1}}{2}\right)^n \quad \text{for all } \ z \neq 0.$$

Therefore, the Laurent series expansion of f in $\mathbb{C}\backslash\{0\}$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta. \tag{4.141}$$

Choosing $C = \{\zeta : |\zeta| = 1\}$, i.e. $\zeta = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, we find that

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{1}{2}(e^{i\theta} - e^{-i\theta})\alpha}}{e^{i(n+1)\theta}} \cdot ie^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(\alpha \sin\theta - n\theta)}\, d\theta.$$

By the change of variable $\theta = 2\pi - \phi$, we note that

$$\begin{aligned}\int_0^{2\pi} \sin(\alpha \sin\theta - n\theta)\, d\theta &= \int_0^{2\pi} \sin(\alpha \sin(2\pi - \phi) - n(2\pi - \phi))\, d\phi \\ &= -\int_0^{2\pi} \sin(2n\pi + (\alpha \sin\phi - n\phi))\, d\phi \\ &= -\int_0^{2\pi} \sin(\alpha \sin\phi - n\phi)\, d\phi,\end{aligned}$$

so that $\int_0^{2\pi} \sin(\alpha \sin\theta - n\theta)\, d\theta = 0$. This observation implies that

$$\exp\left[\frac{1}{2}(z - z^{-1})\alpha\right] = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{with} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha \sin\theta - n\theta)\, d\theta.$$

Now, for each $z \in \mathbb{C}\backslash\{0\}$, we can write

$$f(z) = \exp(\frac{1}{2}\alpha z)\exp(-\frac{1}{2}\alpha z^{-1}). \tag{4.142}$$

Since the Laurent expansion is unique we may compare the coefficients of z^n in (4.141) and (4.142) to obtain (use the Cauchy product of two convergent series)

$$a_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+n)!m!}\left(\frac{\alpha}{2}\right)^{2m+n}.$$

Similarly, it is easy to show the following:

(i) If $\alpha \in \mathbb{R}$, then $\exp\left[\frac{1}{2}(z + z^{-1})\alpha\right] = \sum_{n=-\infty}^{\infty} a_n z^n$ for $z \neq 0$ with

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\alpha\cos\theta}\cos(n\theta)\, d\theta = \sum_{m=0}^{\infty} \frac{1}{(m+n)!m!}\left(\frac{\alpha}{2}\right)^{2m+n} = a_{-n}.$$

(ii) If $\alpha \in \mathbb{R}$, then $\sinh[\alpha(z + z^{-1})] = \sum_{n=-\infty}^{\infty} A_n z^n$ for $z \neq 0$, with

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta)\sinh(2\alpha\cos\theta)\, d\theta.$$

Note that the Laurent coefficients in (i) and (ii) remain unchanged when $-n$ replaces n. Therefore, we can rewrite (i) and (ii) respectively as follows:

$$\exp\left(\frac{1}{2}(z + z^{-1})\alpha\right) = a_0 + \sum_{n\geq 1}^{\infty} a_n(z^n + z^{-n})$$

and

$$\sinh\left(\alpha(z + z^{-1})\right) = A_0 + \sum_{n\geq 1} A_n(z^n + z^{-n}).$$

●

4.143. Example. Consider $f(z) = \text{Log}\,(z^n/(z^n-1))$ for $|z|>1$, where n is a fixed positive integer. Write f as

$$\begin{aligned} f(z) &= \text{Log}\left(\frac{1}{1-(1/z)^n}\right) \\ &= \text{Log}\,1 - \text{Log}\,(1-z^{-n}), \quad \text{since } |1/z|<1, \\ &= \sum_{m=-\infty}^{-1} \frac{1}{m} z^{nm}, \quad |z|>1, \end{aligned}$$

which is the Laurent expansion for f. •

4.13 Exercises

4.144. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) $\gamma(t) = t^2 e^{2\pi i/t}$, $t \in (0,1]$, with $\gamma(0)=0$, is a Jordan arc of class C^1.

(b) $\gamma(t) = t^2 e^{\pi i/4}$, $t \in (0,1]$, is a non-simple smooth contour.

(c) If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [a,b] \to \mathbb{C}$ are two contours, then so is the sum $\gamma_1 + \gamma_2$ only if $\gamma_1(b) = \gamma_2(c)$.

(d) The inequality $|e^z - 1| < |z|$ holds for all $z \in D = \{w : \text{Re}\, w < 0\}$.

(e) The inequality $|e^a - e^b| \le |a-b|$ holds for $a,b \in D = \{w : \text{Re}\, w \le 0\}$.

(f) $I = \displaystyle\int_{|z|=r} |z-r|\,|dz| = 8r^2$.

(g) If f is a complex-valued continuous function on $\mathbb{C}$, then

$$I = \int_{|z|=1} \frac{f(z)-f(1/z)}{z}\,dz = 0.$$

(h) If $p(z)$ is a polynomial of degree n in z with complex coefficients, then

$$I = \int_{|z|=1} p(\overline{z})\,dz = 2\pi i p'(0).$$

(i) There is an entire function $f(z)$ such that $e^{f(z)} = 25(e^{2z}+1)/\cos(iz)$.

(j) There is an entire function $f(z)$ such that $e^{f(z)} = -5(e^{2z}-1)/\sin(iz)$.

(k) If f is analytic and nowhere zero in $\Omega = \{z : \text{Re}\, z < 2003\}$, then $\ln|f|$ is harmonic in Ω.

(l) If $g \in \mathcal{H}(\overline{\Delta})$ and such that $|g(z) - z| \le |z|$ on $|z|=1$, then one has the estimate $|g'(a)| \le 1 + (1-|a|)^{-2}$ for each $a \in \Delta$.

(m) Let Γ be the closed square given by $\{z : -5 \leq \operatorname{Re} z, \operatorname{Im} z \leq 5\}$. Then, there cannot exist a function f which is analytic on a domain that contains Γ such that $\max_{z\in\Gamma}|f(z)| = 5$ and $f''(1) = 1$.

(n) If f is an entire function such that $\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \leq r^{\alpha}$ for some fixed $\alpha > 0$, and for all $r > 0$, then $f(z) \equiv 0$ in $\mathbb{C}$.

(o) Let f be analytic on $\mathbb{C}\backslash\{1\}$ such that $f(\partial\Delta\backslash\{1\}) \subset \mathbb{R}$, then f is a constant function.

(p) The Taylor series $\sum_{n=1}^{\infty} n^{-1}(z-3)^n$ converges to $-\operatorname{Log}(4-z)$ for $|z-3| < 1$.

(q) If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has the property that $\sum_{n=0}^{\infty} f^{(n)}(a)$ converges, then f is necessarily an entire function.

(r) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$ and if $b_n \in \mathbb{C}$ is such that $|b_n| < n^2|a_n|$ for all $n \geq 0$, then $\sum_{n=0}^{\infty} b_n z^n$ converges for $|z| < 1$.

(s) If $\{a_n\}_{n\geq 0}$ is a sequence of real numbers such that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} a_n(z+2)^n,$$

then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is 3.

(t) The power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ $(a \neq 0)$ can converge at $z = 0$ and diverge at $z = b$, whenever $|b-a| < |a|$.

(u) An entire function that takes real values on the real axis and purely imaginary values on the imaginary axis must be an odd function: $f(z) = -f(-z)$ for all $z \in \mathbb{C}$.

(v) If f is entire and $f(z) = f(-z)$ for all z, then there exists an entire function g such that $f(z) = g(z^2)$ for all $z \in \mathbb{C}$.

(w) If f is analytic in a neighborhood Δ_δ of 0 and $f(z) = -f(-z)$ for all $z \in \Delta_\delta$, then there exists an analytic function g in Δ_δ such that $f(z) = zg(z^2)$ for all $z \in \Delta_\delta$.

(x) The Laurent series $\sum_{n=0}^{\infty}\left(\frac{z^n}{n!} + \frac{n^3}{z^n}\right)$ converges only at $z = 0$ and nowhere else.

(y) If f is an analytic function on the closed disk $|z| \leq R$ for some fixed positive number $R > 0$, then f can never satisfy the inequality $\left|f^{(n)}(0)\right| \geq n!n^n$ for all $n \in \mathbb{N}$.

(z) If $f \in \mathcal{H}(D)$ and $a \in D$, then the inequality $|f^{(n)}(a)| \geq n!n^n$ cannot hold for all $n \geq 1$.

4.145. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) $\sin z = 0 \Longleftrightarrow z = k\pi,\ k \in \mathbb{Z}$.

(b) For $z = x + iy$, $\cosh x$ is never zero and $\cosh z$ has infinitely many zeros when $y \neq 0$.

(c) The zeros of $\sin(1/z)$ are $z = 1/n\pi$ $(n \in \mathbb{Z})$ and each zero is isolated.

(d) The Uniqueness theorem does not necessarily hold for harmonic functions.

(e) If the zeros of an analytic function are not isolated then $f(z) \equiv 0$ throughout the domain of analyticity.

(f) If D is a domain and $f \in \mathcal{H}(D)$ vanishes throughout any neighborhood of a point in D, then $f(z) \equiv 0$ in D.

(g) If D is a domain and $f \in \mathcal{H}(D)$ such that $f(z) = 0$ at all points on an arc inside D, then $f(z) \equiv 0$ throughout D.

(h) If $f(z) = \sum_{n \geq 0} a_n z^n$ converges in Δ_R, $R > 0$, such that $f^2(z) = f(z)$ for every z in the open interval $(0, R)$, then $f(z)$ is either 1 or 0 for all $z \in \Delta_R$.

(i) If $f \in \mathcal{H}(\Delta)$ such that $f(x) = -f(-x)$ for every real number x in Δ, then $f(z) = -f(-z)$ for all z in Δ.

(j) There exists no function f that is analytic in the unit disk Δ such that $f(1/n) = f(-1/n) = n^{-2k+1}$ for $n = 2, 3, \dots$, where $k \in \mathbb{N}$ is fixed.

(k) There exists a function f that is analytic in a neighborhood of $z = 0$ such that $f(1/n) = f(-1/n) = n^{-2k}$ for all sufficiently large n, where $k \in \mathbb{N}$ is fixed.

(l) There exists an analytic function in the unit disk Δ such that

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n} \quad \text{for} \quad n \geq 2.$$

(m) There exists an analytic function f in Δ such that $f(1/n)) = (n + 1)/(n-1)$ for $n \geq 2$.

(n) There exists an analytic function f in Δ such that $f(i^n/n) = -n^{-2}$ for $n \geq 2$.

(o) There exists no analytic function f in $\overline{\Delta}$ such that $f(z) \not\equiv 0$ and $f(ni^n/(n+1)) = 0$ for $n \geq 1$.

(p) There exists an analytic function f in Δ such that $f(-1/2) = 3$, $f\left(n^{-2}\right) = 5$ for $n \geq 2$.

(q) Suppose that $f, g \in \mathcal{H}(\Delta)$, and neither f nor g has a zero in Δ. If $(f'/f)(1/n) = (g'/g)(1/n)$ for all $n = 2, 3, \dots$, then $f(z) = cg(z)$ in Δ for some constant c.

(r) If $f \in \mathcal{H}(\Delta)$, $\{z_n\}_{n \geq 1}$ is a sequence of non-zero complex numbers such that $z_n \to 0$ as $n \to \infty$ and $f(z_n) = f(-z_n)$ for all $n \in \mathbb{N}$, then f is even.

(s) If $\{z_n\}_{n\geq 1}$ is a sequence of distinct complex numbers in Δ such that $z_n \to 0$ as $n \to \infty$, there exists an entire function f such that $f(z_n) = z_n$ for all $n \in \mathbb{N}$ and $f(5) = 0$.

(t) If f and g are entire functions which agree on some interval $[a, b] \subset \mathbb{R}$, then $f(z) = g(z)$ in $\mathbb{C}$.

4.146. Compute $I_j = \int_{\gamma_j} |z|^2\, dz,\ j = 1, 2, 3, 4$, where

(a) $\gamma_1(t) = a\cos t + ib\sin t,\ a, b \in \mathbb{R},\ 0 \leq t \leq 2\pi$

(b) $\gamma_2(t) = t^2 + it,\ 0 \leq t \leq 1$

(c) $\gamma_3(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ -ie^{i\pi t/2} & \text{if } 1 \leq t \leq 2 \end{cases}$

(d) $\gamma_4(t) = it,\ 0 \leq t \leq 1$

(e) $\gamma_5(t) = t + it^3,\ 0 \leq t \leq 1$.

4.147. Compute $\int_{\gamma_j} |z|^2\, dz,\ j = 1$ to 7, over the same paths used in Example 4.6.

4.148. For $\gamma = \{z : |z| = 1\}$, evaluate the following integrals

$$\int_\gamma \frac{dz}{|z|^n},\quad \int_\gamma \frac{|dz|}{z^n},\quad \int_\gamma (\overline{z})^n\, dz,\quad \int_\gamma (\overline{z})^n\, |dz|,\quad \int_\gamma \frac{\operatorname{Re} z}{z-\alpha}\, |dz|,\quad \int_\gamma \frac{\operatorname{Im} z}{z-\alpha}\, |dz|$$

where $n \in \mathbb{N}$ is fixed, and $|\alpha| \neq 1$.

4.149. Let $f \in \mathcal{H}(\overline{\Delta})$ and have no zeros in $\overline{\Delta}$. Suppose that

(i) $|f(e^{i\theta})| \leq M_1$ for $-\pi/2 \leq \theta \leq \pi/2$ and,

(ii) $|f(e^{i\theta})| \leq M_2$ for $\pi/2 \leq \theta \leq 3\pi/2$.

Find an upper bound for $|f(0)|$.

4.150. Let f be analytic inside and on the square Q centered at the origin. Suppose that $|f(z)| \leq M_j$ for each $z \in S_j$ $(1 \leq j \leq 4)$, where S_j denotes the enumeration of its sides. Show that $|f(0)| \leq \sqrt[4]{M_1 M_2 M_3 M_4}$.

4.151. Let $f \in \mathcal{H}(\overline{\Delta})$. Using the Cauchy integral formula for derivatives, evaluate the following integrals:

$$I_c = \int_0^{2\pi} f(e^{i\theta}) \cos^2(\theta/2)\, d\theta \quad \text{and} \quad I_s = \int_0^{2\pi} f(e^{i\theta}) \sin^2(\theta/2)\, d\theta.$$

4.152. If $f(a) = \displaystyle\int_{|z|=4} \frac{z^2 + 3z - 7}{(z-a)^2}\, dz$ for $|a| \neq 4$, determine $f(a)$ and also $f'(1+i)$ and $f'(1-i)$.

4.153. Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals:

(a) $\displaystyle\int_{|z-2|=2} \frac{\mathrm{Log}\,(z+1)}{z-3}\,dz$ (e) $\displaystyle\int_{|z|=1} \frac{z+3}{z^4+az^3}$ $(|a|>1)$

(b) $\displaystyle\int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz$ (f) $\displaystyle\int_{|z|=2} \frac{z^n}{z-3}\,dz$

(c) $\displaystyle\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$ (g) $\displaystyle\int_{|z-1-i|=5/4} \frac{z^{1/2}}{z-1}\,dz$

4.154. If $f \in \mathcal{H}(\Delta)$ and $|f(z)| \leq (1-|z|)^{-\alpha}$ for all $z \in \Delta$ and for some $\alpha > 0$, then show that there exists a positive real number M (independent of f) such that $|f'(z)| \leq M(1-|z|)^{-\alpha-1}$ for all $z \in \Delta$. Does it also work for $\alpha < 0$?

4.155. If $f \in \mathcal{H}(\Delta)$ and $|f'(z)| \leq (1-|z|)^{-\alpha-1}$ for all $z \in \Delta$ and for some $\alpha > 0$, then show that $|f(z)| \leq M(1-|z|)^{-\alpha}$ for some $M > 0$, independent of f. Does it also work if $\alpha < 0$?

4.156. Find the radius of convergence of the series on the right hand side of $(1-z)^{-1}\,\mathrm{Log}\,z = \sum_{n=0}^{\infty} a_n(z-3)^n$, where $\mathrm{Log}\,z$ denotes the principal branch of the logarithm. Will your answer change, if we replace $\mathrm{Log}\,z$ by another branch of $\log z$ whose branch cut is the ray $re^{i\pi/4}$ $(r \geq 0)$, rather than the negative real axis.

4.157. Using the Cauchy inequality, discuss the following statements:

(a) There does not exist a function f such that f is analytic on the closed disk $|z+1| \leq 5$, $f''(-1) = i$ and $\displaystyle\max_{|z+1|\leq 5} |f(z)| = 5$.

(b) Answer the same question when $f''(-1) = i$ with $\displaystyle\max_{|z+1|\leq 5} |f(z)| = 25$ as well as when $f''(-1) = 1/3$ with $\displaystyle\max_{|z+1|\leq 5} |f(z)| = 5$.

4.158. Let $f(z) = 1 + z^2 + z^4 + z^6 + \cdots$, $z \in \Delta$, and $\{a_n\}$ be a sequence of real numbers such that $f(z) = \sum_{n=0}^{\infty} a_n(z-5)^n$. Find the radius of convergence of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

4.159. Find the radius of convergence R of the Taylor series, about $z = 1$, of the function $f(z) = (1 + z^3 + z^6 + z^9 + z^{12})^{-1}$.

4.160. Find the Taylor expansions about 0 for

$$f(z) = \frac{z-1}{z^2-z-1}, \quad h(z) = \frac{\sin z}{1+z^2}, \quad \text{and} \quad g(z) = \frac{1}{1-z+z^2}$$

and determine the radius of convergence of the corresponding series.

4.161. **(L'Hôspital rule)** Suppose that f and g are analytic at a, $g^{(k)}(a) = 0 = f^{(k)}(a)$ for $k = 0, 1, 2, \dots, n-1$ but both are not identically equal zero. If $g^{(n)}(a) \neq 0$, show that

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

provided the limit on the right exists.

4.162. What are the zeros of $\sqrt{z}$ (if it has any)?

4.163. Determine all $f \in \mathcal{H}(\Delta)$ such that $f''(1/n) + e^{1/n} = 0$ for all $n = 2, 3, \dots$. Justify your answer.

4.164. Suppose that f is an entire function such that $f'(0) = 0$ and $f''(1 + 1/n) = 7 - 3(1/n)$ for each $n = 1, 2, 3, \dots$. Find all f that satisfy these properties.

4.165. Does Theorem 4.123 provide a Laurent series expansion of any branch of $\log z$ in an annulus $r < |z| < R$?

4.166. Find the Laurent series expansion of each of the following:

(a) $f(z) = \dfrac{1}{z(z^2+z-2)}$ (d) $f(z) = \dfrac{1}{1+z^2} + \dfrac{1}{2+z}$

(b) $f(z) = \dfrac{1}{z} + \dfrac{1}{z-2} + \dfrac{1}{(z+1)^2}$ (e) $f(z) = \dfrac{1}{(z^2-1)(z^2-9)}$

(c) $f(z) = \dfrac{3z-5}{z^2+5z-6}$ (f) $f(z) = \dfrac{z}{z^2+7z-8}$.

In each case, how many such expansions are there? In which region is each of them valid? Find the Laurent coefficients explicitly for each of these expansions.

4.167. Does $\tan(1/z)$ have a Laurent series convergent in a region $0 < |z| < R$?

4.168. Let $\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\csc(\pi z) = \sum_{n=-\infty}^{\infty} b_n z^n$. Define the Laurent series expansion of $\cot(\pi z)$ and $\csc(\pi z)$, on the annulus $1 < |z| < 2$. Evaluate the coefficients a_{-n} and b_{-n} for $n \in \mathbb{N}$.

Chapter 5

Conformal Mappings and Möbius Transformations

There are certain transformations that can be readily described in terms of geometric terms since we often think of a complex-valued function as a mapping from $\Omega \subseteq \mathbb{C}$ onto another subset $\Omega' \subseteq \mathbb{C}$. In this chapter, we are mainly concerned with certain geometric questions although our initial approach will be analytic but can be easily converted into a geometric one. Many important questions among these geometric questions are associated with Möbius transformations (also called *bilinear transformations* or *linear fractional transformations*). The starting point for the discussion follows from a general theory of conformal mappings.

In Section 5.1, we introduce the basic notion of conformal mappings which is a direct application of the complex derivative. In Section 5.2, we discuss a simple but important class of conformal mappings provided by the class of all Möbius transformations. In Section 5.3, we study the fixed point properties of Möbius transformations. In Section 5.4, we show how two different sets of three distinct complex numbers determine Möbius transformations. In Section 5.5, we define cross-ratio and prove that Möbius transformations preserves circles in $\mathbb{C}_\infty$. We close Section 5.6 with a discussion on certain special mappings. More precisely, we discuss the group of (analytic) automorphisms of two domains, namely, disks and half-planes, which indicate the relationship between complex analysis and algebra. These automorphisms are the most frequently used mappings. Section 5.7 is devoted to a discussion on symmetry with respect to circles in $\mathbb{C}_\infty$. In particular, we show that if T is a Möbius transformation, and a and a^* are two points symmetric with respect to a circle K in $\mathbb{C}_\infty$, then their images $T(a)$ and $T(a^*)$ are symmetric with respect to the image circle $K' = T(K)$ in $\mathbb{C}_\infty$. Later in Section 6.3, we also discuss the same problem but by a different method.

5.1 Principle of Conformal Mapping

If z_1 and z_2 are two nonzero complex numbers, then we refer to the quantity

$$\Theta(z_1, z_2) = \arg z_2 - \arg z_1 = \text{Arg}\left(\frac{z_1}{z_2}\right)$$

as the oriented angle from z_1 to z_2 (i.e. the angle from the vector z_1 to the vector z_2), provided a suitable determination is used for their arguments. For instance,

(a) if $z_1 = 1 + i$ and $z_2 = 1$, then $z_2/z_1 = (1 - i)/2$ so that

$$\Theta(z_1, z_2) = \text{Arg}\left(\frac{1-i}{2}\right) = -\frac{\pi}{4} = \arg 1 - \arg(1+i) = -\text{Arg}\,(1+i)\,;$$

(b) if $z_1 = 1 + i$ and $z_2 = -1$, then $z_2/z_1 = (-1 + i)/2$ so that

$$\Theta(z_1, z_2) = \text{Arg}\left(\frac{-1+i}{2}\right) = \frac{3\pi}{4} = \arg(-1) - \arg(1+i) = \pi - \frac{\pi}{4};$$

(c) if $z_1 = -1 - i$ and $z_2 = -1$, then $z_2/z_1 = (1 - i)/2$ so that

$$\Theta(z_1, z_2) = -\frac{\pi}{4} = \arg(-1) - \arg(-1-i) = \pi - \left(\frac{-3\pi}{4}\right) - 2\pi.$$

Consider $f(z) = \text{Log}\, z$ and let $\Omega = \mathbb{C}\backslash\{z = x : x \leq 0\}$. We know that $f \in \mathcal{H}(\Omega)$ and $f'(z) = 1/z \neq 0$ in Ω. Consider the two curves γ_1 and γ_2 in Ω given by

$$\gamma_1 = \{z : |z| = 1,\, \pi/4 \leq \text{Arg}\, z \leq \pi/2\} = \{e^{it} : \pi/4 \leq t \leq \pi/2\}$$

and

$$\gamma_2 = \{z : 1 \leq |z| \leq 3,\, \text{Arg}\, z = \pi/4\} = \{te^{i\pi/4} : 1 \leq t \leq 3\},$$

see Figure 5.1. Clearly, these two curves intersect at $z_0 = e^{i\pi/4}$. The corresponding image curves under $f(z) = \text{Log}\, z$ are

$$\Gamma_1 = \{\,\text{Log}\,(e^{it}) : \pi/4 \leq t \leq \pi/2\} = \{it : \pi/4 \leq t \leq \pi/2\}$$

and

$$\Gamma_2 = \{\,\text{Log}\,(te^{i\pi/4}) : 1 \leq t \leq 3\} = \{\ln t + i\pi/4 : 1 \leq t \leq 3\},$$

respectively. Since $\gamma_1'(t) \neq 0$ for $\pi/4 \leq t \leq \pi/2$ and $\gamma_2'(t) \neq 0$ on $1 \leq t \leq 3$, each has a tangent at z_0. The angle between them from γ_2 to γ_1 (i.e. the angle between the corresponding directed tangent lines) is $90°$. The

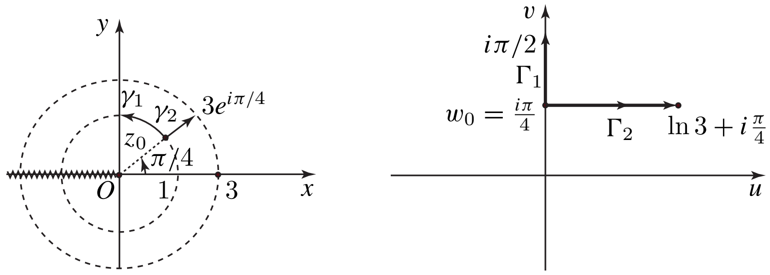


Figure 5.1: Conformality at $z_0 = e^{i\pi/4}$ for $\operatorname{Log} z$.

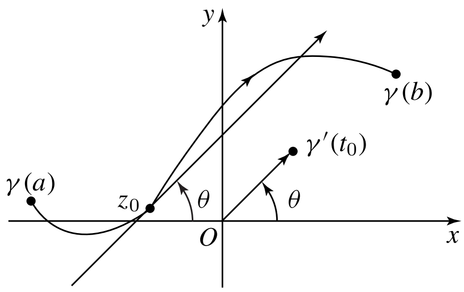


Figure 5.2: Tangent to the arc γ at z_0.

image curves Γ_1 and Γ_2 intersect at $w_0 = f(z_0) = \operatorname{Log}(e^{i\pi/4}) = i\pi/4$ and the angle between them from Γ_2 to Γ_1 is 90°. Thus, the angle between γ_1 and γ_2 is preserved both in sense as well as in size under the mapping $f(z) = \operatorname{Log} z$. Examples of this type help us to formulate the definition of conformal mappings. We start with the

5.1. Definition. Let $\gamma : \gamma(t) = x(t) + iy(t)$, $0 \le t \le 1$, be a smooth parameterized curve with $z_0 = \gamma(t_0)$ for $t_0 \in [0, 1]$. If $\gamma'(t_0) \neq 0$, then we refer to

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = x'(t_0) + iy'(t_0)$$

as the tangent to the curve γ at z_0. Moreover, $\arg \gamma'(t_0)$ represents the direction of the tangent to the curve γ at z_0.

Thus, $\gamma'(t)$ is indeed the complex representation of the usual tangent vector. We define the angle between two curves which intersect at z_0 to be the angle between their tangents.

To be more precise, we need to discuss a tangent line at z_0 to a differentiable curve $\gamma : [0, 1] \to \mathbb{C}$ where $\gamma'(t_0) \neq 0$ at $t_0 \in (0, 1)$, and $z_0 = \gamma(t_0)$, a point on γ. For convenience, we let $z_1 = \gamma(t_0 - h)$ and $z_2 = \gamma(t_0 + k)$.

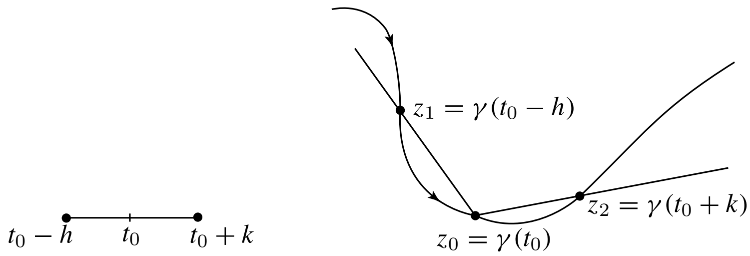


Figure 5.3: Tangent to the arc γ at z_0 as $h, k \to 0$.

Consider the rays

$$\begin{aligned} R_1[z_0, z_1] &= \{z : z = z_0 + \lambda(z_1 - z_0),\ \lambda \geq 0\} \\ &= \left\{z : z = z_0 + \mu\left(\frac{z_1 - z_0}{h}\right),\ \mu \geq 0\right\} \end{aligned}$$

and

$$R_2[z_2, z_0] = \left\{z : z = z_0 + \mu\left(\frac{z_2 - z_0}{k}\right),\ \mu \geq 0\right\}.$$

Now, since $\gamma'(t_0) \neq 0$,

$$\begin{aligned} \lim_{h \to 0^-} \frac{z_0 - z_1}{h} &= \lim_{h \to 0^-} \frac{\gamma(t_0) - \gamma(t_0 - h)}{h} \\ &= \gamma'(t_0) \\ &= \lim_{k \to 0^+} \frac{z_2 - z_0}{k} \\ &= \lim_{k \to 0^+} \frac{\gamma(t_0 + k) - \gamma(t_0)}{k}, \end{aligned}$$

and the rays R_1 (as $h \to 0^-$) and R_2 (as $k \to 0^+$) approach the ray

$$R(z_0) = \{z : z = z_0 + \mu\gamma'(t_0),\ \mu \geq 0\}$$

which is the tangent ray to γ at z_0. Thus if $\gamma'(t_0) \neq 0$, we see that, since z_0 is arbitrary, the direction of the tangent (see Figure 5.3) is determined by $\gamma'(t)$. Note that if $\gamma'(t_0) = 0$, the tangent at z_0 is not defined. On the other hand, if the direction of the tangent at t_0 exists, then it is given by the limit

$$e^{i\theta} = \lim_{t \to t_0} \frac{\gamma'(t)}{|\gamma'(t)|}.$$

The assertion now follows.

5.2. Example. Define $\gamma : [-2, 1] \to \mathbb{C}$ by $\gamma(t) = (1 + i)(1 + t)^2$. Then, $\gamma'(t) = 2(1 + i)(1 + t)$ so that $\gamma'(0) \neq 0$. Therefore, the direction of

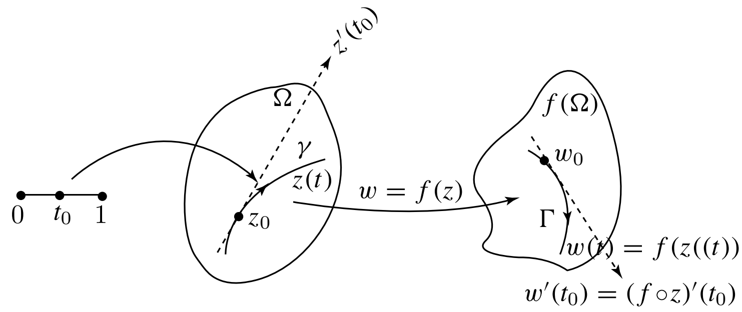


Figure 5.4: Illustration for conformal map.

the tangent at 0 is determined from

$$e^{i\theta} = \frac{\gamma'(0)}{|\gamma'(0)|} = \frac{1+i}{\sqrt{2}}.$$

Thus, the angle between the direction of the tangent and the positive real axis is $\theta = \pi/4$. On the other hand

$$\lim_{t\to -1} \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1+i}{\sqrt{2}} \lim_{t\to -1} \frac{1+t}{|1+t|};$$

but

$$\lim_{\substack{t\to -1\\ t>-1}} \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1+i}{\sqrt{2}} \quad \text{and} \quad \lim_{\substack{t\to -1\\ t<-1}} \frac{\gamma'(t)}{|\gamma'(t)|} = -\frac{1+i}{\sqrt{2}}$$

so that γ does not have a direction at $t = -1$, since $\gamma'(-1) = 0$. ●

5.3. Proposition. *Let $f \in \mathcal{H}(\Omega)$, where Ω is a domain containing a smooth curve*

$$\gamma : \gamma(t), \quad t \in [0,1],$$

passing through a point $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Then the tangent to the curve

$$\Gamma : \Gamma(t) = f(z)|_{z=\gamma(t)} = (f \circ \gamma)(t), \quad t \in [0,1],$$

at $w_0 = f(z_0)$ is

$$(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0) \tag{5.4}$$

(*Note that curves are regarded as mappings so that the transformed curve Γ is simply the composite map $\Gamma = f \circ \gamma$*).

Proof. First we note that γ is a smooth curve that passes through z_0 (when $t = t_0$). If $\gamma'(t_0) \neq 0$, then $\gamma'(t_0)$ determines a well defined tangent vector at z_0 and, because $\gamma'(t_0) \neq 0$, $\gamma(t) \neq \gamma(t_0)$ for t near t_0, $t \neq t_0$. So we may write

$$\frac{f(\gamma(t)) - f(\gamma(t_0))}{t - t_0} = \frac{f(\gamma(t)) - f(\gamma(t_0))}{\gamma(t) - \gamma(t_0)} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

and, if we allow $t \to t_0$, we have

$$(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0).$$

If $\gamma'(t_0) = 0$, we obtain $(f \circ \gamma)'(t_0) = 0$ and the above formula continues to hold. ■

If, in Proposition 5.3, $f'(z_0) \neq 0$, then the formula (5.4) gives

$$(f \circ \gamma)'(t_0) \neq 0$$

which determines a new tangent vector $\Gamma'(t_0)$ at the point $w_0 = f(z_0) = \Gamma(t_0)$ on the transformed curve, as indicated in Figure 5.4. The relation (5.4) shows that the tangent to the image curve Γ depends on the tangent to the original curve γ passing through z_0 and the fixed complex number $f'(z_0)$.

An analytic mapping $w = f(z)$ in a domain Ω that preserves the angle (both in size and in sense) at z_0 is called conformal at z_0. More precisely, $f(z)$ is said to be conformal at $z_0 \in \Omega$ if, whenever γ_1 and γ_2 are two parameterized curves intersecting at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ with non-zero tangents, then the following holds:

(i) the two transformed curves $\Gamma_1 = f \circ \gamma_1$ and $\Gamma_2 = f \circ \gamma_2$ have non-zero tangents at t_0

(ii) the angle from $\Gamma_1'(t_0) = (f \circ \gamma_1)'(t_0)$ to $\Gamma_2'(t_0) = (f \circ \gamma_2)'(t_0)$ is the same as the angle from $\gamma_1'(t_0)$ to $\gamma_2'(t_0)$.

If it is conformal at each point of Ω, then we say that f is conformal in Ω. A function that preserves the size of the angle but not sense (i.e. orientation) is said to be *isogonal*. An example of the latter class of functions is given by $f(z) = \overline{z}$. Indeed, if

$$\gamma_1 = \{t : t \geq 0\} \text{ and } \gamma_2 = \{te^{i\pi/4} : t \geq 0\}$$

are two curves in $\mathbb{C}$, then the image curves under $f(z) = \overline{z}$ are

$$\Gamma_1 = \{t : t \geq 0\} \text{ and } \Gamma_2 = \{te^{-i\pi/4} : t \geq 0\},$$

see Figure 5.5. Although the two curves intersect at an angle $\pi/4$ in each plane, the function $f(z) = \overline{z}$ reverses the angle of orientation. So the mapping in this case is not conformal, but is isogonal.

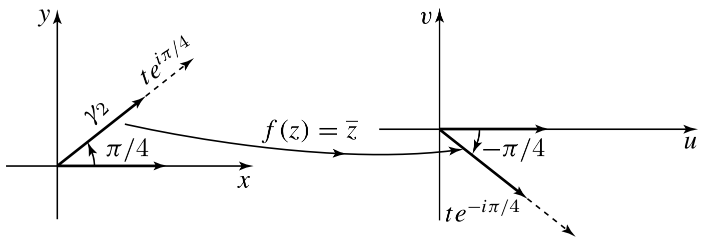


Figure 5.5: Demonstration for isogonal mapping.

Let $f \in \mathcal{H}(\Omega)$, where Ω is a domain containing a smooth curve γ passing through a point $z_0 \in \Omega$ where $f'(z_0) \neq 0$. We wish to show that this condition is sufficient to show that f is conformal at z_0. To do this, we consider two smooth curves

$$\gamma_1 : \gamma_1(t) \text{ and } \gamma_2 : \gamma_2(t), \quad t \in [0,1],$$

that pass through $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ with non-zero tangents at t_0. Then the transformed curves

$$\Gamma_1 = f \circ \gamma_1 \text{ and } \Gamma_2 = f \circ \gamma_2$$

pass through $w_0 = f(z_0)$ in the w-plane when $t = t_0$, and, by Proposition 5.3, the tangents to these curves are given by

$$\Gamma_1'(t_0) = (f \circ \gamma_1)'(t_0) = f'(z_0)\gamma_1'(t_0)$$

and

$$\Gamma_2'(t_0) = (f \circ \gamma_2)'(t_0) = f'(z_0)\gamma_2'(t_0),$$

respectively. Note that the tangents to the transformed curves Γ_1 and Γ_2 are obtained by multiplying the respective tangents to γ_1 and γ_2 by the non-zero factor $f'(z_0)$. Thus, the arguments of both tangents are increased by the same angle, namely the argument of $f'(z_0)$. Consequently,

$$\text{Arg}\left(\frac{\Gamma_2'(t_0)}{\Gamma_1'(t_0)}\right) = \text{Arg}\left(\frac{f'(z_0)\gamma_2'(t_0)}{f'(z_0)\gamma_1'(t_0)}\right) = \text{Arg}\left(\frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}\right)$$

so that the angle between γ_1 and γ_2 at z_0 measured from γ_1 to γ_2 is equal to the angle between Γ_1 and Γ_2 at $f(z_0)$ measured from Γ_1 to Γ_2. We have in fact proved the following result.

5.5. Theorem. *Let $f \in \mathcal{H}(\Omega)$ and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Then f is conformal at z_0.*

Conformality is considered a local property of analytic functions. Further, since

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and $w = f(z)$ maps a curve $\gamma : z(t)$ through z_0 to another curve $\Gamma : w(t) = f(z(t))$ through $w_0 = f(z_0)$, we have

$$\lim_{z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \lim_{z\to z_0} \left| \frac{w - w_0}{z - z_0} \right| = |f'(z_0)|.$$

This equation shows that $|f'(z_0)|$ is a local scaling factor of the function at z_0, and is independent of the curve γ. Moreover, if $|z - z_0|$ is small, then

$$|f(z) - f(z_0)| \approx |f'(z_0)|\,|z - z_0|$$

from which we see that "small" neighborhoods of z_0 are mapped onto roughly the same configuration, magnified by the factor $|f'(z_0)|$. For example, "small triangle" containing z_0 is mapped geometrically onto a similar "curvilinear triangle" magnified by the factor $|f'(z_0)|$. Thus, $\arg f'(z_0)$ measures the rotation while $|f'(z_0)|$ measures (for points nearby) the magnification or distortion of the image.

5.6. Example. To see how conformality may fail at a point z_0 where $f'(z_0) = 0$, we consider the function $f(z) = z^2$. Then $f'(0) = 0$. Now, if γ_1 is the positive real axis from 0 to ∞ and γ_2 is the (positive) imaginary axis in the upper half-plane, then $f(\gamma_1) = \Gamma_1$ is the positive real axis from 0 to ∞, and $f(\gamma_2) = \Gamma_2$ is the negative real axis from 0 to $-\infty$. Note that the angle between γ_1 and γ_2 is $\pi/2$, whereas the angle between their image curves Γ_1 and Γ_2 is π. Thus, $f(z) = z^2$ is not conformal at 0 although it is conformal at every other point of the complex plane. To demonstrate this fact, let $a + ib \neq 0$ and consider two smooth curves γ_1 and γ_2 given by

$$\gamma_1 = \{t + ib : t \geq a\}, \quad \text{and} \quad \gamma_2 = \{a + it : t \geq b\}.$$

They intersect at $z_0 = a+ib$ and the angle between them is $\pi/2$. Further, as $f'(z_0) = 2z_0 \neq 0$, f is conformal at z_0. Then, under the mapping $f(z) = z^2$, we have

$$f(\gamma_1(t)) = (t+ib)^2 = t^2 - b^2 + 2itb \quad \text{and} \quad f(\gamma_2(t)) = (a+it)^2 = a^2 - t^2 + 2iat.$$

Letting $f(\gamma(t)) = u + iv$, it follows that the images Γ_1 and Γ_2 of γ_1 and γ_2 are the parabolas described by

$$\Gamma_1 = \{u + iv : v^2 = 4b^2(u + b^2)\} \quad \text{and} \quad \Gamma_2 = \{u + iv : v^2 = 4a^2(a^2 - u)\},$$

respectively. An inspection of Figure 5.6 indicates that the angle between Γ_1 and Γ_2 at $f(z_0)$ is $\pi/2$. Note that as a and b vary, we obtain two families of parabolas intersecting orthogonally at each point where $a + ib \neq 0$. ●

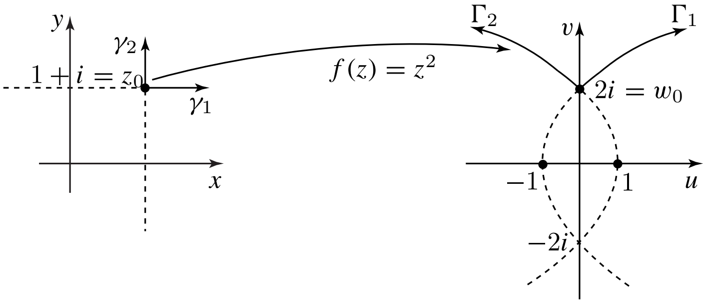


Figure 5.6: Conformality of $f(z)=z^2$ at $1+i$.

If $f(z)=z^n$, then f magnifies the angle at $z=0$ by a factor of n and maps the disk $|z|<r$ onto the disk $|z|<r^n$ in an n-to-one manner. Let us now consider a general situation.

If f is analytic at z_0 such that $f'(z_0)=0$, then the conformal character fails. Such a point z_0 is called a critical point of f. Let us now examine the behavior of an analytic function in a neighborhood of a critical point. More generally, let $w=f(z)$ be analytic at z_0 such that $f^{(k)}(z_0)=0$ for $k=1,2,\ldots,n-1$ and $f^{(n)}(z_0)\neq 0$. We wish to show that angles are not preserved at z_0 but are multiplied by n. By hypotheses, we have

$$f(z)-f(z_0)=(z-z_0)^n[a_n+a_{n+1}(z-z_0)+\cdots]=(z-z_0)^n g(z)$$

where g is analytic at z_0 with $g(z_0)=a_n\neq 0$. Thus,

$$\arg(w-w_0)=\arg(f(z)-f(z_0))=n\arg(z-z_0)+\arg g(z).$$

Suppose α is the angle that the tangent vector to a smooth curve γ at z_0 makes with the positive x-axis, and β is the angle that the tangent to the image curve Γ under $w=f(z)$ at $w_0=f(z_0)$ makes with the positive u-axis. If $z\to z_0$ along γ, then $w=f(z)\to w_0=f(z_0)$ along Γ so that the last equation gives

$$\beta=n\alpha+\arg g(z_0)=n\alpha+\arg a_n.$$

This relation shows that the tangent to the image curve depends on the tangent to original curve as well as the order of the derivatives of f and the argument of the first non-zero coefficient in the series expansion of $f'(z)$ at the point in question.

Let γ_1 and γ_2 be two smooth curves passing through z_0 and let Γ_1 and Γ_2 be their respective images under $w=f(z)$. Suppose that the tangent to the curves γ_k and Γ_k make an angle α_k and β_k with the real axis of the z-plane and of the w-plane, respectively. Then, we have

$$\beta_1=n\alpha_1+\arg a_n \quad\text{and}\quad \beta_2=n\alpha_2+\arg a_n; \quad \text{i.e.} \quad \beta=n\alpha$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$ are respectively the angles between the curves γ_1, γ_2 and the respective image curves Γ_1, Γ_2. Now we have proved

5.7. Theorem. *Suppose that f is analytic at z_0 and $f'(z)$ has a zero of order $n-1$ at z_0. If two smooth curves intersect at an angle α in the z-plane, then their images intersect at an angle $n\alpha$ in the w-plane.*

From Theorem 5.7, we obtain that no analytic function can be conformal at its critical points.

5.8. Example. Consider $f(z) = \sin z$. Then f is entire and $f'(z) = \cos z$ so that $f'(z_n) = 0$ for $z_n = (2n+1)\pi/2, n \in \mathbb{Z}$. Thus, f is conformal on $\Omega = \mathbb{C} \setminus \{(2n+1)\pi/2 : n \in \mathbb{Z}\}$. Note that $f''(z) = -\sin z$ and $f''(z_n) \neq 0$ for each $n \in \mathbb{Z}$. According to Theorem 5.7, the angle between any two smooth curves intersecting at z_n $(n \in \mathbb{Z})$ is increased by a factor of 2 by $w = f(z)$. ●

5.9. The transformation $w = \sin z$. First we note that $\sin z$ is periodic with period 2π, $\sin z = -\sin(-z)$ and $\sin(z+\pi) = -\sin z$. In view of these observations, it suffices to understand the mapping behavior of $\sin z$ on a vertical strip of width π. Further, $\sin z$ is clearly conformal on $D = \{z : |\mathrm{Re}\, z| < \pi/2\}$ and maps D one-to-one onto the domain $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$.

With $z = x + iy$ and $w = u + iv$, $w = \sin z$ gives

$$u = \sin x \cosh y \quad \text{and} \quad v = \cos x \sinh y. \tag{5.10}$$

Let us discuss the behavior of $w = \sin z$ on the horizontal and vertical line segments in $\Omega = \{z \in \mathbb{C} : |\mathrm{Re}\, z| \leq \pi/2\}$. First, we consider the horizontal line segment

$$J_b = \{x + iy : y = b,\ |x| \leq \pi/2\}.$$

By (5.10), the image of J_b is given by

$$u = \sin x \cosh b \quad \text{and} \quad v = \cos x \sinh b \quad (|x| \leq \pi/2). \tag{5.11}$$

If $b = 0$, then the image of the line segment $J_0 = [-\pi/2, \pi/2]$ described by (5.11) reduces to

$$u = \sin x \quad \text{and} \quad v = 0,$$

since $\cosh 0 = 1$ and $\sinh 0 = 0$. As we move x from $-\pi/2$ to $\pi/2$ along the line segment J_0, the image in the w-plane advances from -1 to 1 along the line $v = 0$ and thus, $\sin(J_0) = [-1, 1]$.

Next we fix $b \neq 0$ and consider J_b. In this case, the image set described by (5.11) gives, by eliminating x,

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1 \tag{5.12}$$

which is the equation of an ellipse centered at the origin of the w-plane. Its major and minor axes have lengths $2\cosh b$ and $2\sinh b$, respectively. Further, the major and the minor axes lie on the u- and the v-axes, respectively. If $b > 0$, (5.11) indicates that $v > 0$ showing that the image of I_b for $b > 0$ is the upper-half of the ellipse defined by (5.12). Similarly, the image of J_b for $b < 0$ is the bottom half of the ellipse. The images of the segments corresponding to $\pm b$, $b \neq 0$, fit together to form a complete ellipse.

Let us now discuss the image of the vertical lines. Fix a with $|a| \leq \pi/2$, and let

$$I_a = \{x + iy : x = a, \ y \in \mathbb{R}\}$$

to represent a vertical line. By (5.10), the image of I_a under $w = \sin z$ is described by the conditions

$$u = \sin a \cosh y, \quad v = \cos a \sinh y \quad (y \in \mathbb{R}). \tag{5.13}$$

Elimination of the variable y yields

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1 \tag{5.14}$$

whenever $\sin a \neq 0$ and $\cos a \neq 0$. The last conditions are satisfied when $a \notin \{0, \pi/2, -\pi/2\}$. We discuss separately the case when $a \in \{0, \pi/2, -\pi/2\}$. Clearly, the location of the image of I_0 described by

$$u = 0 \quad \text{and} \quad v = \sinh y \quad (y \in \mathbb{R}),$$

is actually a parametric equation of the imaginary axis of the w-plane (as $\sinh(\mathbb{R}) = \mathbb{R}$). For $a = \pi/2$, the image of the line $I_{\pi/2}$ is the set of points given by

$$u = \cosh y, \quad v = 0 \quad (y \in \mathbb{R}).$$

As we move y from 0 to ∞ along the line $x = \pi/2$, these equations indicate that u moves from 1 to ∞ along the line $v = 0$. As $\cosh y = \cosh(-y)$, it follows that the image of the line $I_{-\pi/2}$ is the set of points $u \leq -1$ on the negative real axis.

Next, we discuss the case when $a \notin \{0, \pi/2, -\pi/2\}$. In this case, the image of I_a is the hyperbola described by (5.14) with vertices at $(\pm \sin a, 0)$ and slant asymptotes $v = \pm(\cot a)u$. If $0 < a < \pi/2$, then the first equation reveals that $u > 0$, and if $-\pi/2 < a < 0$ then $u < 0$. This observation shows that for each fixed a, $0 < a < \pi/2$, the image of I_a is the right branch of the hyperbola containing the point $(\sin a, 0)$ while the image of I_a, for $-\pi/2 < a < 0$, is the left branch of the hyperbola containing the point $(-\sin a, 0)$. Then the image of the pair of vertical lines $x = a$ and $x = -a$ with $|a| < \pi/2$ constitute the full hyperbola given by (5.14). Finally, the mapping $w = \cos z$ can be analyzed using the equation

$$\cos z = \sin(z + \pi/2).$$

5.2 Basic Properties of Möbius Maps

What is a Möbius transformation? They are simply a composition of one, some or all of the following special types of transformations.

- *Translation*: It is a map of the form $z \mapsto z + \alpha$, $\alpha \in \mathbb{C}\setminus\{0\}$. If $\alpha = 0$, then it is an identity map.
- *Magnification*: It is a map of the form $z \mapsto rz$, $r \in \mathbb{R} \setminus \{0\}$. Notice that for $r = 1$, this is the identity map whereas for $r = 0$ it is a constant map. If $r > 0$, then $w = rz$ multiplies the modulus of z by r and leaves its argument unchanged. Thus if $r > 1$, then this is a "*magnification*" and if $0 < r < 1$, it is a "*shrinking/contraction*" rather than saying it a "magnification". If $r < 0$, then $w = rz$ gives the reflection through the origin followed by such a "magnification" or "shrinking" depending on $r < -1$ or $-1 < r < 0$.
- *Rotation*: It is a map of the form $z \mapsto e^{i\theta}z$, $\theta \in \mathbb{R}$. This map produces a rotation through an angle about the origin with positive sense if $\theta > 0$. The rotation coupled with magnification is referred to as *dilation:* $z \mapsto az$ $(a \neq 0)$.
- *Inversion*: It is a map of the form $z \mapsto 1/z$ which produces a geometric inversion (or reciprocal map or the inversion map).

Möbius transformations, named in honor of the geometer A.F. Möbius (1790-1868), are rational functions of the form

$$T(z) = \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{C},\ ad - bc \neq 0) \tag{5.15}$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$. Note that (5.15) does not determine the coefficients a, b, c, d uniquely. If we let $T(z) := T_{abcd}(z)$ and if $\alpha \in \mathbb{C} \setminus \{0\}$, then αa, αb, αc, αd correspond to the same Möbius transformation as

$$T_{abcd}(z) = T_{(a\alpha)(b\alpha)(c\alpha)(d\alpha)}(z)$$

so that if $\alpha^2 = 1/(ad - bc)$ then $(a\alpha)(d\alpha) - (b\alpha)(c\alpha) = 1$. In other words, the behavior of T does not change when a, b, c, d are multiplied by a non-zero constant and thus, we may assume that $ad - bc = 1$ whenever there is a need for this normalization to simplify our studies. Certainly, T is analytic on $\mathbb{C} \setminus \{-d/c\}$. Note that if $c = 0$, then (5.15) reduces to

$$T(z) = \left(\frac{a}{d}\right) z + \left(\frac{b}{d}\right) := \alpha z + \beta \quad (ad \neq 0, \text{ i.e. } \alpha \neq 0).$$

A function of this form is called a *linear map.* Clearly, the identity transformation corresponds to $a = d \neq 0$ and $b = c = 0$. If $c \neq 0$, then we can decompose (5.15) as

$$\begin{aligned} T(z) &= \left[a\left(z+\frac{d}{c}\right)+b-\frac{ad}{c}\right]\frac{1}{c(z+d/c)} \\ &= \frac{a}{c}-\left(\frac{ad-bc}{c^2}\right)\frac{1}{z+d/c},\ c\neq 0. \end{aligned} \tag{5.16}$$

Thus, as mentioned in the beginning, a Möbius transformation is a composition of magnification, rotation and translation, and $T(z)$ reduces to a constant whenever $ad-bc=0$. So, throughout the discussion in this chapter, the cases for which $ad-bc=0$ should be ruled out in order to exclude the trivial case for which $T(z)$ reduces to a constant. It can be also observed that the condition $ad-bc\neq 0$ is required since otherwise $T(-d/c)=0/0$, which is undefined.

Note also that $T'(z)$ exists for all z where $cz+d\neq 0$, and

$$T'(z)=\frac{ad-bc}{(cz+d)^2}\quad (z\neq -d/c),$$

so that the condition $ad-bc\neq 0$ simply guarantees that $T(z)$ is not a constant. Therefore, T defines an analytic function on $\mathbb{C}\setminus\{-d/c\}$. Clearly, $T(z)$ is obtained by successive applications of the following four mappings $w_i=T_i(z)$ for $i=1,2,3,4$, where

$$w_1=z+\frac{d}{c},\ \ w_2=\frac{1}{w_1},\ \ w_3=-\left(\frac{ad-bc}{c^2}\right)w_2,\ \ w_4=T_4(z)=\frac{a}{c}+w_3$$

unless $c=0$ where we write $T(z)=(a/d)z+(b/d)$. It follows that $T(z)$ is actually given by the composition of these simpler transformations. More precisely, we have

$$T=T_4\circ T_3\circ T_2\circ T_1$$

and thus, these four types (translation, rotation, magnification and an inversion) generate the *group* of Möbius transformations. Moreover, the function $w=T(z)$ defined by (5.15) can be written as

$$cwz-az+dw-b=0$$

and the expression on the left is linear in both variables z and w, and so, Möbius transformations are also called *Bilinear transformations.* When $c=0$, the transformation is clearly linear.

5.17. Matrix interpretation and the group structure. There is a strong relationship between Möbius transformations and matrices. Indeed, each Möbius transformation of the form (5.15) can be associated with a 2×2 matrix via the map

$$z\mapsto A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\longleftrightarrow T(z).$$

The Möbius transformation T_0 given by $T_0(z) = z$ is the identity transformation which corresponds to the 2×2 identity matrix. If T and S are two Möbius transformations given by

$$T(z) = \frac{az+b}{cz+d} \quad \text{and} \quad S(z) = \frac{a'z+b'}{c'z+d'},$$

then the composition $T \circ S$ is defined by

$$(T \circ S)(z) = T(S(z)) = \frac{a\left(\dfrac{a'z+b'}{c'z+d'}\right)+b}{c\left(\dfrac{a'z+b'}{c'z+d'}\right)+d} = \frac{(aa'+bc')z+ab'+bd'}{(ca'+dc')z+cb'+dd'}.$$

Note that if A and B are the corresponding matrices associated with the transformations T and S, then

$$AB = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}$$

which implies that $T \circ S$ corresponds to the matrix product AB. Moreover, if $\det A = ad - bc \neq 0$ and $\det B = a'd' - b'c' \neq 0$ then

$$\det(AB) = \det A \cdot \det B = (ad-bc)(a'd'-b'c') \neq 0.$$

In particular, we have the following simple result.

5.18. Proposition. *Composition of two Möbius transformations is a Möbius transformation.*

The fact that matrix multiplication corresponds to composition can be reformulated in the language of group theory. Now, let $GL(2,\mathbb{C})$ denote the general linear group consisting of 2×2 invertible matrices A with complex entries:

$$GL(2,\mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C},\ \det A \neq 0 \right\}.$$

Then, $GL(2,\mathbb{C})$ forms a subgroup of the group of all Möbius transformations under the operation of matrix multiplication. If we assign each $A \in GL(2,\mathbb{C})$ the Möbius transformation defined by (5.15), then the map $A \mapsto T_A$ is an isomorphism; i.e. for all $A, B \in GL(2,\mathbb{C})$, $T_{A \circ B} = T_A \circ T_B$. Clearly, the inverse of A is

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and $T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I$. Thus, all the (non-constant) Möbius transformations are invertible. In particular, they are one-to-one. Indeed if $c = 0$, then $T(z) = (a/d)z + (b/d)$ which is trivially one-to-one in $\mathbb{C}$. If $c \neq 0$ then, by (5.16), for $z_1, z_2 \in \mathbb{C} \setminus \{-d/c\}$, we obtain

$$T(z_1) = T(z_2) \Rightarrow \frac{1}{z_1 + d/c} = \frac{1}{z_2 + d/c} \Rightarrow z_1 = z_2.$$

Therefore, T is 1-1 on $\mathbb{C} \setminus \{-d/c\}$. Is a (non-constant) Möbius transformation a surjection onto $\mathbb{C}$? Is there a point in $\mathbb{C}$ missing from the image under a Möbius transformation? As T is one-to-one, its inverse always exists. The inverse of T is obtained by solving the equation

$$w = T(z) = \frac{az + b}{cz + d} \quad (z \neq -d/c)$$

for z. This gives

$$z = T^{-1}(w) = \frac{dw - b}{-cw + a} \quad (w \neq a/c)$$

where $da - (-b)(-c) = ad - bc \neq 0$. We have

5.19. Proposition. *The inverse of a Möbius transformation is also a Möbius transformation.*

Moreover, $T : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$ is a bianalytic (i.e. T and T^{-1} analytic) mapping with

$$z = T^{-1}(w) = \frac{dw - b}{-cw + a} \quad \text{and} \quad T'(z) = \frac{ad - bc}{(cz + d)^2}. \tag{5.20}$$

This is an important property of Möbius transformations which is quite special for complex-valued functions. The later condition shows that the Möbius transformation T is not only a bianalytic map of $\mathbb{C} \setminus \{-d/c\}$ onto $\mathbb{C} \setminus \{a/c\}$ but is also conformal on $\mathbb{C} \setminus \{-d/c\}$. Is it possible to enlarge the domain of definition of T in order to consider it as a map defined on the extended complex plane? Observe that if $c = 0$, then T maps $\mathbb{C}$ onto $\mathbb{C}$ and, $T(z) \to \infty$ if and only if $z \to \infty$. Thus, it makes sense to define $T(\infty) = \infty$ when $c = 0$. If $c \neq 0$, then $T(z) \to \infty$ as $z \to -d/c$ and $T(z) \to a/c$ if $z = \infty$. In view of these observations, it is natural to introduce the limiting values (in the usual sense)

$$T(\infty) = \lim_{|z| \to \infty} T(z) = \frac{a}{c} \quad \text{and} \quad \begin{cases} T(-d/c) = \infty & \text{for } c \neq 0 \\ T(\infty) = \infty & \text{for } c = 0. \end{cases}$$

Similarly, by the first transformation in (5.20), we define

$$T^{-1}(\infty) = \lim_{|w| \to \infty} T^{-1}(w) = -\frac{d}{c} \quad \text{and} \quad \begin{cases} T^{-1}(a/c) = \infty & \text{for } c \neq 0 \\ T^{-1}(\infty) = \infty & \text{for } c = 0, \end{cases}$$

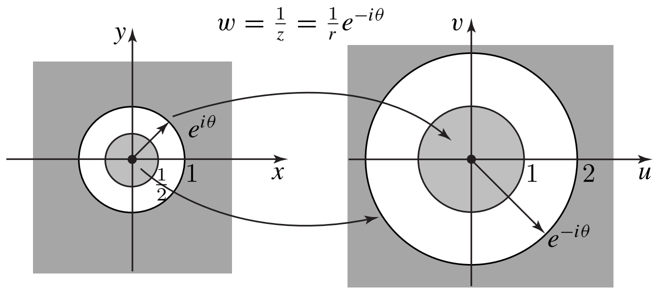


Figure 5.7: Images of disks under inversion.

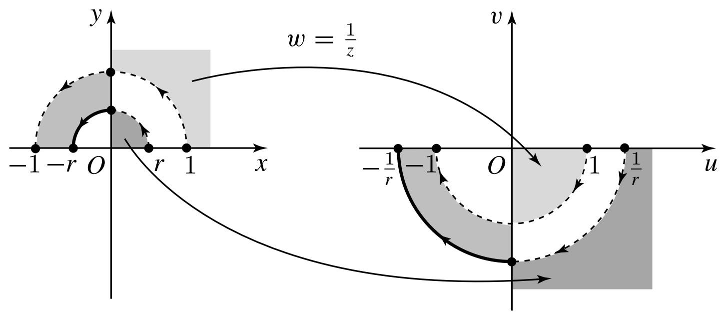


Figure 5.8: Images certain sets under inversion.

so that we conveniently regard a Möbius map T as a one-to-one mapping of the extended complex plane $\mathbb{C}_\infty$ onto itself. Equivalently, we say that T maps the Riemann sphere onto itself; i.e. T defines a conformal self-map of $\mathbb{C}_\infty$. In particular, we conclude that $T(\mathbb{C}_\infty) = \mathbb{C}_\infty$, and for all z and w in $\mathbb{C}_\infty$, we have $T^{-1}(T(z)) = z$ and $T(T^{-1}(w)) = w$.

5.21. Images of circles and lines under Möbius maps. There are many properties of Möbius transformations that have considerable importance in physical applications. Let us first study some basic properties of the inversion function w defined by $w = z^{-1}$ $(z \neq 0)$. Clearly, this function establishes a one-to-one correspondence between the non-zero points of the z, and the w-planes. To start with, we let $z = re^{i\theta}$ $(r > 0,\ \theta = \operatorname{Arg} z)$. Then

$$w = \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{\overline{z}}{|z|^2}.$$

Under this transformation, we easily have the following (see Figures 5.7 and 5.8):

(i) points in the upper half-plane $(\operatorname{Im} z > 0)$ are mapped onto points in the lower half-plane $(\operatorname{Im} w < 0)$ and vice-versa

(ii) the right half-plane ($\mathrm{Re}\, z > 0$) is mapped onto itself

(iii) the left half-plane ($\mathrm{Re}\, z < 0$) is mapped onto itself

(iv) points on the circle $|z| = R$ are mapped onto points on the circle $|w| = 1/R$, the disk $|z| < R$ is mapped onto the disk $|w| > 1/R$, and the points z such that $|z| > R$ are mapped onto points in the disk $|w| < 1/R$.

So, it is natural to ask the following simple question: what happens to general circles and straight lines under inversions, and more generally under Möbius transformations?

Consider the equation in (x, y)-coordinates of the form

$$\alpha(x^2 + y^2) - 2ax - 2by = R^2 - (a^2 + b^2) \quad (\alpha, a, b \in \mathbb{R},\ R > 0)$$

which is a circle or a (open) straight line depending on whether $\alpha \neq 0$ or $\alpha = 0$. In fact $\alpha = 1$ actually corresponds to the circle $|z - (a + ib)| = R$, whereas $\alpha = 0$ gives a line. As

$$2x = z - \overline{z} \ \text{ and } \ 2y = -i(z - \overline{z}),$$

in terms of the complex variable, we can consider all possible circles and straight lines in the form

$$\alpha|z|^2 - (a - ib)z - (a + ib)\overline{z} + a^2 + b^2 - R^2 = 0;$$

or equivalently in the form $\alpha|z|^2 + 2\mathrm{Re}\,(\beta z) + \gamma = 0$, $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$.

5.22. Lemma. (Circle-preserving Property) *Every Möbius transformation maps circles and straight lines in the z-plane into circles or lines.*

Proof. It is quite obvious that among the four elementary transformations, the three of them, namely, translation, magnification (scaling) and rotation do preserve circles and straight lines as it is clear that each of these three transformations sends circles to circles and lines to lines. Therefore, from the equivalent expression for Möbius transformation given by (5.16), it only remains to verify the statement for the inversion given by $w = 1/z$. Every circle or line in $\mathbb{C}$ can be described in the form

$$\alpha|z|^2 + 2\mathrm{Re}\,(\beta z) + \gamma = 0, \quad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C}. \tag{5.23}$$

Obviously, this is an equation of a circle if $\alpha \neq 0$. To obtain the center and the radius, complete the square. If $\alpha = 0$, it is a straight line in $\mathbb{C}$, which is a circle through ∞. Now, if $z \in \mathbb{C}\setminus\{0\}$ and $w = 1/z$ ($w \neq 0$) then $z = 1/w$ so that (5.23) transforms to

$$\frac{\alpha}{|w|^2} + 2\mathrm{Re}\left(\frac{\beta}{w}\right) + \gamma = 0.$$

Multiplying both sides by $w\overline{w} = |w|^2$, this equation assumes the form

$$\gamma|w|^2 + 2\mathrm{Re}\,(\overline{\beta}w) + \alpha = 0. \tag{5.24}$$

Clearly, the desired conclusion follows from (5.23) and (5.24). ■

We observe that Lemma 5.22 neither claims that every circle in $\mathbb{C}$ is mapped to a circle in $\mathbb{C}$ nor does it claim that every line in $\mathbb{C}$ is mapped to a line in $\mathbb{C}$. Further, from Lemma 5.22, it follows that every Möbius transformation, being one of four special transformations, carries the families of circles in $\mathbb{C}_\infty$ onto itself. Here we regarded straight line (as a limiting case of circle) in the extended complex plane as a circle on the Riemann sphere–using the stereographic projection. That is to say that line on the extended complex plane is a circle of infinite radius (meaning that it is a circle through the point at infinity). So, we can think of circles and lines as belonging to the same class, and reformulate Lemma 5.22 as

5.25. Theorem. *Every Möbius transformation maps circles in $\mathbb{C}_\infty$ onto circles in $\mathbb{C}_\infty$.*

It is worth it to have a bit more detail on the two equations (5.23) and (5.24).

Case (i): Assume that $\alpha = 0$ and $\gamma = 0$. In this case, from (5.23) and (5.24), we see that the straight line that passes through the origin given by $\mathrm{Re}\,(\beta z) = 0$ is transformed into a straight line that passes through the origin given by $\mathrm{Re}\,(\overline{\beta}w) = 0$.

Case (ii): Let $\alpha = 0$ and $\gamma \neq 0$. In this case, (5.23) is equivalent to

$$2\mathrm{Re}\,(\beta z) + \gamma = 0$$

which is an equation of a straight line that does not pass through the origin. By (5.24), this straight line is transformed into

$$\gamma|w|^2 + 2\mathrm{Re}\,(\overline{\beta}w) = 0, \quad \text{i.e.} \quad \left|w + \frac{\beta}{\gamma}\right| = \frac{|\beta|}{\gamma},$$

which is a circle passing through the origin. Note that $\beta = 0$ is not possible in both the cases.

Case (iii): Let $\alpha \neq 0$ and $\gamma \neq 0$. In this case, (5.23) is equivalent to

$$\left|z + \frac{\overline{\beta}}{\alpha}\right| = \sqrt{\frac{|\beta|^2 - \alpha\gamma}{\alpha^2}}$$

where $|\beta| > \alpha\gamma$. This circle does not pass through the origin and by (5.24), the image of this circle is given by

$$\left|w + \frac{\beta}{\gamma}\right| = \sqrt{\frac{|\beta|^2 - \alpha\gamma}{\gamma^2}}, \quad |\beta|^2 > \alpha\gamma.$$

Note that this circle does not pass through the origin. More precisely, the above discussion leads to

5.26. Proposition. *Under the function $w = 1/z$, we have*

- *the image of a line through the origin is a line through the origin*
- *the image of a line not through the origin is a circle through the origin*
- *the image of a circle through the origin is a line not through the origin*
- *the image of a circle not through the origin is a circle not through the origin.*

In particular, under the inversion $w = 1/z$, the vertical line $\mathrm{Re}\, z = \alpha$ $(\alpha \neq 0)$ maps onto the circle $|w - 1/(2\alpha)| = 1/(2|\alpha|)$ while the horizontal line $\mathrm{Im}\, z = \alpha$ $(\alpha \neq 0)$ maps onto the circle $|w + i/(2\alpha)| = 1/(2|\alpha|)$.

5.3 Fixed Points and Möbius Maps

Let D be a subset of $\mathbb{C}_\infty$ and $f : D \to \mathbb{C}_\infty$. A point $z_0 \in D$ is said to be a *fixed point* of f if $f(z_0) = z_0$. The set of all fixed points of f is denoted by $\mathrm{Fix}\,(f)$. For example, we have

(i) the function $f(z) = z^2$ has exactly three fixed points, namely, $0, 1$ and ∞ whereas the function $f(z) = z^{-1}$ has two fixed points namely 1 and -1.

(ii) the function $f(z) = z - 1$ has no fixed points in $\mathbb{C}$ whereas it has one fixed point in $\mathbb{C}_\infty$, namely the point at ∞.

(iii) the reflection $z \mapsto \overline{z}$ is not a Möbius transformation but $f(z) = \overline{z}$ has all the points on $\mathbb{R}$ as its fixed points. What are the fixed points of $f(z) = i\overline{z}$?

(iv) the function $f(z) = iz/|z|$, $z \neq 0$, has no fixed points in $\mathbb{C}\backslash\{0\}$.

(v) every non-constant real-valued continuous function $f : (-1, 1) \to (-1, 1)$ has a fixed point in $(-1, 1)$. However, a similar result does not hold for functions $f : \Delta \to \Delta$. For example $\phi_\alpha : \Delta \to \Delta$, $|\alpha| = 1$, defined by
$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$
has no fixed points in Δ.

5.27. Proposition. *Every Möbius transformation $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ has at most two fixed points in $\mathbb{C}_\infty$ unless $T(z) \equiv z$. Equivalently, if a Möbius transformation leaves three points in $\mathbb{C}_\infty$ fixed, then it is none other than the identity function.*

Proof. Suppose that

$$T(z) = \frac{az+b}{cz+d}$$

is such that $T(z) \not\equiv z$. Then

$$T(z) = z \iff cz^2 + (d-a)z - b = 0.$$

If $c \neq 0$, then $T(-d/c) = \infty$ and $T(\infty) = a/c$, so neither $-d/c$ nor ∞ is a fixed point of T. So T has at most two fixed points in this case and they are obtainable from the quadratic equation. If $c = 0$, then

$$T(z) = \frac{a}{d}z + \frac{b}{d}.$$

In this case, T has ∞ and $b/(d-a)$ are the only fixed points. ■

5.28. Corollary. *If S and T are two Möbius transformations which agree at three distinct points of $\mathbb{C}_\infty$, then $S = T$.*

Proof. Suppose that $S(z_j) = T(z_j)$ for z_1, z_2, z_3 in $\mathbb{C}_\infty$. Then,

$$(S^{-1} \circ T)(z_j) = z_j = (T^{-1} \circ S)(z_j) \quad \text{for } j = 1, 2, 3$$

and, by Proposition 5.27, $S^{-1} \circ T = I$. Therefore,

$$T = (S \circ S^{-1}) \circ T = S \circ (S^{-1} \circ T) = S \circ I = S. \qquad ■$$

The following result is actually a reformulation of Proposition 5.27, but we will provide an alternate proof because of its independent interest.

5.29. Proposition. *Every Möbius transformation which fixes $0, 1, \infty$ is necessarily the identity map.*

Proof. Suppose that T is given in the form (5.15), and T fixes $0, 1, \infty$. It follows that

(i) $T(\infty) = \infty$ gives $c = 0$ whereas $T(0) = 0$ gives $b = 0$
(ii) $T(1) = 1$ gives $(a+b)/(c+d) = 1$ so that $a = d$, by (i).

We conclude that T is the identity map. ■

A Möbius transformation which has a unique fixed point in $\mathbb{C}$ is called *parabolic*. If it has exactly two fixed points, then it is called *loxodromic*.

5.30. Characterizations of Möbius maps in terms of their fixed points. Let us characterize a given Möbius transformation according to its fixed points (or point). Let $S(z) \not\equiv z$ be a Möbius map. It might be convenient to use the following equivalent form:

$$(5.31)\qquad S(z) = \begin{cases} \dfrac{a}{d}z + \dfrac{b}{d} = \alpha z + \beta & \text{if } c = 0 \\ \dfrac{a}{c} - \left(\dfrac{ad-bc}{c^2}\right)\dfrac{1}{z+d/c} = \gamma - \delta\left(\dfrac{1}{z+d/c}\right) & \text{if } c \neq 0, \end{cases}$$

where (for convenience) we have used the notation

$$\alpha = \frac{a}{d},\ \beta = \frac{b}{d},\ \gamma = \frac{a}{c}\ (c \neq 0),\ \delta = \frac{ad-bc}{c^2}\ (c \neq 0).$$

Case (i): If $\infty \in \text{Fix}\,(S)$, then $a/c = \infty$ (i.e. $c = 0$), and so

$$(5.32)\qquad S(z) = \alpha z + \beta, \quad 0 \neq \alpha \in \mathbb{C},\ \beta \in \mathbb{C}.$$

If $\alpha = 1$ (i.e. $a = d$), then $S(z) = z + \beta$ and ∞ is the only fixed point. If $\alpha \neq 1$ (i.e. $a \neq d$), then ∞ and $z_0 = \beta/(1-\alpha) = b/(d-a)$ are the only fixed points. Observe that

$$S(z) - z_0 = \alpha(z - z_0) + (\alpha - 1)z_0 + \beta = \alpha(z - z_0).$$

Thus, S can be written in the form $S(z) - z_0 = \alpha(z - z_0)$, where α is a complex number which is neither 0 nor 1, and z_0 is a fixed point of S.

Case (ii): If $\infty \notin \text{Fix}\,(S)$ (i.e. $c \neq 0$), then S has at most two fixed points in $\mathbb{C}$ and are in fact obtained from solving the fixed point equation $S(z) = z$, i.e. $cz^2 + (d-a)z - b = 0$. This gives

$$z_0 = \frac{(a-d) \pm \sqrt{D}}{2c}, \quad D = (a-d)^2 + 4bc = (a+d)^2 - 4(ad-bc).$$

Therefore, we need to deal with two cases, namely, $D = 0$ and $D \neq 0$. If $D = 0$, then S has only one fixed point, say z_0, and it is given by

$$z_0 = \frac{a-d}{2c}, \quad \text{i.e.} \quad cz_0 + d = \frac{a+d}{2}.$$

So, we have $(cz_0 + d)^2 = (a+d)^2/4 = ad - bc$. Using this we can write

$$\begin{aligned} S(z) - S(z_0) &= \delta\left[\frac{1}{z_0 + d/c} - \frac{1}{z + d/c}\right] \\ &= \frac{(ad-bc)(z-z_0)}{(cz_0+d)(cz+d)} \quad (\text{since } \delta = (ad-bc)/c^2) \\ &= \frac{(cz_0+d)(z-z_0)}{cz+d} \quad (\text{since } (cz_0+d)^2 = ad-bc). \end{aligned}$$

Therefore, if $\infty \notin \text{Fix}\,(S)$ and $z_0 \in \text{Fix}\,(S)$ with $(a+d)^2 = 4(ad-bc)$ then, by the last equation, the Möbius transformation S reduces to the expression (note that $S(z_0) = z_0$)

$$\frac{1}{S(z)-z_0} = \frac{cz_0+d+c(z-z_0)}{(cz_0+d)(z-z_0)} = \frac{1}{z-z_0} + \frac{c}{cz_0+d} = \frac{1}{z-z_0} + k,$$

where $k \neq 0$. The discussion in Case (ii) gives

5.33. Theorem. *If z_0 is the coincident fixed point in $\mathbb{C}$ of $w = S(z)$, then*

$$\frac{1}{S(z)-z_0} = \frac{1}{z-z_0} + k, \quad k \neq 0.$$

This result gives

$$S(z) = z_0 + \frac{z-z_0}{1+k(z-z_0)} \quad (k \neq 0)$$

which is the general form of a parabolic transformation that fixes the finite point z_0. Also, we note that, $S(z)$ takes ∞ to $z_0 + 1/k$ and $z_0 - 1/k$ to ∞, which means that z_0, $f(\infty)$ and $f^{-1}(\infty)$ are collinear points in $\mathbb{C}$.

Finally, suppose that $D \neq 0$, i.e. $(a+d)^2 \neq 4(ad-bc)$. Then S has two distinct fixed points in $\mathbb{C}$, say z_1 and z_2. In this case, we may select a Möbius transformation $T(z)$ such that $z_1 \mapsto 0$ and $z_2 \mapsto \infty$. This gives

$$w = T(z) = \frac{z-z_1}{z-z_2}.$$

Then, $z = T^{-1}(w)$ and $f = T \circ S \circ T^{-1}$ has 0 and ∞ as its only fixed points. Hence, by Case (i) with $z_0 = 0$, $f(w) = \alpha w$; or equivalently,

$$T(S(T^{-1}(w))) = \alpha w, \quad \text{i.e. } T(S(z)) = \alpha\left(\frac{z-z_1}{z-z_2}\right),$$

where α is neither 0 nor 1. As $S(z_1) = z_1$ and $S(z_2) = z_2$, the last expression would then complete the proof of the following

5.34. Theorem. *Every Möbius transformation $S(z)$ which has exactly two distinct fixed points z_1 and z_2 in $\mathbb{C}_\infty$ can be written as*

$$\begin{cases} \dfrac{S(z)-z_1}{S(z)-z_2} = \alpha\left(\dfrac{z-z_1}{z-z_2}\right) & \text{if } z_1,\, z_2 \in \mathbb{C} \\ S(z)-z_1 = \alpha(z-z_1) & \text{if } z_2 = \infty, \end{cases}$$

where $\alpha \in \mathbb{C}\backslash\{0,1\}$.

This result may be obtained from another result concerning the invariance property of the cross-ratio that will be discussed later, see Theorem 5.39.

5.4 Triples to Triples under Möbius Maps

The Möbius transformation given by (5.15) has four coefficients a, b, c, d. One of them can be adjusted without changing the transformation. Indeed, as $ad - bc \neq 0$, both a and c cannot be zero simultaneously and so, we can rewrite the homogeneous expression of the coefficients of $T(z)$ as

$$T(z) = \begin{cases} \dfrac{z + (b/a)}{(c/a)z + (d/a)} & \text{if } a \neq 0 \\ \dfrac{(a/c)z + (b/c)}{z + (d/c)} & \text{if } c \neq 0. \end{cases}$$

In either case, T is actually determined by only three constants. Therefore, it is natural to expect that we require only 'three degrees of freedom' to determine T uniquely. As T maps a circle onto another circle and from elementary geometry we know that just three points determine a circle. These observations will be made precise in the following two results which imply that three independent complex parameters in $\mathbb{C}_\infty$ are sufficient to describe Möbius maps uniquely, namely, the image of three prescribed points.

5.35. Theorem. *Given three distinct points z_1, z_2, z_3 in $\mathbb{C}_\infty$, there exists a unique Möbius transformation $T(z)$ such that*

$$T(z_1) = 0,\ T(z_2) = 1, \quad \textit{and} \quad T(z_3) = \infty.$$

Proof. To establish the existence of such a function is easy. Regardless of the choice of the constant k, $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by

$$T(z) = k\left(\frac{z - z_1}{z - z_3}\right)$$

sends $z_1 \mapsto 0$ and $z_3 \mapsto \infty$. Adjust the constant k such that $T(z_2) = 1$. This gives the desired map

$$T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} =: \frac{Az + B}{Cz + D}, \tag{5.36}$$

whenever $z_1, z_2, z_3 \in \mathbb{C}$. Note that

$$AD - BC = (z_2 - z_1)(z_2 - z_3)(z_1 - z_3) \neq 0.$$

How do we define T if any one of the z_j's is not finite? If one of the z_j's is ∞, then the corresponding $T(z)$ is defined by the natural limiting value. For example if $z_1 = \infty$, then we need to write $T(z)$ as

$$T(z) = \lim_{z_1 \to \infty} \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \lim_{z_1 \to \infty} \frac{[(z/z_1) - 1](z_2 - z_3)}{(z - z_3)[(z_2/z_1) - 1]} = \frac{z_2 - z_3}{z - z_3}.$$

Similarly, we define

$$T(z)=\begin{cases}\dfrac{z_2-z_3}{z-z_3} & \text{if } z_1=\infty\\[4pt] \dfrac{z-z_1}{z-z_3} & \text{if } z_2=\infty\\[4pt] \dfrac{z-z_1}{z_2-z_1} & \text{if } z_3=\infty.\end{cases}\tag{5.37}$$

Thus $T(z)$ defined by (5.36) or (5.37) is the desired Möbius transformation which takes z_1 to 0, z_2 to 1 and z_3 to ∞.

To complete the proof, it remains to show that T is unique. Suppose that $S(z)$ is a Möbius transformation satisfying the condition

$$S(z_1)=0,\ S(z_2)=1, \text{ and } S(z_3)=\infty.$$

It follows that $f=S\circ T^{-1}$ is again a Möbius transformation and that f fixes $0,1,\infty$. By Proposition 5.29, $f(z)=z$ which means that $S=T$.

Note that $T^{-1}\circ S$ fixes three distinct points z_1,z_2,z_3 and so, by Proposition 5.27, it is the identity transformation. ∎

Theorem 5.35 immediately implies an important mapping property which asserts that three distinct points uniquely determine a Möbius transformation.

5.38. Theorem. *If $\{z_1,z_2,z_3\}$ and $\{w_1,w_2,w_3\}$ are two sets of triplets of distinct points in $\mathbb{C}_\infty$, then there exists a unique Möbius transformation taking z_j to w_j $(j=1,2,3)$ and that it is given by*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Proof. According to Theorem 5.35, there exists two Möbius transformations S and R such that

$$S(z_1)=R(w_1)=0,\ S(z_2)=R(w_2)=1, \text{ and } S(z_3)=R(w_3)=\infty.$$

They are given by

$$S(z)=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad\text{and}\quad R(w)=\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)},$$

respectively. If any one of the z_j's and w_j's is ∞, then the above are formally treated as a limiting case. Now the existence of an f sending z_j to w_j $(j=1,2,3)$ is clear, as the Möbius transformation $f=R^{-1}\circ S$ does the job.

As for the uniqueness, suppose that there are two Möbius transformations, say f and g, which enjoy the stated property; that is (see Figure 5.9), $f(z_j)=g(z_j)=w_j$ for $j=1,2,3$. Then it follows that $R\circ g\circ S^{-1}$ fixes $0,1,\infty$ and so it is the identity map. Thus, $g=R^{-1}\circ S$, which proves that f is unique. ∎

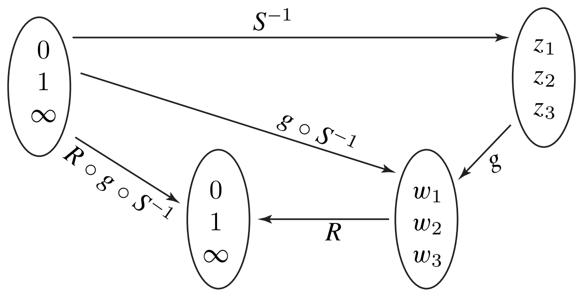


Figure 5.9: Uniqueness of $f = R^{-1} \circ S$.

5.5 The Cross-Ratio and its Invariance Property

For the set of three distinct points z_1, z_2, z_3 of $\mathbb{C}_\infty$, the expression

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(z - z_1)/(z - z_3)}{(z_2 - z_1)/(z_2 - z_3)}$$

is called the *cross-ratio* of the four points z, z_1, z_2, z_3 and is denoted by (z, z_1, z_2, z_3), where if one of the four points is ∞, the factors containing it should be omitted (see the limiting form in (5.37)). Considering z as a variable point and treating the three distinct points z_1, z_2, z_3 as fixed complex numbers in $\mathbb{C}_\infty$, we obtain that the cross-ratio (z, z_1, z_2, z_3) is precisely the Möbius transformation which sends z_1, z_2, z_3 into $0, 1, \infty$, respectively. In view of the uniqueness of the map as described in Theorem 5.35, the cross-ratio is well defined. Also, we observe that

$$(z, z_1, z_2, z_3) = (z_1, z, z_3, z_2) = (z_2, z_3, z, z_1) = (z_3, z_2, z_1, z).$$

Note that there are $4! = 24$ cross-ratios corresponding to the permutations performed on the four points z, z_1, z_2, z_3. But it can be easily seen that only six of these are different. In addition, we stress that a cross-ratio (z_4, z_1, z_2, z_3) associated with four distinct points z_4, z_1, z_2, z_3 in $\mathbb{C}_\infty$ is a finite quantity different from 0 and 1.

The invariance of cross-ratios under Möbius transformations is the subject of our next result which provides a way to represent a Möbius transformation that carries three distinct points to prescribed image points w_1, w_2 and w_3.

5.39. Theorem. *The cross-ratio is invariant under Möbius transformations.*

Proof. Let $w = S(z)$ be a Möbius transformation defined by (5.31). Let $\{z_1, z_2, z_3\}$ be a set of three distinct points in $\mathbb{C}_\infty$, and let $\{w_1, w_2, w_3\}$

be their images under this map, i.e. $w_j = S(z_j)$ for $j = 1, 2, 3$. Now, for each $j = 1, 2, 3$, we have

$$w - w_j = \begin{cases} \alpha(z - z_j) & \text{if } c = 0 \\ \dfrac{\delta(z - z_j)}{(z + d/c)(z_j + d/c)} & \text{if } c \neq 0, \end{cases}$$

where $\alpha = a/d$ and $\delta = (ad - bc)/c^2$. Therefore, it follows easily that

$$\begin{aligned} (w, w_1, w_2, w_3) &= \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} \\ &= \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \\ &= (z, z_1, z_2, z_3) \end{aligned}$$

as asserted. ∎

5.40. Example. Let us find the Möbius transformation which sends 0 to 1, i to 0 and ∞ to -1. To do this, for $z \notin \{0, i, \infty\}$, we may appeal to

$$(z, 0, i, \infty) = (w, 1, 0, -1), \quad \text{i.e.} \quad \frac{z - 0}{i - 0} = \frac{(w - 1)(0 + 1)}{(w + 1)(0 - 1)}$$

and so, we arrive at the formula $w = (1 + iz)/(1 - iz)$. There is a direct approach to problems of this type. Since $i \mapsto 0$, we can normalize a to be 1 and write the Möbius transformation in the form

$$w = \frac{z - i}{cz + d}.$$

The conditions $0 \mapsto 1$ and $\infty \mapsto -1$ yield

$$1 = -\frac{i}{d} \quad \text{and} \quad -1 = \frac{1}{c}, \ \text{i.e.} \ \ d = -i \text{ and } c = -1.$$

The desired formula follows. •

5.41. Example. To demonstrate Theorem 5.34, we consider $w = T(z) = 1/z$. Then the fixed points of $T(z)$ are $z_1 = 1$ and $z_2 = -1$. Note that $w_1 = T(z_1) = 1$, $w_2 = T(z_2) = -1$. Select the third point z_3 distinct from z_1 and z_2 such that $T(z_3) \neq \infty$. Because $1 \mapsto 1$, $-1 \mapsto -1$ and $z_3 \mapsto w_3$, by the invariance property of the cross-ratio,

$$[w, 1, w_3, -1] = [z, 1, z_3, -1]$$

so that

$$\frac{(w - 1)(w_3 + 1)}{(w + 1)(w_3 - 1)} = \frac{(z - 1)(z_3 + 1)}{(z + 1)(z_3 - 1)}, \quad \text{i.e.} \quad \frac{w - 1}{w + 1} = \mu \left(\frac{z - 1}{z + 1} \right)$$

for some $\mu \in \mathbb{C}$. To find μ, substitute $z = i$ (so that $w = -i$) into this equation. This gives $\mu = -1$. Note that the choice $z = 0$, and $w = \infty$ quickly gives $\mu = -1$. ●

5.42. Remark. Suppose that z, z_1, z_2, z_3 are four distinct points in $\mathbb{C}_\infty$ and one of them is ∞, say z_1. Then,

$$\begin{aligned}(z, \infty, z_2, z_3) \text{ is real } &\iff \frac{z_2 - z_3}{z - z_3} = \frac{1}{t} \text{ for some real } t \\ &\iff (1-t)z_3 + tz_2 = z\end{aligned}$$

showing that, (z, ∞, z_2, z_3) is real iff z, ∞, z_2 and z_3 are on a line through ∞.

Next, we recall that the equation $\arg(z - b) = \alpha$ (a constant) defines a half straight line issued from the point $b \in \mathbb{C}$. If $a \in \mathbb{C}$ also lies on this line, then $\arg(a - b) = \alpha$ and therefore,

$$\frac{z-b}{a-b} = t, \quad \text{or} \quad z = ta + (1-t)b, \text{ for some real } t.$$

This observation implies that if z_1, z_2, z_3 are distinct and lie on a line L in $\mathbb{C}$, then we see that the quantity $(z_2 - z_3)/(z_2 - z_1)$ is a real number. Consequently,

$$\begin{aligned}(z, z_1, z_2, z_3) \text{ is real } &\iff \frac{1 - z_1/z}{1 - z_3/z} \text{ is real} \\ &\iff \text{either } z = \infty \text{ or } \frac{z - z_1}{z - z_3} = 1 - \frac{1}{s} \text{ for some } s \in \mathbb{R} \\ &\iff \text{either } z = \infty \text{ or } z = sz_1 + (1-s)z_3.\end{aligned}$$

The last implication clearly shows that if we assume that z_1, z_2, z_3 are distinct and lie on a line L in $\mathbb{C}$, then the cross-ratio (z, z_1, z_2, z_3) is a real number iff either $z = \infty$ or z is on the line L. ●

5.43. Theorem. *The four distinct points z, z_1, z_2, z_3 in $\mathbb{C}_\infty$ all lie on a circle or on a line iff their cross-ratio (z, z_1, z_2, z_3) is a real number.*

Proof. Suppose that z, z_1, z_2, and z_3 are four distinct points in $\mathbb{C}_\infty$ and

$$w = T(z) = (z, z_1, z_2, z_3).$$

As $(w, 0, 1, \infty) = w$, the last equation is equivalent to writing

$$(w, 0, 1, \infty) = (z, z_1, z_2, z_3).$$

But we know that T maps generalized circles into generalized circles and the points $0, 1, \infty$ are collinear points so that the real axis in the w-plane is the image of the line or circle through the points z_1, z_2, z_3, respectively. Therefore, the point z is on this line or circle iff $w = T(z)$ is real. In other words, $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff z_1, z_2, z_3, and z_4 lie on a generalized circle. ■

5.6 Conformal Self-maps of Disks and Half-planes

Our results in this section mainly will be of the following type: Given two simply connected domains Ω and Ω', construct an analytic conformal mapping from Ω onto Ω'. An analytic map $f : \Omega \to \Omega'$ is said to be homeomorphic if it has an analytic inverse map $g : \Omega' \to \Omega$, i.e. $f \circ g = I_\Omega$ and $g \circ f = I_{\Omega'}$. In addition, if $\Omega = \Omega'$ then we say that f is an automorphism of Ω. More precisely, a conformal self-map of a domain Ω is an analytic function from Ω into Ω that is one-to-one and onto. Every conformal self-map of a domain is called an *automorphism* of that domain. We denote by Aut (Ω), the set of all automorphisms of Ω. The set Aut (Ω) forms what is called a "group", with composition as the group operation. The identity element is the identity map given by $I(z) = z$.

5.44. Mappings of half-planes onto disks. There are a number of ways to characterize all conformal mappings of the upper half-plane $\mathbb{H}^+$ onto the (open) unit disk Δ and the extended real line $\mathbb{R}_\infty$ onto the unit circle $\partial\Delta$.

Suppose that $\theta \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are fixed numbers such that $\operatorname{Im}\beta > 0$. Then, it is a simple exercise to see that f, defined by

$$w = f(z) = e^{i\theta}\left(\frac{z-\beta}{z-\overline{\beta}}\right),$$

maps $\mathbb{H}^+$ onto Δ and $\mathbb{R}_\infty$ onto $\partial\Delta$. Indeed,

$$\begin{aligned} |w| < 1 &\iff |z-\overline{\beta}|^2 - |z-\beta|^2 > 0 \\ &\iff -2\operatorname{Re}\,(z(\beta-\overline{\beta})) = 4(\operatorname{Im}\beta)(\operatorname{Im} z) > 0. \end{aligned}$$

For example, for $\beta = i$, we see that $g(z) = e^{i\theta}\,((z-i)/(z+i))$ is a mapping of $\mathbb{H}^+$ onto Δ. Consequently, as $\phi(z) = e^{i\pi/2}z$ maps the right half-plane $\{z : \operatorname{Re} z > 0\}$ onto $\mathbb{H}^+$, the composition $g \circ \phi$ given by

$$(g \circ \phi)(z) = g(e^{i\pi/2}z) = e^{i\theta}\left(\frac{z-1}{z+1}\right)$$

maps the right half-plane $\{z : \operatorname{Re} z > 0\}$ onto the unit disk Δ and the imaginary axis onto the unit circle $|w| = 1$. We are actually interested in the following:

5.45. Problem. Do all one-to-one, onto analytic maps of $\mathbb{H}^+$ to Δ precisely have the form of f for some $\beta \in \mathbb{H}^+$ with some $\theta \in \mathbb{R}$?

The answer is yes. First, let us give a direct proof. Consider

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

First, we recall that $f(\infty) = \infty$ iff $c = 0$. Since we require a map which takes $\mathbb{R}_\infty$ into $\partial\Delta$, we must have $c \neq 0$ (and hence $a \neq 0$). Consequently, we write

$$f(z) = \frac{a}{c}\left(\frac{z + b/a}{z + d/c}\right).$$

To obtain the explicit map, we translate our requirements into the equations

$$|f(\infty)| = 1 = |f(0)|, \quad \text{i.e.} \quad \left|\frac{a}{c}\right| = 1 = \left|\frac{b}{d}\right|. \tag{5.46}$$

Note that both b and d cannot be zero, since $|b/d| = 1$. Now, since $|a/c| = 1$, we can write $a/c = e^{i\theta}$ for some $\theta \in \mathbb{R}$ so that

$$f(z) = e^{i\theta}\left(\frac{z - \beta}{z - \alpha}\right)$$

with $\alpha = -d/c$ and $\beta = -b/a$. Moreover, $|f(1)| = 1$ gives

$$|1 - \beta| = |1 - \alpha|, \quad \text{i.e.} \quad 1 + |\beta|^2 - 2\mathrm{Re}\,\beta = 1 + |\alpha|^2 - 2\mathrm{Re}\,\alpha.$$

Since $|\alpha| = |\beta|$, by (5.46), the last equation yields $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$. Therefore, either $\alpha = \beta$ or $\alpha = \overline{\beta}$. Consequently, $\alpha = \overline{\beta}$ as the alternate is not possible because otherwise $ad - bc = 0$. Thus, the desired map f turns out to be of the form

$$f(z) = e^{i\theta}\left(\frac{z - \beta}{z - \overline{\beta}}\right)$$

for some $\theta \in \mathbb{R}$. Finally, we note that $f(\beta) = 0 \in \Delta$. Therefore, if $\mathrm{Im}\,\beta > 0$, i.e. $\beta \in \mathbb{H}^+$, then we have proved the following result.

5.47. Theorem. *All conformal mappings which map $\mathbb{H}^+$ onto the unit disk Δ such that $\beta \in \mathbb{H}^+$ maps onto 0 are given by*

$$f(z) = e^{i\theta}\left(\frac{z - \beta}{z - \overline{\beta}}\right) \tag{5.48}$$

for some $\theta \in \mathbb{R}$. The inverse mapping $f^{-1} : \Delta \to \mathbb{H}^+$ of f is given by

$$f^{-1}(w) = \frac{\beta e^{i\theta} - \overline{\beta}w}{e^{i\theta} - w}.$$

Using this result, we can find the most general Möbius transformation sending the unit disk Δ onto itself (see also Theorem 5.59). We recall that every Möbius transformation is one-to-one on $\mathbb{C}_\infty$ and the inverse function exists and is also one-to-one on $\mathbb{C}_\infty$. Observe that f defined by (5.48) belongs to $\mathcal{H}(\mathbb{C} \setminus \{\overline{\beta}\})$. Once again, as $\left|z - \overline{\beta}\right|^2 - |z - \beta|^2 = 4(\mathrm{Im}\,\beta)(\mathrm{Im}\,z)$, it follows that

"$|f(z)| < 1 \iff (\mathrm{Im}\,\beta)\,(\mathrm{Im}\,z) > 0$" and "$|f(z)| = 1 \iff (\mathrm{Im}\,\beta)\,(\mathrm{Im}\,z) = 0$".

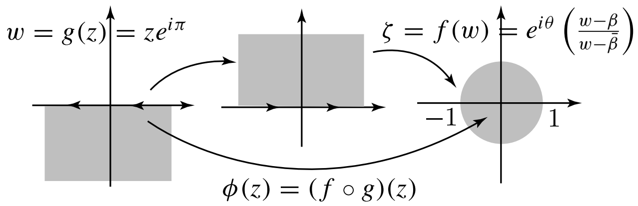


Figure 5.10: Mapping of lower half-plane onto unit disk.

Thus, it is often interesting to study the mapping properties between lines or circles.

5.49. Corollary. *All conformal mappings which map the lower half-plane* $\mathbb{H}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ *onto the unit disk* Δ *such that* $b \in \mathbb{H}^-$ *maps onto* 0 *are given by (see Figure 5.10)*

$$f(z) = e^{i\theta}\left(\frac{z-b}{z-\overline{b}}\right), \quad \theta \in \mathbb{R}.$$

5.50. Theorem. *All conformal mappings which map the right half-plane* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *onto the unit disk* Δ *such that* $\gamma\,(\operatorname{Re}\gamma > 0)$ *maps onto* 0 *are given by*

$$f(z) = e^{i\theta}\left(\frac{z-\gamma}{z+\overline{\gamma}}\right), \quad \theta \in \mathbb{R}.$$

Proof of this theorem is easy if one proceeds exactly as in Theorem 5.47. Alternatively, if $\phi(z) = iz$ and f is as in Theorem 5.47, then the composition $(f \circ \phi)(z)$ gives the desired map.

5.51. Example. Choosing $\theta = 0$ and $\beta = i$ in (5.48) we have

$$w = f(z) = \frac{z-i}{z+i}, \tag{5.52}$$

which is called a *Cayley mapping* of $\mathbb{H}^+$ onto Δ and the inverse is given by

$$f^{-1}(w) = i\left(\frac{1+w}{1-w}\right) = e^{i\pi/2}\left(\frac{1+w}{1-w}\right).$$

(i) By rotation, we see that the function g given by

$$g(w) = e^{-i\pi/2}f^{-1}(w) = \frac{1+w}{1-w}$$

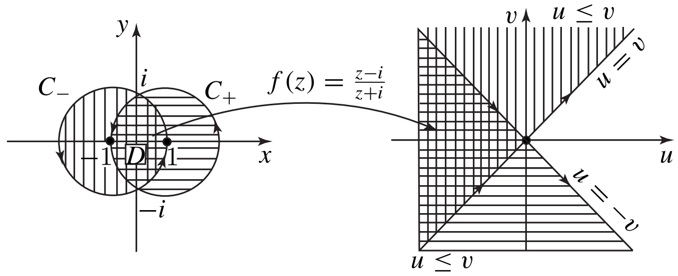


Figure 5.11: Image of D under Cayley map.

defines an analytic map of the unit disk Δ onto the right half-plane $\{\zeta : \operatorname{Re}\zeta > 0\}$. Further, the image of $|w| = r$ ($r \in (0,1)$) under $\zeta = g(w)$ may be computed as the circle

$$\left|\zeta - \frac{1+r^2}{1-r^2}\right| = \frac{2r}{1-r^2}.$$

Thus, $|w| < r$ for $r \in (0,1)$ is mapped onto the interior of this circle. On the other hand, if $r \in (1,\infty)$ then the image of $|w| < r$ is seen to be the domain given by

$$\left|\zeta - \frac{1+r^2}{1-r^2}\right| > \frac{2r}{|1-r^2|}.$$

What is the image of $1 < |w| < r$ under $\zeta = g(w)$?

(ii) Consider the two circles defined by

$$C_+ = \{z : |z-a| = R\} \text{ and } C_- = \{z : |z+a| = R\},$$

where $a > 0$ is fixed and $R = \sqrt{1+a^2}$. Clearly, these two circles intersect at $z = i$ and at $z = -i$. Set

$$D_+ = \operatorname{int} C_+ = \Delta(a;R), \quad D_- = \operatorname{int} C_- = \Delta(-a;R)$$

and let $D = D_+ \cap D_-$ denote their common region of intersection, see Figure 5.11. Let us find the image of D under the Cayley map given by (5.52). Observe the following:

- $f(-i) = \infty$ and so, each circle that passes through the point $z = -i$ is mapped onto a straight line. In particular, the image of each of the circles C_+ and C_- is a straight line
- these two straight lines necessarily pass through the origin, since both the circles pass through $z = i$, where $f(i) = 0$

- $f(i\beta) = (1-\beta)/(1+\beta) > 0$ whenever $-1 < \beta < 1$. In particular, the open vertical line segment connecting $-i$ and i is mapped onto the positive real axis $(0, \infty)$.

Finally, by (5.52), it follows that

$$\begin{aligned}
|z-a| \gtreqless R \iff & \left| i\left(\frac{1+w}{1-w}\right) - a \right| \gtreqless R \\
\iff & |w(a+i) - (a-i)|^2 \gtreqless R^2|1-w|^2 \\
\iff & \mathrm{Re}\,[w(R^2 - (a+i)^2)] \gtreqless 0 \quad (\text{since } R^2 = 1 + a^2) \\
\iff & \mathrm{Re}\,[w(1-ia)] \gtreqless 0 \\
\iff & \mathrm{Re}\,[we^{-i\alpha}] \gtreqless 0 \quad (\text{with } \mathrm{Arg}\,(1+ia) = \alpha).
\end{aligned}$$

Similarly, $|z+a| \gtreqless R \iff \mathrm{Re}\,[w(1+ia)] \gtreqless 0 \iff \mathrm{Re}\,[we^{i\alpha}] \gtreqless 0$. Now, it is clear that the domain D is mapped onto the infinite wedge bounded within the two lines $\mathrm{Arg}\,w = \alpha$ and $\mathrm{Arg}\,w = -\alpha$. For instance, if $a = 1$ then $\mathrm{Arg}\,(1+ia) = \pi/4 = -\mathrm{Arg}\,(1-ia)$. ●

5.53. Example. Consider $w = f(z) = (i-z)/(i+z)$. Then we have

(i) f maps the unit disk Δ onto the right half-plane $\{w : \mathrm{Re}\,w > 0\}$

(ii) f maps the upper half-plane $\{z : \mathrm{Im}\,z > 0\}$ onto the unit disk Δ

(iii) f maps the open first quadrant $\{z : \mathrm{Im}\,z > 0, \mathrm{Re}\,z > 0\}$ onto the upper semi-disk $\{w : |w| < 1, \mathrm{Im}\,w > 0\}$.

Indeed, we see that

$$w = \frac{1+iz}{1-iz} \iff -iz = \frac{1-w}{1+w} = \frac{1-|w|^2 - 2i\mathrm{Im}\,w}{|1+w|^2}$$

so that

- $|z| < 1 \iff |w-1| < |w+1| \iff \mathrm{Re}\,w > 0$
- $\mathrm{Im}\,z > 0 \iff \mathrm{Re}\,(-iz) > 0 \iff |w| < 1$
- $\mathrm{Im}\,z > 0$ and $\mathrm{Re}\,z > 0 \iff |w| < 1$ and $\mathrm{Im}\,w > 0$. ●

5.54. Example. Let us try to find a map which takes the circle $\{z : |z-1| = 1\}$ onto the line $\{w = u + iv : v = u\}$, see Figure 5.12. Our problem does not insist which points on the circle should map which points on the line. Since a circle is to map to a line, we must have $c \neq 0$ and so, we can normalize c to be 1. This suggests to consider the Möbius transformation in the form

$$f(z) = \frac{az+b}{z+d}.$$

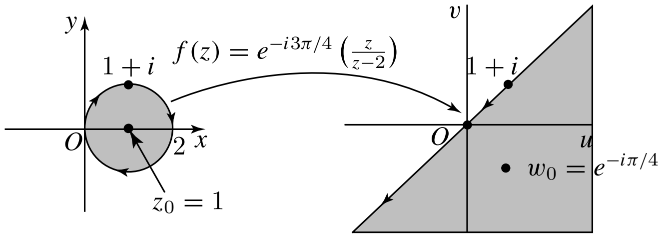


Figure 5.12: Mapping from $\{z : |z-1| = 1\}$ into the line $\{w = u + iv : v = u\}$.

It follows that our requirements translate into the equations (for example) by considering $0 \mapsto 0$, $2 \mapsto \infty$, and $1+i \mapsto 1+i$:

$$0 = f(0) = \frac{b}{d}, \quad \infty = f(2) = \frac{2a+b}{2+d}, \quad f(1+i) = 1+i.$$

The first two equations show that $b = 0$ and $2 + d = 0$. So

$$f(z) = \frac{az}{z-2}.$$

Finally, the last requirement $f(1+i) = 1+i$ yields that $a = -1+i = e^{3\pi i/4}$. Hence, the desired map is given by

$$f(z) = e^{3\pi i/4}\left(\frac{z}{z-2}\right). \tag{5.55}$$

Note that $f(1) = 1 - i$ and therefore, because of the principle of conformal map (preservation of sense and magnitude), the map f given by (5.55) takes the disk $\{z : |z-1| < 1\}$ onto the half-plane $\{w : \operatorname{Im} w < \operatorname{Re} w\}$. •

5.56. Automorphisms of the unit disk. Let $\theta \in \mathbb{R}$, $z_0 \in \Delta$ be fixed and ϕ be given by

$$\phi(z) = e^{i\theta}\left(\frac{z - z_0}{1 - \overline{z}_0 z}\right). \tag{5.57}$$

Note that ϕ is one-to-one in $\mathbb{C} \setminus \{1/\overline{z}_0\}$. If $|z| = 1$, then $z^{-1} = \overline{z}$ so that $|\phi(z)| = 1$.

In addition, since every Möbius transformation maps a circle onto a circle or a straight line, ϕ must map the unit circle $|z| = 1$ *onto* itself. Also, $\phi(z_0) = 0$ and $\phi(\Delta) = \Delta$, because

$$\begin{aligned} |\phi(z)| < 1 &\iff |z - z_0|^2 < |1 - z\overline{z}_0|^2 \\ &\iff (1 - |z_0|^2)(1 - |z|^2) > 0 \\ &\iff |z| < 1 \quad (\text{since } |\overline{z}_0| < 1) \end{aligned}$$

so that Δ must be mapped onto itself by $\phi(z)$. Now ϕ has the inverse given by

$$\phi^{-1}(w) = e^{-i\theta}\left(\frac{w + e^{i\theta}z_0}{1 + e^{-i\theta}\overline{z}_0 w}\right).$$

which has a similar form as ϕ and so, ϕ^{-1} shares similar properties as that of ϕ. Organizing these observations together, we can assert that for each z_0 with $|z_0| < 1$ and $\theta \in \mathbb{R}$, ϕ is a bijective self mapping of the unit disk Δ. This result raises the following

5.58. Problem. Do all one-to-one, onto analytic maps of Δ to itself precisely have the form of ϕ for some $|z_0| < 1$ with some $\theta \in \mathbb{R}$?

The answer to this problem is yes (see also Example 5.81 and Theorem 6.45). The question of mapping the upper half-plane $\mathbb{H}^+$ onto itself can also be treated easily (see Theorem 5.69).

5.59. Theorem. *All conformal mappings which map the unit disk* Δ *onto itself and the point* z_0, $|z_0| < 1$, *onto* 0 *must be of the form* (5.57) *for some* $\theta \in \mathbb{R}$. *Equivalently,*

$$\mathrm{Aut}\,(\Delta) = \left\{e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z} : z_0 \in \Delta,\ 0 \le \theta \le 2\pi\right\}.$$

Proof. We start with the Möbius transformation

$$T(z) = \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{C},\ ad - bc \neq 0).$$

For each $\theta \in (-\pi, \pi]$, the condition $|T(e^{i\theta})| = 1$ implies that $|ae^{i\theta} + b|^2 = |ce^{i\theta} + d|^2$. That is

$$|a|^2 + |b|^2 + 2\mathrm{Re}\,(a\overline{b}e^{i\theta}) = |c|^2 + |d|^2 + 2\mathrm{Re}\,(c\overline{d}e^{i\theta}) \tag{5.60}$$

which must be true for each θ in $(-\pi, \pi]$. Choosing $\theta = 0, \pi$, it follows that

$$|a|^2 + |b|^2 + 2\mathrm{Re}\,(a\overline{b}) = |c|^2 + |d|^2 + 2\mathrm{Re}\,(c\overline{d})$$

and

$$|a|^2 + |b|^2 - 2\mathrm{Re}\,(a\overline{b}) = |c|^2 + |d|^2 - 2\mathrm{Re}\,(c\overline{d}).$$

Adding the last two equations, we find that $|a|^2 + |b|^2 = |c|^2 + |d|^2$ so that

$$|a|^2 - |c|^2 = |d|^2 - |b|^2 \tag{5.61}$$

and therefore, by (5.60), we get that

$$\mathrm{Re}\,(a\overline{b}e^{i\theta}) = \mathrm{Re}\,(c\overline{d}e^{i\theta}) \quad \text{for each } \theta \in (-\pi, \pi].$$

Again, choosing $\theta = 0$ or π, and $\theta = \pi/2$, we see that

$$a\overline{b} = c\overline{d}. \tag{5.62}$$

Now using the two equalities (5.61) and (5.62), we find that

$$\begin{aligned} |w|^2 - 1 &= \frac{|az+b|^2 - |cz+d|^2}{|cz+d|^2} \\ &= \frac{(|a|^2 - |c|^2)|z|^2 - (|d|^2 - |b|^2) + 2\mathrm{Re}\,(z(a\overline{b} - c\overline{d}))}{|cz+d|^2} \\ &= \frac{(|z|^2 - 1)(|a|^2 - |c|^2)}{|cz+d|^2} \end{aligned}$$

and therefore the requirement that "$|z| < 1 \;\Rightarrow\; |w| < 1$" gives rise to the inequality

$$0 < |a|^2 - |c|^2, \text{ i.e. } |c| < |a|$$

(Note that a cannot be zero, but c could be zero). Therefore, from (5.61), we also have $d \neq 0$ (but b could be zero). Thus, by (5.62), we write

$$\frac{c}{a} = \overline{\left(\frac{b}{d}\right)} = k, \quad \text{say}, \tag{5.63}$$

with $|k| < 1$ and so

$$1 - \left|\frac{c}{a}\right|^2 = 1 - \left|\frac{b}{d}\right|^2; \text{ i.e. } \frac{|a|^2 - |c|^2}{|a|^2} = \frac{|d|^2 - |b|^2}{|d|^2}$$

so that by (5.61), we have $|a| = |d|$ which, by (5.61), again gives $|b| = |c|$. Thus, we can write $a/d = e^{i\theta}$ for some θ so that

$$T(z) = \frac{a}{d} \cdot \frac{z + b/a}{1 + (c/d)z} = e^{i\theta}\left(\frac{z - z_0}{1 - \overline{z}_0 z}\right)$$

where, by (5.61) and the fact that $|a| = |d|$,

$$z_0 = -\frac{b}{a} = -\frac{b\overline{a}}{|a|^2} = -\frac{\overline{c}d}{|d|^2} = -\overline{\left(\frac{c}{d}\right)}.$$

By (5.61), we also observe that if $c = 0$, then b must be zero since $a \neq 0$. In that case, we have $S(z) = e^{i\theta}z$. On the other hand, if $c \neq 0$ then $b \neq 0$ and therefore,

$$z_0 = -\frac{b}{a} = -\frac{c}{a} \cdot \frac{b}{c} = -k\frac{b}{c}, \text{ i.e. } |z_0| = |k| < 1. \qquad \blacksquare$$

There are several alternate proofs of Theorem 5.59, which gives a characterization of all conformal self mappings of the unit disk Δ. For a better understanding on these techniques, we mention two more such proofs. The second one follows from the principle of symmetry (see Example 5.81). The third proof is a consequence of Schwarz' lemma (see Theorem 6.45).

5.64. Example. Let us discuss the question of finding the image of the open upper semi-unit disk $\Omega = \{z : |z| < 1, \text{ Im}\, z > 0\}$ under the Möbius map defined by

$$w = \phi_a(z) = \frac{a-z}{1-\overline{a}z}. \tag{5.65}$$

In our discussion, we restrict our attention to a special situation $a = i\alpha$, $-1 < \alpha \neq 0 < 1$, and proceed to discuss the desired mapping properties. Note that

$$w = \frac{a-z}{1-\overline{a}z} \Longleftrightarrow z = \frac{a-w}{1-\overline{a}w} = \frac{(a-w)(1-a\overline{w})}{|1-\overline{a}w|^2} = \frac{a-w+a|\overline{w}|^2-a^2\overline{w}}{|1-\overline{a}w|^2}$$

and $|z| < 1 \Longleftrightarrow |w| < 1$. With $a = i\alpha$,

$$\begin{aligned}
\text{Im}\, z > 0 &\Longleftrightarrow \text{Im}\,[a - w + a|w|^2 - a^2\overline{w}] > 0 \\
&\Longleftrightarrow \alpha|w|^2 - (1+\alpha^2)\text{Im}\, w + \alpha > 0 \\
&\Longleftrightarrow \alpha\left[\left|w - i\left(\frac{1+\alpha^2}{2\alpha}\right)\right|^2 - \left(\left(\frac{1+\alpha^2}{2\alpha}\right)^2 - 1\right)\right] > 0 \\
&\Longleftrightarrow \alpha\left[\left|w - i\left(\frac{1+\alpha^2}{2\alpha}\right)\right|^2 - \left(\frac{1-\alpha^2}{2\alpha}\right)^2\right] > 0 \\
&\Longleftrightarrow \begin{cases} \left|w - i\left(\dfrac{1+\alpha^2}{2\alpha}\right)\right| > \dfrac{1-\alpha^2}{2\alpha} & \text{if } \ 0 < \alpha < 1 \\[2ex] \left|w - i\left(\dfrac{1+\alpha^2}{2\alpha}\right)\right| < \dfrac{1-\alpha^2}{2|\alpha|} & \text{if } \ -1 < \alpha < 0. \end{cases}
\end{aligned}$$

For example, if $a = i/2$ then (see Figure 5.13) 5 $\alpha = 1/2$ so that the region $\text{Im}\, z > 0$ under $\phi_{i/2}(z)$ corresponds to the set of points given by

$$\{w : |w - (5/4)i| > 3/4\}.$$

Thus, the image of the open upper semi-disk $\{z : |z| < 1, \text{ Im}\, z > 0\}$ under $\phi_{i/2}(z)$ is the shaded region in Figure 5.13 and the image of the lower semi-disk $\{z : |z| < 1, \text{ Im}\, z < 0\}$ under the same map is also indicated in Figure 5.13. Clearly, the image of the open segment $(-1, 1)$ under $\phi_{i/2}(z)$ is the open arc of the circle $|w - (5/4)i| = 3/4$ indicated in the same figure. What happens when $\alpha = \pm 1$? ●

5.66. Example. Let us start constructing a bijective analytic map ϕ taking the upper semi-unit disk $D = \{z : |z| < 1, \text{Im}\, z > 0\}$ onto the

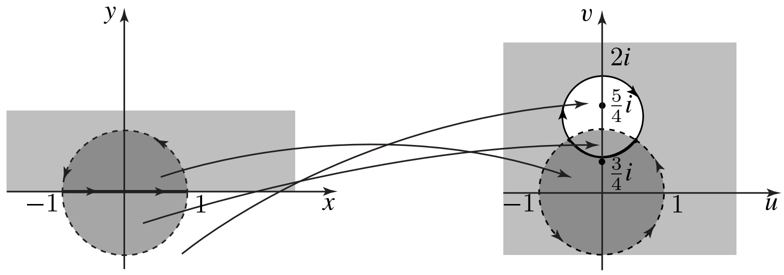


Figure 5.13: Mapping from Δ under $\phi_{i/2}(z) = (i - 2z)/(2 + iz)$.

unit disk Δ, leaving the points $1, -1, i$ fixed, for example. Can there be a Möbius transformation carrying D onto Δ? As

$$f(z) = \frac{1+z}{1-z} = \frac{(1+z)(1-\overline{z})}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} + i\left(\frac{2\mathrm{Im}\, z}{|1-z|^2}\right),$$

f maps $|z| < 1$ onto $\mathrm{Re}\, w > 0$, and $\mathrm{Im}\, z > 0$ onto $\mathrm{Im}\, w > 0$. This observation implies that (because $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a bijective conformal map) the upper semi-unit disk maps onto the open first quadrant Ω,

$$\Omega = \{w : \mathrm{Im}\, w > 0,\, \mathrm{Re}\, w > 0\}.$$

This also follows from the fact that

$$z = x \Longleftrightarrow f(x) = \frac{1+x}{1-x}, \quad \text{and} \quad |z| = 1 \Longleftrightarrow f(z) = \frac{iy}{|1-z|^2}.$$

If we let $g(z) = z^2$, then $g(\Omega) = \mathbb{H}^+ = \{w : \mathrm{Im}\, w > 0\}$. We know that

$$h(z) = \frac{z-i}{z+i}$$

is an analytic bijection of $\mathrm{Im}\, z > 0$ onto the unit disk Δ. Thus, the composition

$$\phi(z) = (h \circ g \circ f)(z) = \frac{(1+z)^2 - i(1-z)^2}{(1+z)^2 + i(1-z)^2}$$

transforms the (open) upper semi-unit disk $\Delta \cap \mathbb{H}^+$ conformally onto the unit disk Δ, leaving the points $1, -1, i$ fixed, see Figure 5.14. This idea can be used to show that the transformation

$$\phi(z) = \frac{(1+z^n)^2 - i(1-z^n)^2}{(1+z^n)^2 + i(1-z^n)^2}, \quad n \in \mathbb{N}, \tag{5.67}$$

maps the domain $\{z : |z| < 1,\, 0 < \mathrm{Arg}\, z < \pi/n\}$ conformally onto the unit disk Δ. What are its fixed points? Are all points z such that $z^n = 1$ or $z^n = -1$ fixed points, in particular? •

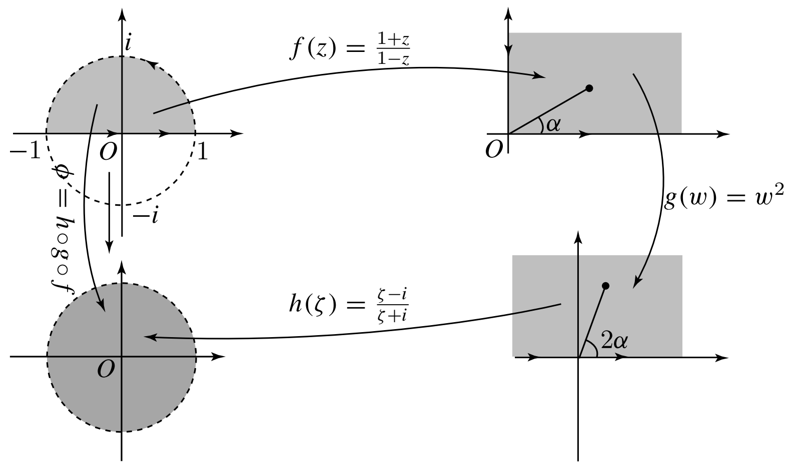


Figure 5.14: Conformal map of upper semi-unit disk onto the unit disk.

5.68. Automorphisms of the upper half-plane $\mathbb{H}^+$. There are several different proofs of the following result and all these proofs follow if we proceed with the idea of the proof of the results concerning automorphisms of disks.

5.69. Theorem. *Every Möbius transformation of the form*

$$T(z) = \frac{az+b}{cz+d}$$

is a conformal self-map of the upper half-plane $\mathbb{H}^+$ iff a, b, c, d are real numbers satisfying the condition $ad - bc > 0$. Equivalently,

$$\mathrm{Aut}\,(\mathbb{H}^+) = \left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}.$$

Proof. The required map should take the real axis in the z-plane onto the real axis in the w-plane. As every Möbius map is one-to-one, the desired map should carry three distinct real numbers on the x-axis onto three distinct real numbers on the u-axis. Then there always exist three distinct real numbers x_1, x_2, x_3 such that $0, 1, \infty$ are their images. By the definition of the cross-ratio and its invariance property (see Theorem 5.39), we get

$$(5.70) \quad (T(z), 0, 1, \infty) = (z, x_1, x_2, x_3), \quad \text{i.e.} \quad T(z) = \frac{(z - x_1)(x_2 - x_3)}{(z - x_3)(x_2 - x_1)}$$

(If one of the x_j's is ∞, then use the limit process). Since the required map

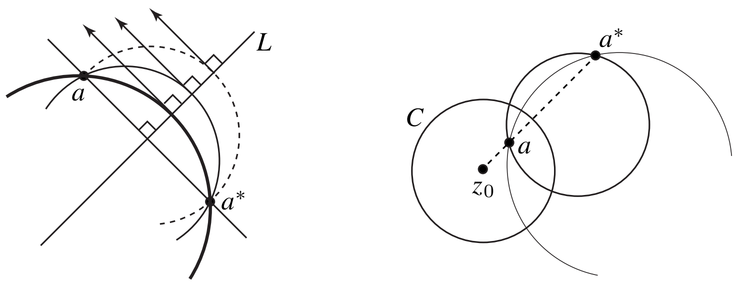


Figure 5.15: Symmetric with respect to a line/circle.

must send $\mathbb{H}^+$ onto itself, $\operatorname{Im} T(i) > 0$. This gives

$$\operatorname{Im} T(i) = \operatorname{Im}\left(\frac{i - x_1}{i - x_3}\,\frac{x_2 - x_3}{x_2 - x_1}\right) = \frac{x_1 - x_3}{1 + x_3^2}\left(\frac{x_2 - x_3}{x_2 - x_1}\right) > 0$$

which shows that $(x_1 - x_3)(x_2 - x_3)(x_2 - x_1) > 0$. Now, rewriting (5.70) in the form

$$T(z) = \frac{az + b}{cz + d}$$

we see that

$$\begin{aligned} ad - bc &= (x_2 - x_3)[(x_2 - x_1)(-x_3)] - [(x_2 - x_3)(-x_1)](x_2 - x_1) \\ &= (x_2 - x_3)(x_2 - x_1)(x_1 - x_3) > 0 \end{aligned}$$

and this completes the proof. ■

5.7 Principle of Symmetry and Möbius Maps

Let L be a line in $\mathbb{C}$. Two points a and a^* in $\mathbb{C}$ are said to be *symmetric* with respect to L if L is the perpendicular bisector of $[a, a^*]$–the line segment connecting a and a^*. Clearly, every circle or line passing through both a and a^* intersect the line L at a right angle, see Figure 5.15. For example, two points z and z^* are symmetric with respect to the real axis precisely when $z^* = \overline{z}$. Similarly, two points z and z^* are symmetric with respect to the imaginary axis precisely iff $z^* = -\overline{z}$. When we do say two points z and z^* are symmetric with respect to the line $\{t(1 - i) : t \in \mathbb{R}\}$? A Möbius transformation $w = T(z)$ with real coefficients maps the real axis in the z-plane onto the real axis in the w-plane and, z and $\overline{z}$ onto the points w and $\overline{w}$, respectively, which are again symmetric with respect to the real axis. This observation motivates us to formulate the definition of symmetric points with respect to a circle in $\mathbb{C}_\infty$.

Suppose that K is a circle $|z - z_0| = r$ in $\mathbb{C}$. Two points a and a^* are said to be *symmetric with respect to the circle* K (or *inverse points with*

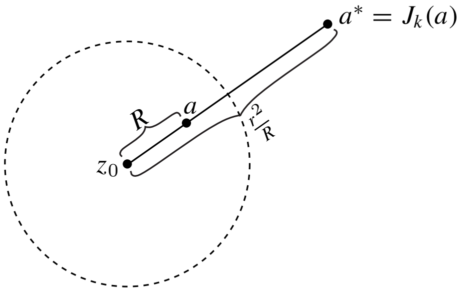


Figure 5.16: The reflection/inversion map J_K.

respect to the circle K) iff

(5.71) $\quad |a - z_0|\,|a^* - z_0| = r^2$ and $\operatorname{Arg}(a - z_0) = \operatorname{Arg}(a^* - z_0)$.

That is, a and a^* lie on the same ray emanating from the center z_0 of K, and the product of their distances from the center of the circle K is equal to the square of the radius of the circle. If we let $|a - z_0| = R$, then $a = z_0 + Re^{i\alpha}$ for some $\alpha \in \mathbb{R}$ so that (5.71) is equivalent to

$$a = z_0 + Re^{i\alpha} \quad \text{and} \quad a^* - z_0 = \frac{r^2 e^{i\alpha}}{R}.$$

That is,

(5.72) $\quad (\overline{a} - \overline{z}_0)(a^* - z_0) = r^2$ or $a^* = z_0 + \dfrac{r^2}{\overline{a} - \overline{z}_0} = z_0 + \dfrac{r^2(a - z_0)}{|a - z_0|^2}$.

It follows from (5.71) that if a approaches the circle, then a^* also approaches the circle. In other words, $a^* = a$ iff $a \in K$. If a approaches the center z_0, then the point a^* moves away to infinity. This fact is expressed by saying that z_0 and ∞ are symmetric with respect to the circle. This allows us to define the symmetric point a^* of a with respect to the circle K in $\mathbb{C}$ by the map $J_K : \mathbb{C}_\infty \to \mathbb{C}_\infty$:

(5.73)
$$a^* = J_K(a) = \begin{cases} z_0 + \dfrac{r^2}{\overline{a} - \overline{z}_0} & \text{if } a \neq z_0, \infty \\ \infty & \text{if } a = z_0 \\ z_0 & \text{if } a = \infty. \end{cases}$$

The map described by (5.73) is often called *the reflection/inversion* in the circle $K = \{z : |z - z_0| = r\}$ or reflection with respect to the circle, and the pair of points a and $a^* = J_K(a)$ are said to be *symmetric* with respect to K, see Figure 5.16. For example, if we let $z_0 = 0$ then $K = \{z : |z| = r\}$

and so, we have

$$J_K(0) = \infty, \quad J_K(\infty) = 0, \text{ and } J_K(a) = \frac{r^2}{\overline{a}} \text{ for } a \neq 0, \infty$$

Then, we call the map $a \mapsto r^2/\overline{a}$, *a reflection/inversion* in the circle $|z| = r$. For $r = 1$, this gives $a^* = 1/\overline{a}$ and so a and $1/\overline{a}$ are symmetric points with respect to the unit circle $|z| = 1$. In particular, 0 and ∞ are symmetric points with respect to the unit circle $|z| = 1$.

Moreover, it is easy to see that if $K = L \cup \{\infty\}$ for a line L with the equation

$$\overline{\alpha}z + \alpha\overline{z} + \gamma = 0 \quad (0 \neq \alpha \in \mathbb{C},\ \gamma \in \mathbb{R}),$$

then the reflection map J_K in a line K in $\mathbb{C}_\infty$ is obtained by replacing z and $\overline{z}$ respectively by a and $\overline{a^*}$:

$$J_K(a) = \begin{cases} -\left(\dfrac{\alpha}{\overline{\alpha}}\right)\overline{a} - \dfrac{\gamma}{\overline{\alpha}} & \text{if } a \neq \infty \\ \infty & \text{if } a = \infty. \end{cases} \tag{5.74}$$

By (5.73) and (5.74), J_K fixes every point of K and $J_K \circ J_K$ is the identity transformation of $\mathbb{C}_\infty$ so that $J_K^{-1} = J_K$. If we let M_c to be the set of all transformations of the form

$$\frac{A\overline{z} + B}{C\overline{z} + D} \quad (A, B, C, D \in \mathbb{C},\ AD - BC \neq 0),$$

then we note that every reflection J_K defined by (5.73) and (5.74) belongs to M_c. Note that

$$M_c = \{(T \circ J)(z) : T \in M \text{ and } J(z) = \overline{z}\},$$

where M denotes the set of all Möbius transformations. The set M_c shares many properties of M in common.

For example, if C is a circle in $\mathbb{C}_\infty$, then, for $f \in M_c$, $f(C)$ is again a circle in $\mathbb{C}_\infty$ and the composition of a pair of two transformations in M_c produces a Möbius transformation.

5.75. Theorem. (Symmetry Principle for Möbius Maps) *Let T be a Möbius transformation and K be a circle in $\mathbb{C}_\infty$ with $K' = T(K)$. Then*

$$T \circ J_K = J_{K'} \circ T, \quad \text{i.e.} \quad T(J_K(a)) = J_{K'}(T(a)) \quad \text{for each } a \in \mathbb{C}_\infty. \tag{5.76}$$

In particular, T maps any pair of symmetric points with respect to K onto a pair of symmetric points with respect to the image circle K' under T.

Proof. Consider the auxiliary function g defined by $g = J_{K'} \circ T \circ J_K$. Then g, being a composition of Möbius maps, is a Möbius map and so, g is one-to-one on $\mathbb{C}_\infty$. Also, we know that

$$J_K(a) = a \text{ and } J_{K'}(b) = b \text{ for each } a \in K \text{ and } b \in K'$$

which gives that $T(z) = g(z)$ for each $z \in K$. The uniqueness principle shows that $g = T$ so that

$$T = J_{K'} \circ T \circ J_K, \quad \text{i.e.} \quad T \circ J_K^{-1} = J_{K'} \circ T.$$

Since J_K is its own inverse, (5.76) follows.

Now, we let a and a^* be two points that are symmetric with respect to the circle K. Then, by (5.76), the image points $b = T(a)$ and $b^* = T(a^*)$ satisfy

$$J_{K'}(b) = J_{K'}(T(a)) = T(J_K(a)) = T(a^*) = b^*$$

so that b and b^* are symmetric with respect to the image circle $T(K)$. ■

5.77. Remark. Theorem 5.75 may also be proved by using the following two facts:

(i) Two points are symmetric with respect to a circle in $\mathbb{C}_\infty$ if every circle containing the points intersect the given circle orthogonally.

(ii) Möbius transformations are conformal and preserve circles, and so preserve the orthogonality; hence they preserve the symmetry condition. ●

5.78. Theorem. *Let K be a circle passing through three points in $\mathbb{C}_\infty$. Then the reflection J_K satisfies the relation*

$$(J_K(a), z_1, z_2, z_3) = \overline{(a, z_1, z_2, z_3)} \tag{5.79}$$

for $a \neq z_1, z_2, z_3$. Conversely, if

$$(a^*, z_1, z_2, z_3) = \overline{(a, z_1, z_2, z_3)} \tag{5.80}$$

then a and a^* in $\mathbb{C}_\infty$ are symmetric with respect to the circle K.

Proof. Define $T(z) = (z, z_1, z_2, z_3)$. Then T maps z_1 to 0, z_2 to 1, and z_3 to ∞, and so $T(K) = \mathbb{R} \cup \{\infty\}$. By the invariance property of the cross-ratio, one has

$$T(J_K(a)) = (J_K(a), z_1, z_2, z_3).$$

If $a \in K$ then, as J_K fixes each point of K and $T(K)$ is real, we have

$$T(J_K(a)) = (a, z_1, z_2, z_3) = \overline{(a, z_1, z_2, z_3)}.$$

If $a \notin K$ then, by the principle of symmetry, $T(a)$ and $T(J_K(a))$ are symmetric with respect to the circle $T(K) = \mathbb{R} \cup \{\infty\}$. That is, by (5.76), we have

$$T(J_K(a)) = T(a) = \overline{T(a)}$$

which is indeed (5.79).

Conversely, suppose that a and a^* are a pair of points in $\mathbb{C}_\infty$ such that (5.80) holds.

Case (i): Let $K = L \cup \{\infty\}$ for a straight line L in $\mathbb{C}$. Then, we choose $z_3 = \infty$ and the condition (5.80) gives

$$\frac{a^* - z_1}{z_2 - z_1} = \frac{\overline{a} - \bar{z}_1}{\overline{z}_2 - \overline{z}_1}$$

and so, $|a^* - z_1| = |a - z_1|$. But, since z_1 is arbitrary, it follows that a and a^* are equidistant from the line L. Moreover,

$$\operatorname{Im}\left(\frac{a^* - z_1}{z_2 - z_1}\right) = -\operatorname{Im}\left(\frac{a - z_1}{z_2 - z_1}\right)$$

showing that a and a^* lie in different half-planes determined by L. This is obviously a reflection with respect to L.

Case (ii): Let $K = \{z : |z - z_0| = r\}$ in $\mathbb{C}$ and let K pass through $z_1, z_2, z_3 \in \mathbb{C}$, i.e. $|z_j - z_0| = r$ for $j = 1, 2, 3$. A symmetric use of the invariance property of the cross-ratio under Möbius transformation gives

$$\begin{aligned}
\overline{(a, z_1, z_2, z_3)} &= \overline{(a - z_0, z_1 - z_0, z_2 - z_0, z_3 - z_0)} \quad (f(z) = z - z_0) \\
&= \left(\overline{a - z_0}, \frac{r^2}{z_1 - z_0}, \frac{r^2}{z_2 - z_0}, \frac{r^2}{z_3 - z_0}\right) \\
&\qquad ((z_j - z_0)\overline{(z_j - z_0)} = r^2) \\
&= \left(\frac{r^2}{\overline{\overline{a - z_0}}}, z_1 - z_0, z_2 - z_0, z_3 - z_0\right) \quad \left(f(z) = \frac{r^2}{z}\right) \\
&= \left(z_0 + \frac{r^2}{\overline{a} - \overline{z}_0}, z_1, z_2, z_3\right).
\end{aligned}$$

Hence, in view of (5.80) (as the cross-ratio $(z, z_1, z_2, z_3) = f(z)$ is univalent in $\mathbb{C}_\infty$), we find that

$$a^* = z_0 + \frac{r^2}{\overline{a} - \overline{z}_0} \quad \text{or} \quad (a^* - z_0)(\overline{a} - \overline{z}_0) = r^2.$$

This means that a and a^* are symmetric with respect to K. ■

A practical application of the symmetry principle is to find a Möbius transformation $w = T(z)$ which maps a given circle onto another circle. Next we present an example; many similar problems may be solved using the same idea.

5.81. Example. Suppose we wish to present an alternate proof of Theorem 5.59 (see also Theorem 6.45). To do this we consider $\phi(z)$, a general Möbius transformation, which maps the unit disk Δ onto itself. Then there must exist a point $a \in \Delta$ such that $\phi(a) = 0$. If a and a^*

are symmetric with respect to the unit circle $\partial\Delta$, then $\phi(a)$ and $\phi(a^*)$ are symmetric with respect to $\phi(\partial\Delta) = \partial\Delta$. As $\phi(a) = 0$, we have $\phi(a^*) = \infty$ (because 0 and ∞ are symmetric with respect to $|w| = 1$). As $a\overline{a^*} = 1$, we have $a^* = 1/\overline{a}$ so that $\phi(z)$ has the form

$$w = \phi(z) = \lambda \left(\frac{z-a}{1-\overline{a}z} \right)$$

for some $\lambda \in \mathbb{C}$. Also, as $\phi(1)$ is a point on the unit circle $|w| = 1$, we have $|\phi(1)| = 1$ which gives that $|\lambda| = 1$, i.e. $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Consequently, $\phi(z)$ has the desired form.

Finally, if $|a| > 1$ then $\phi(z)$ maps $|z| \geq 1$ onto $|w| \leq 1$ so that $\phi(z)$ maps $|z| \leq 1$ onto $|w| \geq 1$. ●

We end this section with some remarks. We know that every Möbius transformation T is conformal and it maps a circle C_1 in $\mathbb{C}_\infty$ onto a circle C_2 in $\mathbb{C}_\infty$. Moreover, T can be found by requiring three points z_1, z_2, z_3 in C_1 to map onto three prescribed points w_1, w_2, w_3 in C_2. Is it possible to find a Möbius Transformation such that $a_1 \in C_1$ maps onto $b_1 \in C_2$, and $a_2 \notin C_1$ onto $b_2 \notin C_2$? Yes, it is. Such a transformation is given by

$$(w, b_1, b_2, b_2^*) = (z, a_1, a_2, a_2^*)$$

where a_2^* is symmetric to a_2 with respect to C_1 while b_2^* is symmetric to b_2 with respect to C_2.

5.8 Exercises

5.82. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The function $f(z) = \sin z$ is not conformal on the infinite strip $\Omega = \{z \in \mathbb{C} : |\mathrm{Re}\, z| < \pi/2\}$.

(b) The composition of two Möbius transformations is again a Möbius transformation.

(c) The set M of all Möbius transformations $T_{abcd}(z) = (az+b)/(cz+d)$ with $ad - bc \neq 0$, forms a group with respect to the composition as a binary operation.

(d) The subset M_1 of all Möbius transformations $T_{abcd}(z)$ with $ad - bc > 0$, forms a subgroup in the group M of all Möbius transformations.

(e) The subset M_2 of all Möbius transformations given by

$$M_2 = \left\{ z, \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{z-1}{z}, \frac{z}{z-1} \right\}$$

forms a group with respect to the composition as a binary operation.

(f) The set M of all Möbius transformations does not have the commutative property.

(g) Any two Möbius transformations that have the same fixed points are commutative.
Note: If S and T are two Möbius transformations which commute (i.e. $S \circ T = T \circ S$), will they have the same fixed points?

(h) If a Möbius transformation T carries z_1 and z_2 into a same number w_1, then either $z_1 = z_2$ or else T is a constant map.

(i) Every Möbius transformation $T_{abcd}(z) = (az + b)/(cz + d)$ such that $|c| = |d|$ carries the unit circle $\partial\Delta$ onto a straight line.

(j) A Möbius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ taking $0, 1, 6$ into $2, 3, 4$, respectively, is $f(z) = [z + (2/3)]/[z + (3/2)]$.

(k) Given three distinct points z, z_1, z_2 and a number $\alpha \in \mathbb{C}$ such that $\alpha \notin \{0, 1, (z - z_2)/(z_2 - z_1)\}$, there always exists a unique z_3 with $(z, z_1, z_2, z_3) = \alpha$.

(l) If z_1, z_2, z_3 are three distinct points in $\mathbb{C}_\infty$ such that z, z' satisfy $(z, z_1, z_2, z_3) = (z', z_1, z_2, z_3)$, then $z = z'$.

(m) The cross-ratios (z_4, z_1, z_2, z_3) and (w_4, w_1, w_2, w_3) are equal iff there exists a Möbius transformation f such that $f(z_j) = w_j$ for $j = 1, 2, 3, 4$.

(n) Given a pair of (generalized) circles, there always exists a Möbius transformation carrying one circle onto another.

(o) If a circle C is mapped under the inversion $w = 1/z$ onto another circle C', then the center of the circle C need not be mapped onto the center of the circle C' unless the center of the circle C is zero.

(p) A Möbius transformation, which maps the upper half-plane $\{z : \operatorname{Im} z > 0\}$ onto itself and fixing $0, \infty$ and no other points, must be of the form $T(z) = \alpha z$ for some $\alpha > 0$ and $\alpha \neq 1$.

(q) A Möbius transformation which maps the upper half-plane $\{z : \operatorname{Im} z > 0\}$ onto itself which fixes ∞ and no other points must be of the form $T(z) = z + \beta$ for some $\beta \neq 0$ with $\operatorname{Im} \beta > 0$.

(r) Let T be a Möbius transformation such that $\infty \in \operatorname{Fix}(T)$. Then T carries $\mathbb{R}$ onto itself iff $T(z) = \alpha z + \beta$ for some $\alpha \in \mathbb{R}\backslash\{0\}$, $\beta \in \mathbb{R}$.

(s) There exist transcendental entire functions having no fixed points.

(t) A Möbius transformation which carries the upper half-plane $\mathbb{H}^+$ onto the unit disk Δ such that $z = 2i$ is mapped onto $w = 0$ while $z = \infty$ is mapped onto $w = -1$ is precisely $(2i - z)/(z + 2i)$.

(u) A Möbius transformation takes $\mathbb{R}$ into $\mathbb{R}$ iff it can be represented with real coefficients.

(v) A transformation which carries an infinite sector of angle π/n ($n \in \mathbb{N}$) onto the unit disk Δ is $(z^n - i)/(z^n + i)$.

(w) The Jacobian of a Möbius transformation T is identically equal to 1 iff $T(z) = e^{i\theta}z + b$.

(x) If $a, b \in \mathbb{C}$ and $r, R > 0$ are fixed, then a Möbius transformation that maps the disk $\Delta(a; r)$ onto $\mathbb{C}_\infty \setminus \Delta(b; R)$ is given by $T(z) = b + Rr/(z - a)$.

(y) The transformation $w = (1 - iz)/(z - i)$ maps $|z| = r$, where $r < 1$, into a circle in the w-plane, whose center is on the imaginary axis.

(z) A mapping which transforms $\Omega = \{z : 0 < \operatorname{Arg} z < \pi/6\}$ onto the unit disk Δ is given by $f(z) = e^{i\varphi}(z^6 - \alpha)/(z^6 - \overline{\alpha})$, $\operatorname{Im} \alpha > 0$.

5.83. What is the angle between images of the curves $x^2 + y^2 = 1$ and $y = x$ at their point of intersections, under the map $f(z) = z^2$? Answer the same if the circle is replaced by the ellipse $x^2 + 4y^2 = 4$.

5.84. Determine points where each of the following mappings is conformal:

(i) $z + e^{-z} - 5$ (ii) ze^{z^3+1} (iii) $\cos z$ (iv) $z + az^2$ (v) $z + az^3$.

5.85. Determine points where each of the following mappings fails to be conformal:

(i) $z^5 + 1$ (ii) $z^2 - \exp(\pi z^2)$ (iii) $\cosh z$ (iv) $\sinh z$.

5.86. Determine a, b, c, d such that the Möbius transformation $T(z)$ defined by (5.15) coincides with its inverse given by (5.20).

5.87. Find all Möbius transformations that map the unit disk Δ onto the left half-plane $\mathbb{H}^- = \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$.

5.88. Set $S_1(z) = \dfrac{z}{2z - 1}$, $S_2(z) = \dfrac{2z}{3z - 1}$, and $S_3(z) = \dfrac{(2 + i)z - 2}{z + i}$. Show that 0 and 1 are the fixed points for S_1 as well as for S_2, whereas $1 + i$ and $1 - i$ are the fixed points of S_3, and S_3 is loxodomic.

5.89. Find the image of the circle $|z| = r$ $(r \neq 1)$ under the Cayley mapping $w = f(z) = (z - i)/(z + i)$, see Example 5.51. What happens when $r = 1$?

5.90. Using the invariance property of the cross-ratio, find a Möbius transformation f in each of the following cases:

(a) $\{\infty, i, 0\}$ onto $\{1, i, -1\}$

(b) $\{0, 1, -1\}$ onto $\{0, 1, \infty\}$

(c) $\{1, i, -1\}$ onto $\{2i, -2, -2i\}$

(d) $\{0, 2 - 2i, 4\}$ onto $\{-3/4, 11i/4, \infty\}$

Chapter 6

Maximum Principle, Schwarz' Lemma, and Liouville's Theorem

In Section 6.1, we derive the Maximum and the Minimum modulus principles/theorems as an important application of the Cauchy integral formula. The Maximum modulus principle is a powerful tool in obtaining an explicit estimate for the size of the absolute value of an analytic function. In Section 6.2, we use the Maximum principle to derive Hadamard's three lines/circles theorem. Section 6.3 is devoted to Schwarz' lemma which is one of the most important consequences of the Maximum principle and the power series expansion. Later in this section, we use the classical form of Schwarz' lemma to characterize the conformal self-maps of the unit disk in the form of the Schwarz-Pick lemma which is a basic tool in the introduction of the hyperbolic metric and hyperbolic geometry in the unit disk. In Section 6.4, we discuss Liouville's theorem and its various generalizations which give rise to fascinating and surprisingly practical results such as the celebrated discovery of Gauss, the so called fundamental theorem of algebra (see Section 6.6).

6.1 Maximum Modulus Principle

When we deal with a function of one variable we frequently speak about the concept of maxima and minima. On the other hand, we cannot speak of maxima and minima of a complex function f since $\mathbb{C}$ is not an ordered field. However, it is meaningful to consider maximum and minimum values of the modulus of the complex function f, the real part of f and the imaginary part of f.

6.1. Definition. Let D be a subset of $\mathbb{C}$. A complex function defined on D is said to have a (local) *maximum modulus* at $a \in D$ if there exists a $\delta > 0$ such that $\Delta(a;\delta) \subset D$ and $|f(z)| \le |f(a)|$ for all $z \in \Delta(a;\delta)$; a (local) minimum of $|f|$ is similarly defined.

For example, on the closed disk $|z| \le r$, we have $|e^z| = e^{\mathrm{Re}\, z} \le e^r$ and at the boundary point $z = r$, $e^z = e^r$. Thus, e^z attains its maximum modulus at $z = r$. Consequently, $M(r, e^z) = \max_{|z| \le r} |e^z| = e^r$. Analogously, for $|z| \le r$, we easily see that

$$|\cos z| = \left|\frac{e^{iz} + e^{-iz}}{2}\right| \le \frac{|e^{iz}| + |e^{-iz}|}{2} = \frac{e^{-y} + e^{y}}{2} \le \frac{e^r + e^{-r}}{2} = \cos(ir)$$

which shows that $M(r, \cos z) = \max_{|z| \le r} |\cos z| = \cosh r$. Although, in real variable theory, many functions such as $\sin x$, $\cos x$ are bounded on $\mathbb{R}$, we will see that neither $\sin z$ nor $\cos z$ are bounded in $\mathbb{C}$.

6.2. Theorem. (Maximum Modulus Principle) *Suppose that f is analytic in a domain D and a is a point in D such that $|f(z)| \le |f(a)|$ holds for all $z \in D$. Then, f is a constant.*

For the proof of Theorem 6.2, we need the following basic fact from Real Analysis:

6.3. Theorem. *Let $h(\theta)$ be a continuous real-valued function on $[a, b]$ with $h(\theta) \ge 0$ for all $\theta \in [a, b]$. If $\int_a^b h(\theta)\, d\theta = 0$, then $h(\theta) = 0$ for all $\theta \in [a, b]$.*

Proof. Geometrically, the proof of this theorem is obvious. Alternately, we fix $t \in [a, b]$. Then we have

$$0 \le H(t) := \int_a^t h(\theta)\, d\theta \le \int_a^b h(\theta)\, d\theta = 0$$

which gives $H(t) = 0$ on $[a, b]$ so that $0 = H'(\theta) = h(\theta)$ on $[a, b]$. ∎

Proof of Theorem 6.2. Since $a \in D$ and D is open, there exists an r such that $\overline{\Delta}(a;r) \subset D$. Then, f is analytic inside and on the circle $\gamma = \partial\Delta(a;r)$. So, by the Cauchy integral formula (see Theorem 4.66),

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta. \tag{6.4}$$

By hypothesis, $|f(a + re^{i\theta})| \le |f(a)|$ and, by (6.4), we note that

$$|f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, d\theta = |f(a)|$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi}\left[|f(a)| - |f(a+re^{i\theta})|\right]\,d\theta = 0. \tag{6.5}$$

Since the integrand in (6.5) is continuous and non-negative, Theorem 6.3 applies and (6.5) implies that

$$|f(a+re^{i\theta})| = |f(a)| \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

This equation holds on all circles $|\zeta - a| = s$, $0 < s \le r$ and therefore, $|f(z)|$ is constant on $\Delta(a;r)$. By the Uniqueness theorem, f is constant on the whole of D. ■

6.6. Remark. It is important to note that Theorem 6.2 does not necessarily hold for open sets. Further, one could also prove Theorem 6.2 without using Theorem 6.3. For example, as $f \in \mathcal{H}(\overline{\Delta(a;r)})$, we have

$$f(z) = \sum_{n\ge 0} a_n(z-a)^n \quad \text{for } |z-a| < r' \text{ with } r' > r.$$

In particular, for each circle $|z-a| = s$ with $0 < s \le r$, we have

$$f(a+se^{i\theta}) = \sum_{n\ge 0} a_n s^n e^{in\theta}.$$

Now, as $|f(a+se^{i\theta})| \le |f(a)| = |a_0|$, we see that $|f(a+se^{i\theta})|^2 \le |a_0|^2$ which gives

$$\sum_{n\ge 0} |a_n|^2 s^{2n} = \frac{1}{2\pi}\int_0^{2\pi} f(a+se^{i\theta})\overline{f(a+se^{i\theta})}\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |a_0|^2 d\theta = |a_0|^2;$$

that is, $a_n = 0$ for each $n \in \mathbb{N}$. Thus, $f(z) = a_0$ on every circle $|\zeta - a| = s$ in $\overline{\Delta}(a;r)$. Consequently, $f(z) = a_0 = f(a)$ on $\Delta(a;r)$. By the Uniqueness theorem, f is constant on the whole of D. ●

6.7. Example. Consider $f(z) = e^z$ for $z \in \overline{\Delta}(a;r)$. Then,

$$\max_{|z-a|\le r} |e^z| = \max_{0\le\theta\le 2\pi} \left|e^{a+re^{i\theta}}\right| = e^{\mathrm{Re}\, a + r}$$

and the maximum modulus of e^z is attained at the boundary point $z = a+r$. Similarly, to determine $M = \max_{|z|\le 1} |z^3 + 3z - 1|$, we may let $z = e^{i\theta}$ so that

$$|z^2 + 3z - 1| = |e^{2i\theta} + 3e^{i\theta} - 1| = |e^{i\theta} + 3 - e^{-i\theta}| = \sqrt{9 + 4\sin^2\theta}.$$

Thus, we have $M = \sqrt{13}$ and the maximum is attained at $z = \pm i$, i.e. when $\theta = \pm\pi/2$. ●

It is interesting to observe that the minimum value of $|f|$ may be attained at an interior point of D without f being constant. For instance, consider $f(z) = z$ for $z \in \Delta_r$. Then, $|f(z)| = |z| \geq 0 = |f(0)|$ and so the minimum value of $|f(z)|$ is attained at the origin. The maximum value of $|f(x+iy)| = \sqrt{x^2+y^2}$ is attained at the boundary points $|z| = r$. Note that f has a zero at the origin.

The Maximum modulus principle is often used in the following form known as the Maximum modulus theorem:

6.8. Theorem. (Maximum Modulus Theorem) *Suppose that f is analytic in a bounded domain D and continuous on $\overline{D}$. Then, $|f(z)|$ attains its maximum at some point on the boundary ∂D of D.*

Proof. Recall that a continuous function on a compact set attains a maximum. So, by hypothesis, f is bounded on $\overline{D}$ and the maximum value of $|f(z)|$ is attained at some point of $\overline{D}$. By Theorem 6.2, it cannot be in D, so it must be on the boundary ∂D. ∎

6.9. Example. Consider the function $f(z) = z^2$ defined on the closed disk $D = \{z : |z-1-i| \leq 1\}$. Let us show that the maximum value of $|f(z)|$ is attained at $z = (1+1/\sqrt{2})(1+i)$. To do this, set

$$z = 1 + i + e^{i\theta} = (1+\cos\theta) + i(1+\sin\theta), \quad \theta \in [0, 2\pi).$$

Then, $|f(z)| = 3 + 2(\cos\theta + \sin\theta)$. It follows that the maximum value of $|f(z)|$ is attained at $\theta = \pi/4$ and the maximum value is $3 + 2\sqrt{2}$. The maximum is attained at $z = 1 + i + e^{i\pi/4}$. ●

6.10. Corollary. *Suppose that f is analytic in a bounded domain D and continuous on $\overline{D}$. Then, each of* $\text{Re}\, f(z)$, $-\text{Re}\, f(z)$, $\text{Im}\, f(z)$ *and* $-\text{Im}\, f(z)$ *attains its maximum at some point on the boundary ∂D of D.*

Proof. Let $u(x,y) = \text{Re}\, f(z)$ and $g(z) = e^{f(z)}$. By the Maximum modulus theorem, $|g(z)| = e^{u(x,y)}$ cannot assume the maximum value in D. Since e^u is maximized when u is maximized, we obtain that $u(x,y)$ cannot assume its maximum value in D. The remaining cases follow similarly. ∎

6.11. Remark. The concept of a local maximum at a point $a \in S \subseteq D$ is meaningful only if a is a limit point. If a is isolated, we do not have $|f(z)| \leq |f(a)|$ in neighborhood of a. Thus, we observe that an interesting application occurs when D is a closed region. Therefore, another way of stating Theorem 6.8 is that "*if f is analytic inside and on a closed curve C, then $|f(z)|$ attains its maximum value only on the boundary C.*" ●

Another direct application of Theorem 6.8 is the following corollary which is helpful in practice for identifying the maximum modulus of functions by their boundary values.

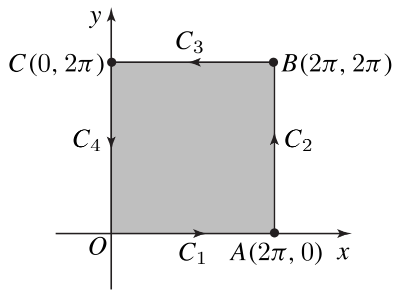


Figure 6.1: $[0, 2\pi] \cup [2\pi, 2\pi + 2\pi i] \cup [2\pi + 2\pi i, 2\pi i] \cup [i2\pi, 0]$.

6.12. Corollary. *Let f be analytic on Δ_R and continuous on its closure $\overline{\Delta}_R$. If $|f(z)| \leq M$ for some $M > 0$ on $\partial\Delta_R$, then $|f(z)| \leq M$ on $\overline{\Delta}_R$.*

6.13. Example. Let $D = \{z = x + iy : 0 < x, y < 2\pi\}$. We wish to find $\max_{z \in \overline{D}} |\cos z|$. To do this, we first note that

$$|\cos z| = \sqrt{\sinh^2 y + \cos^2 x}.$$

By the Maximum modulus theorem, the maximum is attained on the boundary (see Figure 6.1)

$$\partial D = [0, 2\pi] \cup [2\pi, 2\pi + 2\pi i] \cup [2\pi + 2\pi i, 2\pi i] \cup [i2\pi, 0].$$

For $z = x + i0$ with $0 \leq x \leq 2\pi$, $|\cos z|$ has the maximum value 1 at $z = 0, 2\pi$. For $z = 2\pi + iy$ with $0 \leq y \leq 2\pi$, $|\cos z|$ has the maximum $\sqrt{1 + \sinh^2(2\pi)}$ at $z = 2\pi + 2\pi i$, since $\sinh y$ is an increasing function of y. For $z = x + 2\pi i$ with $0 \leq x \leq 2\pi$, $|\cos z|$ has the maximum $\sqrt{1 + \sinh^2(2\pi)}$ at the points $z = 0 + 2\pi i, \pi + 2\pi i$. Finally, for $z = 0 + iy$ with $y \in [0, 2\pi]$, the corresponding maximum is seen to be $\sqrt{1 + \sinh^2(2\pi)}$. Hence,

$$\max_{z \in \overline{D}} |\cos z| = \sqrt{1 + \sinh^2(2\pi)} = \cosh 2\pi. \qquad \bullet$$

6.14. Theorem. (Minimum Modulus Theorem) *If f is a non-constant analytic function in a bounded domain D and $f(z) \neq 0$ on D then, $|f(z)|$ cannot attain its minimum in D.*

Proof. Suppose that $f \in \mathcal{H}(D)$ and $f(z) \neq 0$ in D. Then, $1/f(z)$ is analytic throughout D. The assertion now follows on applying Theorem 6.8 to $1/f(z)$. ∎

Theorem 6.14 is often stated in the following form: *"Suppose that f is analytic in a domain D, $f(z) \neq 0$ on D and a is a point in D such that $|f(z)| \geq |f(a)|$ holds for all $z \in D$. Then, f is a constant."*

6.15. Example. Suppose that f is an analytic function in a neighborhood of the closed unit disk $\overline{\Delta}$ such that there exists an $M > 0$ with $|f(z)| > M$ for $|z| = 1$ and $f(0) = a + ib$ with $|a + ib| < M$. Under these assumptions, we wish to show that f has a zero in Δ.

Suppose on the contrary that f has no zeros in Δ. Then, by assumption, $1/f$ would be a non-vanishing analytic function in the neighborhood of $\overline{\Delta}$ and therefore, would attain its maximum value on the circle $|z| = 1$. The Maximum modulus theorem gives that $|1/f(z)| < 1/M$ on $|z| \leq 1$. In particular,

$$1/|a+ib| = |1/f(0)| < 1/M, \quad \text{i.e.} \quad M < |a+ib|,$$

which is a contradiction. Thus, f must have a zero in Δ. ●

Theorem 3.31 and the Uniqueness theorem (see Theorems 3.75 and 4.103) show that a non-constant analytic function in a domain cannot map an open set into a point or an arc. From these two observations, we note that the Open mapping theorem (see Theorem 12.1) about mapping properties of analytic functions is a considerable strengthening of these facts arising out of the Maximum modulus theorem.

6.16. Corollary. (Maximum/Minimum Modulus Theorem for Harmonic Functions) *Suppose that $u(x,y)$ is a real-valued non-constant harmonic function on a bounded domain D. Then, $u(x,y)$ cannot attain its maximum or minimum value in D. That is, if there exists a point $(x_0,y_0) \in D$ such that $u(x_0,y_0) = \sup_{z\in D} u(x,y)$ or $u(x_0,y_0) = \inf_{z\in D} u(x,y)$, then $u(x,y)$ is constant on D.*

Proof. We shall prove only the maximum case as the proof for the minimum case follows by simply applying the maximum case for $-u$. First, we assume that D is simply connected. Then (see Theorem 3.39) there exists an $f \in \mathcal{H}(D)$ with $u = \operatorname{Re} f$. The desired conclusion follows if we apply the Maximum modulus theorem to $g(z) = e^{f(z)}$ (see Corollary 6.10).

Now we assume that D is a multiply connected domain. Suppose on the contrary that $u(x,y)$ attains its maximum value at some point $z_0 = (x_0,y_0) \in D$. Since D is open, there exists a closed disk $\overline{\Delta}(z_0;\delta) \subset D$. Then, in $\Delta(z_0;\delta)$, $u(x,y)$ attains its local maximum at the interior point (x_0,y_0) which is a contradiction to the previous case. Consequently, $u(x,y)$ cannot attain its maximum value on D. ∎

6.17. Example. Consider $u(x,y) = 2(x^2 - y^2) + 3$ for $|z| \leq 2$. Clearly, u is harmonic on the disk $|z| \leq 2$. To find $\max u$ and $\min u$ on

$|z| \le 2$, it suffices to find these on the boundary $|z| = 2$. Setting $x = 2\cos\theta$ and $y = 2\sin\theta$, we have

$$u = 8\cos 2\theta + 3 \quad (0 \le \theta \le 2\pi)$$

and so, $\max_{|z|=2} = 13$ and $\min_{|z|=2} = -5$. Note that the maximum occurs when $\theta = 0$ and π while the minimum occurs when $\theta = \pi/2$ and $3\pi/2$. ●

6.18. Example. Suppose that f and g are analytic on the closed unit disk $|z| \le 1$ such that

(i) $|f(z)| \le M$ for all $|z| \le 1$
(ii) $f(z) = z^n g(z)$ for all $|z| \le 1/3$ and for some $n \in \mathbb{N}$.

We wish to use the Maximum modulus principle to find the maximum value of $|f(z)|$ on $|z| \le 1/3$. To do this, we proceed as follows. On $|z| = 1$, we have

$$M \ge |f(z)| = |z^n g(z)| = |g(z)|$$

and so, $|g(z)| \le M$ for $|z| \le 1$. Now, for $|z| = 1/3$, we have

$$|f(z)| = |z^n g(z)| = |z^n|\,|g(z)| = 3^{-n}|g(z)| \le 3^{-n}M.$$

It follows that $|f(z)| \le 3^{-n}M$ for all $|z| \le 1/3$. ●

6.2 Hadamard's Three Circles/Lines Theorems

We notice that the hypothesis that D is bounded in the Maximum modulus theorem (see Theorem 6.8) cannot be dropped. Therefore, the Maximum modulus theorem is not always true on unbounded domains. To illustrate this, we present three different examples.

(i) Define $f(z) = e^{-iz}$ on $D = \{z : \operatorname{Im} z > 0\}$. Then $|f(\zeta)| = 1$ on the boundary $\partial D = \{\zeta : \operatorname{Im}\zeta = 0\}$, the real axis. But for $z = x+iy \in D$,

$$|f(x+iy)| = e^y \to \infty \quad \text{as } y \to +\infty;$$

that is, f itself is not bounded. Note also f is a periodic function of period 2π. In particular, for $z = 2\pi + iy \in D$,

$$\operatorname{Re} f(2\pi + iy) = e^y \to \infty \quad \text{as } y \to +\infty;$$

Similarly, if $g(z) = -iz$ on $D = \{z : \operatorname{Im} z > 0\}$ then $\operatorname{Re} g(z) = 0$ on ∂D, yet g is not constant on ∂D. Note that for $z = x + iy \in D$,

$$\operatorname{Re} g(z) = y \to \infty \quad \text{as } y \to +\infty.$$

(ii) Define $f(z) = e^{-iz^2}$ on $D = \{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$ or $D = \{z : \operatorname{Re} z < 0,\ \operatorname{Im} z < 0\}$. Then, $|f(\zeta)| = 1$ on the boundary ∂D. But for $x + iy \in D$, $|f(x+iy)| = e^{2xy}$ so that f itself is not bounded on D.

(iii) Finally, we consider $f(z) = e^{e^z}$ for $z \in D = \{z : |\mathrm{Im}\, z| < \pi/2\}$. Then, for $a + ib \in \partial D = \{\zeta : |\mathrm{Im}\, \zeta| = \pi/2\}$,

$$|f(a+ib)| = \left| e^{e^{a \pm i\pi/2}} \right| = \left| e^{\pm i e^a} \right| = 1.$$

Again the conclusion of Theorem 6.8 fails since for $z = x \in \mathbb{R} \subset D$, $f(x) = e^{e^x} \to \infty$ as $x \in \mathbb{R},\ x \to +\infty$.

The failure of the Maximum principle on certain unbounded domains raises the following

6.19. Problem. If $f(z)$ is bounded in an unbounded domain Ω, can we then conclude that $\sup_{z\in\overline{\Omega}} |f(z)| = \sup_{z\in\partial\Omega} |f(z)|$?

Here we speak of $\sup |f(z)|$ rather than $\max |f(z)|$ because, although both $\overline{\Omega}$ and $\partial\Omega$ are closed, they are not compact. The following result provides an affirmative answer in the case of parallel strips.

6.20. Theorem. (Phragmen-Lindelöf Theorem) *Let* $\Omega = \{z : 0 < \mathrm{Re}\, z < 1\}$. *Suppose that* f *is analytic and bounded on* Ω, *and continuous on its closure* $\overline{\Omega}$. *Then,* $\sup_{z\in\partial\Omega} |f(z)| = \sup_{z\in\overline{\Omega}} |f(z)|$. *In particular, if* $|f(z)| \le M$ *for* $z \in \partial\Omega$ *then* $|f(z)| \le M$ *for all* $z \in \overline{\Omega}$.

Proof. Set $M_1 = \sup_{z\in\partial\Omega} |f(z)|$ and $M_2 = \sup_{z\in\overline{\Omega}} |f(z)| < \infty$. Then, $M_1 \le M_2$. We claim that $M_1 \ge M_2$. We shall show that $|f(z)| \le M_1$ for all $z \in \overline{\Omega}$. To do this we fix $a \in \mathbb{R} \cap \Omega$. Then, $0 < a < 1$. Let $\epsilon > 0$ be given. Define an auxiliary function g defined by

$$g(z) = \frac{f(z)}{1 + \epsilon z}$$

and consider the rectangular domain $R = \{z \in \Omega : |\mathrm{Im}\, z| < A\}$, where $A > 0$ will be chosen later. Then, as $|1 + \epsilon z| \ge \mathrm{Re}\,(1 + \epsilon z) = 1 + \epsilon \mathrm{Re}\, z \ge 1$ for $z \in \overline{\Omega}$, we have

- g is continuous on $\overline{\Omega}$, analytic on Ω and $|g(z)| \le |f(z)|$ for $z \in \overline{\Omega}$. In particular, $|g(z)| \le |f(z)| \le M_1$ on the vertical sides of ∂R.
- for z on the horizontal sides of the closed rectangle ∂R, we have $z = r \pm iA$ with $0 \le r \le 1$ and so

$$|1 + \epsilon z| \ge |1 + \epsilon(r \pm iA)| = \sqrt{(1 + \epsilon r)^2 + \epsilon^2 A^2} \ge \sqrt{1 + \epsilon^2 A^2}.$$

Therefore, as $|f(z)| \le M_2$ on $\overline{\Omega}$,

$$|g(z)| \le \frac{|f(z)|}{\sqrt{1 + \epsilon^2 A^2}} \le \frac{M_2}{\sqrt{1 + \epsilon^2 A^2}}$$

which holds for all z on the horizontal sides of the rectangle ∂R.

Now choose A large enough that $M_2/\sqrt{1+\epsilon^2A^2} < M_1$ which is possible as M_2 is finite. Thus, $|g(z)| \leq M_1$ on ∂R . We apply the Maximum principle to g on R to get $|g(z)| \leq M_1$ on R. Therefore, $|f(z)| \leq M_1|1+\epsilon z|$ for $z \in \overline{R}$. Allowing $\epsilon \to 0$, the last inequality leads to $|f(a)| \leq M_1$. Since a was an arbitrary point of Ω, $|f(z)| \leq M_1$ holds for all $z \in \Omega$. ■

6.21. Theorem. (Hadamard's Three Lines Theorem) *Let $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$. Suppose that f is analytic and bounded on Ω, and continuous on its closure $\overline{\Omega}$. Suppose that there exist two constants M_0 and M_1 such that*

$$|f(z)| \leq M_0 \ \text{ for } z = 0+iy \in \partial\Omega, \text{ and } |f(z)| \leq M_1 \text{ for } z = 1+iy \in \partial\Omega.$$

Then, $|f(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $z \in \Omega$.

Proof. Without loss of generality we may assume that $M_0 > 0$ and $M_1 > 0$. Define an auxiliary function $F : \overline{\Omega} \to \mathbb{C}$ by $F(z) = e^{\lambda z} f(z)$, where λ is a real number to be fixed later. This function satisfies the hypotheses of Theorem 6.20, and

$$\begin{aligned}
\sup_{z\in\partial\Omega} |F(z)| &= \max\left\{ \sup_{z=0+iy\in\partial\Omega} |F(z)|, \sup_{z=1+iy\in\partial\Omega} |F(z)| \right\} \\
&= \max\left\{ \sup_{z=0+iy\in\partial\Omega} |f(z)|, e^{\lambda} \sup_{z=1+iy\in\partial\Omega} |f(z)| \right\} \\
&\leq \max\left\{ M_0, e^{\lambda} M_1 \right\}.
\end{aligned}$$

Choose λ such that $M_0 = e^{\lambda}M_1$, i.e. $\lambda = \ln(M_0/M_1)$. By Theorem 6.20,

$$|F(z)| \leq e^{\lambda} M_1 \quad \text{for } z \in \overline{\Omega}$$

which means that $e^{\lambda \operatorname{Re} z}|f(z)| \leq e^{\lambda} M_1$ on $\overline{\Omega}$. Substituting $\lambda = \ln(M_0/M_1)$, this gives $|f(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $z \in \overline{\Omega}$. ■

6.22. Corollary. *Let $\Omega = \{z : a < \operatorname{Re} z < b\}$. Suppose that f is analytic and bounded on Ω, and continuous on its closure $\overline{\Omega}$. Suppose that there exist two constants M_0 and M_1 such that*

$$|f(z)| \leq M_0(a) \ \text{ for } z = a+iy \in \partial\Omega$$

and $|f(z)| \leq M_1(b)$ for $z = b+iy \in \partial\Omega$. Then, for all $z \in \Omega$,

$$|f(z)| \leq [M_0(a)]^{1-\theta}[M_1(b)]^{\theta}, \quad \theta = \frac{\operatorname{Re} z - a}{b-a}.$$

Proof. Define $w = \phi(z) = a + (b-a)z$. Then, ϕ takes the vertical strip $\{z : 0 < \operatorname{Re} z < 1\}$ into $\{z : a < \operatorname{Re} z < b\}$. Now, apply Theorem 6.21 to $F = f \circ \phi$. ■

A striking consequence of the Maximum modulus principle is the following result which can be easily proved using the last corollary and the fact that e^w maps the strip $\{w : \ln a < \operatorname{Re} w < \ln b\}$ onto the open annulus $\{z : a < |z| < b\}$ and the boundary maps into the corresponding boundary. However, because of its independent interest, we include a direct proof without referring to the corollary.

6.23. Theorem. (Hadamard's Three Circles Theorem) *Let* $\Omega = \{z : a \leq |z| \leq b\}$, $f \in \mathcal{H}(\Omega)$ *and* $M(r) = \max_{|z|=r} |f(z)|$. *If* $a < r < b$, *then*

$$M(r) \leq [M(a)]^{1-\theta}[M(b)]^{\theta}, \quad \theta = \frac{\ln(r/a)}{\ln(b/a)}.$$

Here θ *depends on* a, b *and* r, *but is independent of* $f(z)$, *and is between* 0 *and* 1.

Proof. We examine the behavior of $g(z) = z^{\lambda} f(z)$, where λ is a real number to be fixed later. Unless λ is an integer, $g(z)$ is clearly a multiple-valued function in the annulus Ω (as z^{λ} is multiple-valued). So we cannot claim that $g \in \mathcal{H}(\Omega)$ and, as a consequence, we cannot apply the Maximum modulus principle directly. But if $D = \Omega \setminus [-b, -a]$, then $g(z)$ becomes analytic on D as the principal power function given by $z^{\lambda} = e^{\lambda \operatorname{Log} z}$ becomes analytic on D. Consequently, $\max |g(z)|$ occurs on this new boundary. Note that, as $\lambda \in \mathbb{R}$, there is no maximum for $|z|^{\lambda}$ at any interior point on the interval $(-b, -a)$ on the real axis. It follows that

$$|z|^{\lambda}\, |f(z)| \leq \max\{a^{\lambda} M(a), b^{\lambda} M(b)\}$$

and so,

$$r^{\lambda}\, M(r) \leq \max\{a^{\lambda} M(a), b^{\lambda} M(b)\}.$$

Ignoring the trivial case $f(z) \equiv 0$, we see that $M(a)$, $M(r)$ and $M(b)$ are all positive, and we may choose λ uniquely (which is at our disposal) such that

$$a^{\lambda} M(a) = b^{\lambda} M(b), \quad \text{i.e.} \quad \lambda = -\left(\frac{\ln(M(b)/M(a))}{\ln(b/a)}\right).$$

We then have

$$\begin{aligned} M(r) &\leq M(a)\left(\frac{a}{r}\right)^{\lambda} \\ &= M(a) \exp\{-\lambda \ln(r/a)\} \\ &= M(a) \exp\left\{\frac{\ln(r/a)}{\ln(b/a)} \ln(M(b)/M(a))\right\} \\ &= M(a)\left(\frac{M(b)}{M(a)}\right)^{\frac{\ln(r/a)}{\ln(b/a)}} \\ &= [M(a)]^{1-\theta}[M(b)]^{\theta}. \end{aligned}$$

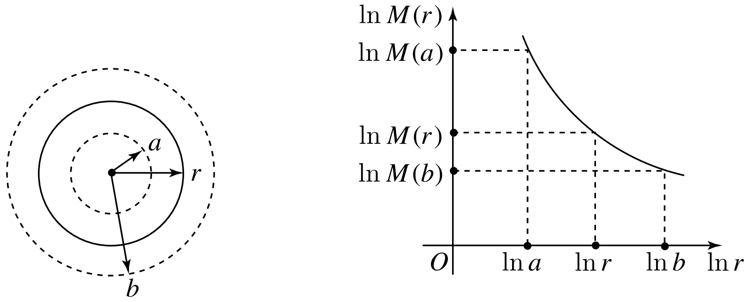


Figure 6.2: Geometry of Hadamard's three circles theorem.

This may be rewritten as

$$[M(r)]^{\ln(b/a)} \leq [M(a)]^{\ln(b/r)}[M(b)]^{\ln(r/a)},$$

or equivalently as

$$\ln M(r) \leq \left(\frac{\ln b - \ln r}{\ln b - \ln a}\right) \ln M(a) + \left(\frac{\ln r - \ln a}{\ln b - \ln a}\right) \ln M(b).$$

The last inequality shows that we may simply express Hadamard's three circles theorem as $\ln M(r)$ is a convex function of $\ln r$ (Figure 6.2). . ■

6.3 Schwarz' Lemma and its Consequences

In this section we start with a simple but one of the classical theorems in complex analysis, namely, Schwarz' lemma, which states that if f is analytic and satisfies $|f(z)| < 1$ in Δ and $f(0) = 0$, then $|f(z)| \leq |z|$ for each $z \in \Delta$ with the sign of equality iff f has the form

$$f(z) = e^{i\alpha} z \tag{6.24}$$

for some $\alpha \in \mathbb{R}$. Furthermore, $|f'(0)| \leq 1$ with the equality iff f has the form (6.24). This is referred to as an infinitesimal version (or simply a classical version) of Schwarz' lemma and is interesting on its own sake. This result has an important role in the proof of the Riemann mapping theorem which is an important theorem concerning the conformal equivalence of two simply connected domains (see Section 12.4).

Let us now begin by proving the sharp version of the classical Schwarz lemma which plays a significant role in geometric function theory. In fact, there are now many extensions of this result. The following form is an important application of the Maximum principle.

6.25. Theorem. **(Schwarz' Lemma)** *Let $f : \Delta \to \overline{\Delta}$ be analytic having a zero of order n at the origin. Then*

(i) $|f(z)| \leq |z|^n$ *for all* $z \in \Delta$,

(ii) $|f^{(n)}(0)| \leq n!$

and the equality holds either in (i) *for some point* $0 \neq z_0 \in \Delta$ *or in* (ii) *occurs iff* $f(z) = \epsilon z^n$ *with* $|\epsilon| = 1$.

Proof. Let $f : \Delta \to \overline{\Delta}$ be analytic on Δ and has n-th order zero at the origin. Then, we have $f(0) = 0 = f'(0) = \cdots = f^{(n-1)}(0)$ and $f^{(n)}(0) \neq 0$. So, we can write

$$f(z) = \sum_{k=n}^{\infty} a_k z^k = z^n g(z) \quad \text{for} \quad z \in \Delta,$$

where $a_k = f^{(k)}(0)/k!$ and $g(z) = \sum_{k=n}^{\infty} a_k z^{k-n}$. The function $g(z) = f(z)/z^n$ has a removable singularity at the origin so that if

$$g(z) = \begin{cases} z^{-n} f(z) & \text{for } z \in \Delta \setminus \{0\} \\ a_n & \text{for } z = 0, \end{cases}$$

then g is analytic in $\Delta \setminus \{0\}$ and continuous on Δ. Referring to the Riemann removability theorem (see Theorem 4.88), we conclude that g is analytic in Δ.

(i) We claim that $|g(z)| \leq 1$ for all $z \in \Delta$. Now, for $0 < r < 1$,

(a) g is analytic on the bounded domain $\Delta_r = \{z : |z| < r\}$

(b) g is continuous on its closure $\overline{\Delta}_r = \{z : |z| \leq r\}$.

Therefore, the Maximum modulus principle is applicable. As $|f(z)| \leq 1$ for every $z \in \Delta$, it follows that for $|\zeta| = r$

$$|g(\zeta)| = \frac{|f(\zeta)|}{|\zeta|^n} \leq \frac{1}{r^n}.$$

By the Maximum modulus principle, $|g(z)| \leq r^{-n}$ for all z with $|z| \leq r$. Since r is arbitrary, by letting $r \to 1$, we find that $|g(z)| \leq 1$; that is

$$|g(z)| \leq 1 \quad \text{for all} \quad z \in \Delta \tag{6.26}$$

and this is same as (i). Equality in (i) holds for some point $z_0 \in \Delta \setminus \{0\}$ implies that $|g(z_0)| = 1$. It follows that g achieves its maximum modulus at an interior point z_0. Consequently, by the Maximum modulus theorem, g must reduce to a constant, say ϵ. Then $f(z) = \epsilon z^n$, where $|\epsilon| = 1$.

(ii) Note that $|g(z)| \leq 1$ throughout the disk Δ. Since $|a_n| = |g(0)|$, (6.26) implies that $|g(0)| \leq 1$ and so (ii) follows.

Again, if $|f^{(n)}(0)| = n!$ then $|g(0)| = 1$ showing that g achieves its maximum modulus 1 at the interior point '0'. Consequently, g is a constant function of absolute value 1 and as before, this means that $f(z) = \epsilon z^n$, where $|\epsilon| = 1$. ■

6.27. Remark. Note that the case $n = 1$ of Theorem 6.25 is the original form of Schwarz' lemma stated at the outset of Section 6.3. ●

For instance, if $f \in \mathcal{H}(\Delta)$ with $|f(z)| \leq 1$ and $f(0) = 0$ then what kind of function is f when $f(1/3) = 1/3$? It must be none other than the identity function because the equality in Theorem 6.25(i) holds with $n = 1$ and $z = 1/3 \in \Delta$.

If f is known to satisfy the conditions of Theorem 6.25 in Δ_R instead of the unit disk Δ, the original form of the theorem can be applied to the function $f(Rz)$ (see also Exercise 6.82). More generally, Theorem 6.25 immediately yields the following result.

6.28. Corollary. *If f is analytic and satisfies $|f(z)| \leq M$ in $\Delta(a;R)$ and $f(a) = 0$, then*

(i) $|f(z)| \leq M|z-a|/R$ *for every* $z \in \Delta(a;R)$,

(ii) $|f'(a)| \leq M/R$

with the sign of equality iff f has the form $f(z) = M\epsilon(z-a)/R$ for some constant ϵ with $|\epsilon| = 1$.

Proof. Use Schwarz' lemma with $g(z) = f(Rz+a)/M,\ |z| < 1$. ■

Does Schwarz' lemma hold for the case of real-valued functions of a real variable? Consider

$$u(x) = \frac{2x}{x^2+1}.$$

Then u is infinitely differentiable on $\mathbb{R}$. In particular, $u'(x)$ is continuous on $[-1,1]$, $u(0) = 0$ and $|u(x)| \leq 1$. But $|u(x)| > |x|$ for $0 < |x| < 1$.

6.29. Example. Let $\omega = e^{2\pi i/n}$ be an n-th root of unity, where $n \in \mathbb{N}$ is fixed. Suppose that $f : \Delta \to \Delta$ is analytic such that $f(0) = 0$. We wish to apply Schwarz' lemma to show that

$$|f(z) + f(\omega z) + f(\omega^2 z) + \cdots + f(\omega^{n-1} z)| \leq n|z|^n \tag{6.30}$$

and equality for some point $0 \neq z_0 \in \Delta$ occurs iff $f(z) = \epsilon z^n$ with $|\epsilon| = 1$. To do this, we define $F : \Delta \to \Delta$ by

$$F(z) = \frac{1}{n}\sum_{k=0}^{n-1} f(\omega^k z).$$

Clearly, F is analytic on Δ, $F(0) = 0$ and, for $1 \le m \le n-1$,

$$F^{(m)}(z) = \frac{1}{n}\sum_{k=0}^{n-1}(\omega^k)^m f^{(m)}(\omega^k z)$$

so that (as $\omega^n = 1$)

$$F^{(m)}(0) = \frac{1}{n}\sum_{k=0}^{n-1}(\omega^m)^k f^{(m)}(0) = \frac{f^{(m)}(0)}{n}\left(\frac{1-(\omega^m)^n}{1-\omega^m}\right) = 0.$$

By Schwarz' lemma (see Theorem 6.25), it follows that $|F(z)| \le |z|^n$ for all $z \in \Delta$ which is the same as (6.30). The equality in this inequality for some point $z_0 \ne 0$ occurs iff $F(z) = \epsilon z^n$ with $|\epsilon| = 1$, or equivalently

$$\sum_{k=0}^{n-1}[f(\omega^k z) - \epsilon z^n] = 0. \tag{6.31}$$

We claim that the above equation implies that $f(z) = \epsilon z^n$. If we let $f(z) = \sum_{m=1}^{\infty} a_m z^m$, then (6.31) becomes

$$\sum_{m=1}^{\infty} a_m \left(\sum_{k=0}^{n-1}\omega^{km}\right) z^m = n\epsilon z^n.$$

In view of the identity

$$\sum_{k=0}^{n-1}\omega^{km} = \begin{cases} n & \text{if } m \text{ is a multiple of } n \\ 0 & \text{otherwise} \end{cases},$$

the last equation implies that $a_n = \epsilon$ and $a_{2n} = a_{3n} = \cdots = 0$. On the other hand, as $|f(z)| < 1$ on Δ and $|a_n| = 1$, we have

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta = \sum_{m=1}^{\infty}|a_m|^2 \le 1$$

which shows that all the Taylor's coefficients of f (except a_n) must vanish and so, $f(z) = e^{i\theta}z^n$. ●

Our first application of Schwarz' lemma is the following theorem which gives an interesting relationship that exists between the maximum modulus of an analytic function and the maximum of its real part. This result will be used to generalize Theorem 6.60 (see Exercise 6.89).

6.32. Theorem. **(Borel Caratheodary)** *Let f be analytic in $|z| \le R$. For each $0 < r < R$, let*

$$M(r) = \max_{|z|=r}\{|f(z)|\} \quad \textit{and} \quad A(r) = \max_{|z|=r}\{\operatorname{Re} f(z)\}.$$

Then

$$M(r) \le \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r}A(R). \tag{6.33}$$

Proof. The result clearly holds if f is constant. Indeed, if $f(z) = \alpha$ (a complex constant), then $M(r) = |\alpha|$, $A(r) = \operatorname{Re}\alpha$, $|f(0)| = |\alpha|$. Substituting these values in the inequality (6.33), then it becomes $-|\alpha| \le \operatorname{Re}\alpha$ which is trivially true. For a non-constant f, we first assume $f(0) = 0$. Then, $A(R) > A(0) = \operatorname{Re} f(0) = 0$. Define

$$F(z) = \frac{f(z)}{2A(R) - f(z)}.$$

Then, F is analytic in $|z| \le R$ and $F(0) = 0$. Also, $\operatorname{Re}[2A(R) - f(z)] \ne 0$; otherwise f would be a real-valued function of complex variable contradicting the analyticity of f. Further, we also note that

$$-2A(R) + \operatorname{Re} f(z) \le \operatorname{Re} f(z) \le 2A(R) - \operatorname{Re} f(z).$$

This means that

$$|F(z)|^2 = \frac{|f(z)|^2}{[2A(R) - \operatorname{Re} f(z)]^2 + [\operatorname{Im} f(z)]^2} \le 1,$$

and so we have $|F(z)| < 1$ in $|z| < R$. Thus, by Schwarz' lemma (take $a = 0$ in Corollary 6.28), we have $|F(z)| \le |z|/R$ for $|z| = r < R$. From the definition of $F(z)$, we have

$$f(z) = \frac{2A(R)F(z)}{1 + F(z)} \quad \text{and} \quad |f(z)| \le \frac{2A(R)|F(z)|}{1 - |F(z)|} \le \frac{2r}{R-r}A(R)$$

which gives (6.33) for the case $f(0) = 0$. If $f(0) \ne 0$, then applying the result to $f(z) - f(0)$, we find that

$$\begin{aligned} |f(z)| - |f(0)| &\le |f(z) - f(0)| \\ &\le \frac{2r}{R-r} \max_{|z|=R} \{\operatorname{Re}(f(z) - f(0))\} \\ &\le \frac{2r}{R-r}[A(R) + |f(0)|], \end{aligned}$$

that is,

$$|f(z)| \le \frac{2r}{R-r}A(R) + |f(0)| \left(\frac{R+r}{R-r}\right). \qquad \blacksquare$$

Now we state and prove the invariant form of Schwarz' lemma.

6.34. Lemma. (Schwarz-Pick Lemma) *Suppose that f is analytic on the unit disk Δ and satisfies the following two conditions*

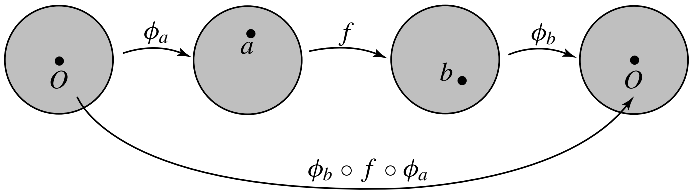


Figure 6.3: Illustration for Schwarz-Pick Lemma.

(i) $|f(z)| \leq 1$ *for all* $z \in \Delta$

(ii) $f(a) = b$ *for some* $a, b \in \Delta$.

Then

$$|f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}. \tag{6.35}$$

Moreover, for a pair of elements a, a' *in* Δ, *the following inequality holds:*

$$\rho(f(a), f(a')) \leq \rho(a, a') \tag{6.36}$$

where $\rho(z, a) = |(z - a)/(1 - \overline{a}z)|$ *for* $z, a \in \Delta$. *Equality is obtained in* (6.35) *at a point* $a \in \Delta$, *or equality is obtained in* (6.36) *for a pair of points* a, a' *with* $a \neq a'$ *in* Δ, *iff* $f \in \mathrm{Aut}\,(\Delta)$; *otherwise, there is strict inequality in* $|z| < 1$

Proof. First, we observe that if $|f(z)| = 1$ for some point $z \in \Delta$, then f is a constant and (6.35) as well as (6.36) are trivial. Thus, we may assume that $|f(z)| < 1$ for all $z \in \Delta$. For a fixed $\alpha \in \Delta$, we recall certain facts that are already familiar to us (see Figure 6.3):

- ϕ_α defined by $\phi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha}z)$ is analytic on $\mathbb{C} \setminus \{1/\overline{\alpha}\}$ if $\alpha \neq 0$, and $\phi_0(z) = -z$. In particular, for each $\alpha \in \Delta$, ϕ_α is analytic in a neighborhood of $\overline{\Delta}$ with $\phi_\alpha(\alpha) = 0$.
- one checks that

$$1 - |\phi_\alpha(z)|^2 = \frac{|1 - \overline{\alpha}z|^2 - |\alpha - z|^2}{|1 - \overline{\alpha}z|^2} = \frac{(1 - |\alpha|^2)(1 - |z|^2)}{|1 - \overline{\alpha}z|^2}$$

so that

(i) $|\phi_\alpha(z)| = 1$ if $|z| = 1$.

(ii) $|\phi_\alpha(z)| < 1$ if $|z| < 1$. Indeed, the fact that $\phi_\alpha(\partial\Delta) = \partial\Delta$ immediately implies that $|\phi_\alpha(z)| < 1$ on Δ, by the Maximum modulus theorem.

(iii) ϕ_α is one-to-one and onto, since

$$\phi_\alpha(\phi_\alpha(z)) = \frac{\alpha - \phi_\alpha(z)}{1 - \overline{\alpha}\phi_\alpha(z)} = \frac{\alpha - \left(\frac{\alpha - z}{1 - \overline{\alpha}z}\right)}{1 - \overline{\alpha}\left(\frac{\alpha - z}{1 - \overline{\alpha}z}\right)} = z$$

so that ϕ_α is invertible and ϕ_α itself is the inverse for ϕ_α. Note that, since $\phi_\alpha^{-1} = \phi_\alpha \in \mathcal{H}(\Delta)$, $\phi_\alpha \in \text{Aut}\,(\Delta)$.

Now, we define $g = \phi_b \circ f \circ \phi_a$ and apply Schwarz' lemma for g. First, we observe that g satisfies the hypothesis of Schwarz' lemma. Thus, by Schwarz' lemma, we conclude that $|g'(0)| \leq 1$ and $|g(z)| \leq |z|$ on Δ. Now, we compute

$$g'(0) = \phi_b'(b) f'(a) \phi_a'(0). \tag{6.37}$$

Since

$$\phi_\alpha'(z) = -\left(\frac{1-|\alpha|^2}{(1-\overline{\alpha}z)^2}\right),$$

it follows that $\phi_\alpha'(0) = -(1-|\alpha|^2)$ and $\phi_\alpha'(\alpha) = -1/(1-|\alpha|^2)$. Using these in (6.37), we find that

$$g'(0) = \left(\frac{1-|a|^2}{1-|b|^2}\right) f'(a)$$

and therefore, since $|g'(0)| \leq 1$ and $b = f(a)$, the conclusion (6.35) follows. For the proof of (6.36), we use the second condition $|g(z)| \leq |z|$ which is equivalent to the inequality

$$|(\phi_b \circ f \circ \phi_a)(z)| \leq |z|, \quad z \in \Delta.$$

Since $\phi_\alpha(\phi_\alpha(z)) = z$ for each $z \in \Delta$, setting z for $\phi_a(z)$, it follows that

$$|(\phi_b \circ f)(z)| \leq |\phi_a(z)|, \quad z \in \Delta.$$

In particular, if we take $z = a'$ then the last inequality is equivalent to the inequality $|\phi_b(b')| \leq |\phi_a(a')|$ which, by the definition of ϕ_α, is the same as the inequality (6.36).

By Schwarz' lemma, equality in (6.35) or in (6.36) holds if and only if $g(z) = \epsilon z =: I_\epsilon(z)$, $|\epsilon| = 1$; that is, $f(z) = \phi_b^{-1} \circ I_\epsilon \circ \phi_a^{-1} \in \text{Aut}\,(\Delta)$. ∎

6.38. Remark. Clearly, (6.35) may be obtained directly from (6.36). Indeed, rewriting the inequality (6.36) in the form

$$\left|\frac{f(a) - f(a')}{a - a'}\right| \leq \left|\frac{1 - f(a)\overline{f(a')}}{1 - a\overline{a'}}\right|$$

and letting $a' \to a$, we obtain (6.35). Thus, every analytic function $w = f(z)$ from Δ into Δ satisfies the condition

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \Delta,$$

or equivalently,

$$\frac{|dw|}{1-|w|^2} \leq \frac{|dz|}{1-|z|^2} \quad (w = f(z),\ z \in \Delta)$$

and the equality holds iff $f \in \mathrm{Aut}\,(\Delta)$.

Also, we observe that the inequality (6.36) for analytic functions $f : \Delta \to \Delta$ reveals the contraction property of f with respect to the metric ρ (in fact, it is a non-expansive map with respect to the metric ρ). Further, f is an isometry, i.e. $\rho(f(a), f(a')) = \rho(a, a')$, iff $f \in \mathrm{Aut}\,(\Delta)$. •

6.39. Example. Suppose that $f : \Delta \to \overline{\Delta}$ is analytic such that $f(a) = 0$ and $f(a-1) = b$ for some $a \in (0,1)$ and $b \in \Delta$. Then, by (6.36),

$$|b| \leq \frac{1}{1-a(a-1)} = \frac{1}{1+a-a^2}.$$

In particular, this observation implies the following assertions:

(i) There exists no function f that is analytic in Δ such that $|f(z)| \leq 1$, $f(1/3) = 0$ and $f(-2/3) = 5/6$.

(ii) There exists no analytic function $f : \Delta \to \overline{\Delta}$ such that $f(1/4) = 0$ and $f(-3/4) = 17/20$. •

6.40. Example. Suppose we are given that $f : \Delta \to \overline{\Delta}$ is analytic and $f(a) = 0$ for some point $a \in \Delta$. Suppose that we are asked to find an estimate for $|f(b)|$ for some $b \in \Delta$. To do this, by (6.36), we see first that (since $f(a) = 0$),

$$|f(z)| \leq |\phi_a(z)|\,, \quad \phi_a(z) = \frac{a-z}{1-\overline{a}z} \ \text{ for } z \in \Delta.$$

In particular, $|f(b)| \leq |\phi_a(b)|$ and note that the maximum is achieved when $f(z) = \phi_a(z)$. For instance, if $f(1/2) = 0$ then the estimate for $f(1/4)$ is given by

$$|f(1/4)| \leq \left|\frac{(1/2)-(1/4)}{1-(1/2)\,(1/4)}\right| = \frac{2}{7}$$

and the maximum is attained for $\phi_{1/2}(z)$. •

6.41. Example. Suppose that $f : \Delta \to \overline{\Delta}$ is an analytic function such that $f(0) = a$ for some $a \in \Delta$. We wish to verify whether there exists such an f with the property that $f'(0) = c$ for a complex constant c. If so, under what condition on c, such a function exists? To do this, according to (6.35), we first observe that $|f'(0)| \leq 1 - |a|^2$ which means that c must satisfy the inequality $|c| \leq 1 - |a|^2$. For instance, consider the $\phi_a(z)$ defined above. Then, $\phi_a(0) = a$, $\phi_a'(0) = -(1-|a|^2)$ and so, the function f defined by $f(z) = k\phi_a(z)$ with $|k| \leq 1$ does the job. •

6.42. Example. Suppose that $f : \overline{\Delta} \to \overline{\Delta}$ is an analytic function, $|f(z)| \leq 1$ on $|z| = 1$, $f(a) = 0$ and $f(-a) = b$ for some $a \in (0, 1)$ and $b \in (0, 1]$. Then, by (6.36), it follows that $|b| \leq 2a/(1+a^2)$. This observation is useful to conclude the following:

(i) There exists no analytic function $f : \overline{\Delta} \to \overline{\Delta}$ such that

$$|f(z)| \leq 1 \ \text{ on } \ |z| = 1, \quad f(1/2) = 0 \ \text{ and } \ f(-1/2) = 19/20.$$

Note that the existence of such a function is guaranteed, for example, if we replace the condition $f(-1/2) = 19/20$ by $f(-1/2) = 4/5$.

(ii) There exists an analytic function $f : \overline{\Delta} \to \overline{\Delta}$ such that $f(1/3) = 0$ and $f(-1/3) = 3/5$. Such an f is given by $\phi_{1/3}(z)$. ●

6.43. Example. Let $f \in \mathcal{H}(\Delta)$ with $f(a) = 0$ and $|f(z)| \leq |e^{iz}|$ for all $|z| = 1$. How large can $|f(-a)|$ be? To do this, we rewrite the given condition on f as $|F(z)| \leq 1$ for $|z| = 1$, where $F(z) = e^{-iz} f(z)$. Define g by $g = F \circ \phi_a$, where $\phi_a(z)$ is defined as above. Then, $g(a) = 0$ and $|g(z)| \leq 1$ for $|z| \leq 1$ (by the Maximum modulus principle) so that, by Schwarz' lemma,

$$|g(z)| \leq |z|, \ \text{ i.e. } \ |e^{-i\phi_a(z)} f(\phi_a(z))| \leq |z|,$$

or equivalently,

$$|e^{-iw} f(w)| \leq \left| \frac{a - w}{1 - \overline{a}w} \right|, \ \text{ i.e. } \ |f(w)| \leq \left| e^{iw} \left(\frac{a - w}{1 - \overline{a}w} \right) \right| \quad \text{for all } \ |w| \leq 1$$

and the equality holds if $f(w) = e^{i\theta} e^{iw} \phi_a(w)$ for some θ. ●

6.44. Example. A finite Blaschke product is defined to be a rational function of the form $B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a}_k z}$, where $a_1, a_2, \dots, a_n$ are in Δ and $\alpha \in [0, 2\pi]$. If f is analytic in the unit disk Δ, continuous on $\overline{\Delta}$ and $|f(z)| = 1$ for $|z| = 1$, then it can be easily shown that f is a finite Blaschke product. If, in addition, f is entire then $f(z) = e^{i\alpha} z^n$ for some $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$.

In fact, to exclude the trivial case, we assume that f is non-constant. Since f is analytic in the unit disk $|z| < 1$ and f is continuous in $|z| \leq 1$ and $|f(z)| = 1$, it can have only finitely many zeros; otherwise, the limit point of the zeros must lie on $|z| = 1$ (since the zeros are isolated), which is a contradiction. Denote these zeros by $a_1, a_2, \dots, a_n$ (multiple zeros being repeated). Define

$$F(z) = f(z) \, / \left(\prod_{k=1}^{n} \phi_{a_k}(z) \right), \quad \phi_{a_k}(z) = \frac{a_k - z}{1 - \overline{a}_k z}.$$

Then, F is analytic for $|z| < 1$, continuous on $|z| \leq 1$, $|F(z)| = 1$ for $|z| = 1$ and $F(z) \neq 0$ in Δ. The same conclusion holds for $1/F(z)$ also. Applying the Maximum modulus principle, both to $F(z)$ and $1/F(z)$, we get $|F(z)| \leq 1$ and $|1/F(z)| \leq 1$ for $|z| \leq 1$. Thus, $|F(z)| = 1$ on $|z| \leq 1$ and so,

$$f(z) = e^{i\alpha} \prod_{k=1}^{n} \phi_{a_k}(z).$$

If f is entire, then $a_k = 0$ for each k, and the second part follows. ●

We have seen a number of variants of Schwarz' lemma which are useful. Incidentally, this result can be used to characterize all conformal self mappings (i.e. one-to-one, onto and analytic) of the unit disk Δ and which map the unit circle $\partial\Delta$ onto $\partial\Delta$ (see also Theorem 5.59).

6.45. Theorem. *Every univalent analytic function f from Δ onto itself that has an analytic inverse must be of the form*

$$f(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z},$$

where z_0 is a complex number, $|z_0| < 1$, and $0 \leq \theta \leq 2\pi$.

Proof. Let $w = f(z)$ be such an analytic univalent function, and let $f(a) = b$. Then, according to (6.36), we have

$$\rho(w, b) \leq \rho(z, a), \quad \rho(z, a) = \left|\frac{z - a}{1 - \overline{a}z}\right|. \tag{6.46}$$

Applying the same argument to the inverse f^{-1}, for which $f^{-1}(b) = a$, we obtain

$$\rho(f^{-1}(w), f^{-1}(b)) \leq \rho(w, b), \quad \text{i.e. } \rho(z, a) \leq \rho(w, b) \tag{6.47}$$

for all $z \in \Delta$. Equations (6.46) and (6.47) imply that for each $z \in \Delta$

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| = \left|\frac{z - a}{1 - \overline{a}z}\right|;$$

that is

$$f(z) = e^{i\theta}\frac{[(z - a)/(1 - \overline{a}z)] + f(a)e^{-i\theta}}{1 + \overline{f(a)}e^{-i\theta}[(z - a)/1 - \overline{a}z)]} \quad \text{for some real } \theta$$

which has the desired form. ■

An equivalent formulation of Theorem 6.45 is the following:

$$\begin{aligned} \operatorname{Aut}\Delta &= \left\{\frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1\right\} \\ &= \left\{e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z} : z_0 \in \Delta,\ 0 \leq \theta \leq 2\pi\right\}. \end{aligned}$$

From Theorem 6.45, it is clear that

$$\operatorname{Aut}(\Delta_R) = \left\{ R^2 e^{i\theta} \frac{z - z_0}{\overline{z}_0 z - R^2} : z_0 \in \Delta_R,\ 0 \le \theta \le 2\pi \right\}.$$

For a fixed θ, $f(z) = e^{i\theta} z$ is a conformal self-map of Δ that fixes the origin. We now show that these are the only conformal self-maps that fix the origin.

6.48. Corollary. *Every automorphism $f : \Delta \to \Delta$ with $f(0) = 0$ is given by $f(z) = e^{i\theta} z$.*

Proof. The corollary follows if we choose $a = 0$ and $b = 0$ in Theorem 6.45. Alternatively, Schwarz' lemma applied to f and f^{-1}, because each of f and f^{-1} maps Δ onto itself such that $f(0) = 0$ and $f^{-1}(0) = 0$, yields

$$|f(z)| \le |z| \quad \text{and} \quad |z| = |f^{-1}(f(z))| \le |f(z)|.$$

Hence, $|f(z)| = |z|$; that is, $f(z) = e^{i\theta} z$. ∎

6.49. Theorem. *Let p be analytic in Δ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in Δ. Then, $|p'(0)| \le 2$ and*

$$\frac{1 - |z|}{1 + |z|} \le |p(z)| \le \frac{1 + |z|}{1 - |z|}, \quad z \in \Delta.$$

Equality holds in each of these inequalities for $p(z) = (1 + z)/(1 - z)$.

Proof. Define $\phi(w) = (w - 1)/(w + 1)$. Then, ϕ maps $\{w : \operatorname{Re} w > 0\}$ conformally onto Δ and so $f = \phi \circ p$ maps Δ conformally onto itself with $f(0) = 0$. From Schwarz' lemma it follows then that

(i) $|\phi(p(z))| = |f(z)| \le |z|$ for $z \in \Delta$

(ii) $|\phi'(p(0))p'(0)| = |f'(0)| \le 1$.

The assertion (i) implies that

$$\left. \frac{|p(z)| - 1}{|p(z)| + 1},\ \frac{1 - |p(z)|}{1 + |p(z)|} \right\} \le \left| \frac{p(z) - 1}{p(z) + 1} \right| \le |z|, \text{ i.e. } \frac{1 - |z|}{1 + |z|} \le |p(z)| \le \frac{1 + |z|}{1 - |z|}.$$

As $|\phi'(p(0))| = |\phi'(1)| = 1/2$, (ii) gives that $|p'(0)| \le 2$. ∎

6.50. Remark. If, in Theorem 6.49, $p(0) = \alpha + i\beta$, $\alpha > 0$, we may apply the above result to the function $(p(z) - i\beta)/\alpha$. ●

In $\mathbb{R}$, for a differentiable function f of one variable to have maximum or minimum at x_0 it is necessary that $f'(x_0) = 0$. However, the analog of this does not hold when we deal with functions of a complex variable. This surprising fact in the behavior of an analytic function at a point where it assumes its maximum modulus will be reflected in the next theorem.

6.51. Theorem. *Suppose that f is analytic in a neighborhood of $\overline{\Delta}$ and $z_0 \in \partial\Delta$ satisfies $|f(z_0)| = \max_{z \in \overline{\Delta}} |f(z)|$. Then, $f'(z_0) \neq 0$ unless f is constant.*

Proof. Suppose that f is a non-constant analytic function. Without loss of generality, we suppose $1 = z_0 = |f(z_0)|$, multiplying f by a constant, if necessary. Then, by the Maximum modulus principle, we have $f(\Delta) \subset \Delta$. Assume first that $f(0) = 0$. Then, for all $t \in (0,1)$, applying the triangle inequality and Schwarz' lemma, we find that $|1 - f(t)| \geq 1 - |f(t)| \geq 1 - t$ and so

$$\left| \frac{f(1) - f(t)}{1 - t} \right| \geq 1 \quad \text{for } 0 \leq t < 1.$$

Thus, we obtain $|f'(1)| \geq 1$ by making $t \to 1^-$. Suppose $f(0) = a \neq 0$ and consider the function g defined by $g(z) = (\phi_a \circ f)(z)$, where

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad |a| < 1.$$

Then, $g : \Delta \to \Delta$ and $g(0) = 0$. Applying the preceding argument to g, we get $|g'(1)| \geq 1$. A direct calculation shows that

$$g'(z) = -\frac{(1 - |a|^2) f'(z)}{[1 - \overline{a} f(z)]^2}$$

so that

$$f'(1) = -\frac{g'(1)[1 - \overline{a} f(1)]^2}{1 - |a|^2} = -\frac{g'(1)(1 - \overline{a})^2}{1 - |a|^2}.$$

Since $|g'(1)| \geq 1$, this gives that

$$|f'(1)| \geq \frac{(1 - |a|)^2}{1 - |a|^2} = \frac{1 - |a|}{1 + |a|} > 0.$$

This completes the proof. ■

From the proof of Theorem 6.51, because of its independent interest, we notice the following:

6.52. Corollary. *If f is analytic on Δ, $f(0) = 0$, $|f(z)| < 1$ in Δ and if f is analytic at $z = 1$ with $f(1) = 1$, then $|f'(1)| \geq 1$.*

6.4 Liouville's Theorem

We wish to address, in detail, the following questions:

- Which entire functions are bounded?
- Which entire functions omit two distinct complex values? Recall that a function $f : \Omega \to \mathbb{C}$ is said to omit a value a if $a \in \mathbb{C}\backslash f(\Omega)$. For example, the entire functions e^z and $1+e^z$ omit 0 and 1, respectively.
- Which meromorphic functions omit three distinct complex values? For example, the meromorphic function $1/(1-e^{2\pi iz})$ in $\mathbb{C}$ omits two different values, namely 0 and 1.

These questions have elegant and complete answers in the form of Liouville's theorem and Picard's little theorem. In Picard's great theorem, we shall actually answer a more general question. Let us discuss the first question and the remaining will be done in Section 12.7.

Recall that a function is entire iff it is analytic on the whole complex plane $\mathbb{C}$. The simplest examples of entire functions are polynomials. Entire functions which are not polynomials (e.g. e^z, $\sin z$, $\cos z$, $\cosh z$, $\sinh z$ $\exp(\sin z)$ etc.) are called *entire transcendental functions.*

Suppose now that f is entire and $a \in \mathbb{C}$ is arbitrary. Then, f admits a Taylor expansion around a:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

If $a_n = 0$ for all $n \geq 1$, then $f(z) = f(a) = a_0$; that is, f is a constant function. Note that a non-zero constant function is the only (rational) entire function having no zeros. If $a_k \neq 0$ for some integer $k \geq 1$ but $a_n = 0$ for all $n > k$, then f is a polynomial of degree $k \geq 1$. Finally if $a_k \neq 0$ for an infinite number of values of k, then f becomes an *entire transcendental function*. Recall that entire functions are just the Taylor's series about any point a with infinite radius of convergence. For instance, the power series such as

$$\sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha} \ (\alpha > 0), \quad \sum_{n=0}^{\infty} e^{-n^2} z^n, \quad \sum_{n=0}^{\infty} \frac{z^n}{2^{-n}n!}, \quad \sum_{n=0}^{\infty} \frac{z^n}{2^n n!},$$

which are convergent for all $z \in \mathbb{C}$. In conclusion, a non-constant entire function is either a polynomial of degree n so that it has a pole of order n at infinity; or else an entire transcendental function in which case it will have an essential singularity at infinity (see Chapter 7). With this observation, elementary examples of the last type are the exponential function e^z, the trigonometric functions $\sin z$, $\cos z$, and the hyperbolic functions $\sinh z$, $\cosh z$ (which are, of course, well-known simple combinations of exponential functions). From our earlier results, it can be easily observed that the sum, difference and product of a finite number of entire functions are also entire; and the quotient of two entire functions is also entire provided the denominator is nowhere zero.

Now, our aim is to examine the behavior of entire functions for sufficiently large $|z|$. Let us start with the following simple problem.

6.53. Problem. Are there non-constant functions of a real variable x which are both bounded and differentiable on $\mathbb{R}$?

We shall first see that familiar functions on $\mathbb{R}$ such as e^x, $\sin x$, $\cos x$ and $\log(1+x^2)$ exhibit surprising behavior when we allow x to be complex rather than just real. To do this, we shall look at three different types of simple examples before stating the theorem. Let us look at the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$. Then

- f is non-constant and differentiable on $\mathbb{R}$
- f is unbounded on $\mathbb{R}$ and $f^{(n)}$ exists on $\mathbb{R}$ for each $n \in \mathbb{N}$
- f is one-to-one on $\mathbb{R}$.

In any subject, just with one example, it may not be possible to conclude that there exists no function with the desired property as in Problem 6.53. Note that $f(z) = e^z$ is not one-to-one on $\mathbb{C}$. Now, we look at the second example. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sin x$ or $\cos x$ for $x \in \mathbb{R}$. Clearly,

- g is non-constant and differentiable on $\mathbb{R}$
- g is bounded by 1 and $g^{(n)}$ exists on $\mathbb{R}$ for each $n \in \mathbb{N}$
- g is not one-to-one on $\mathbb{R}$ (because it is a periodic function).

As the third example, let us consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f_a(x) = (x^2 + a^2)^{-1},$$

where a is a fixed non-zero real number. Then, $0 < f_a(x) \leq 1/a^2$ for each $x \in \mathbb{R}$. However, we see that neither of these functions, namely, $f_a(x)$, $\cos x$ and $\sin x$, is bounded if *"complex values are permitted for the variable x."* Indeed, for

$$f_a(z) = \frac{1}{z^2 + a^2},$$

the absolute value of $f_a(z)$ approaches ∞ whenever z approaches $\pm ia$, $a \in \mathbb{R}\backslash\{0\}$. Note that $f_a(z)$ is *not* differentiable on the whole of $\mathbb{C}$. On the other hand, $\sin x$ and $\cos x$ are non-constant bounded functions of real variable and each of them is differentiable on $\mathbb{R}$. Observe that $\cos z$ and $\sin z$ are the well-known non-constant entire functions defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

It follows that for $z = iy$, $y \in \mathbb{R}$,

$$\cos(iy) = \frac{e^{-y} + e^y}{2} \quad \text{and} \quad |\sin(iy)| = \frac{|e^{-y} - e^y|}{2}$$

which implies that both $|\cos z|$ and $|\sin z|$ increase indefinitely when z approaches infinity along the imaginary axis. Thus, the entire functions $\cos z$ and $\sin z$ are unbounded on $\mathbb{C}$; i.e. no K can exist such that $|\cos z| \leq K$ on $\mathbb{C}$. Similarly, no K can exist such that $|\sin z| \leq K$ on $\mathbb{C}$. In particular, we say that the range of each of the functions $\exp z$, $\cos z$ and $\sin z$ is an unbounded set in $\mathbb{C}$. As a consequence, it follows that both $\sin z$ and $\cos z$ are bounded for $|z| < 2004$ whereas both $\sin z$ and $\cos z$ are unbounded for $|z| > 2004$. More generally, for sufficiently large R, both $\sin z$ and $\cos z$ maps $|z| > R$ into any prescribed neighborhood of infinity.

6.54. Example. Let α, β, γ and δ be some fixed real numbers. Then, we have the following:

- if $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > \alpha\}$, then the entire function $f(z) = e^{-iz}$ is unbounded on the half-plane Ω (note that $f(iy) = e^y$ which approaches $+\infty$ as $y \to +\infty$);
- if $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > \beta\}$, then the entire function $g(z) = e^z$ is unbounded on the half-plane Ω;
- the entire function $h(z) = e^{-z^2}$ is unbounded on any open set which contains either part of the imaginary axes $\{iy : y > \gamma\}$ or $\{iy : y < \delta\}$. Note that $h(iy) = e^{y^2}$. ●

Finally, for $n \in \mathbb{N}$, we consider $f_n(z) = z/(1 + n|z|)$ or $\sin(|z|^n)$. Then, for each n, $f_n(z)$ is continuous on $\mathbb{C}$ and $|f_n(z)| < 1$ on $\mathbb{C}$. This observation shows that there exist non-constant bounded continuous functions on $\mathbb{C}$. Thus, it is natural to expect a simple characterization for functions of complex variable that are both bounded and analytic in $\mathbb{C}$. This is precisely given by Liouville's theorem in the following form which implies that the range of every non-constant entire function is an unbounded set in $\mathbb{C}$.

6.55. Theorem. **(Liouville's Theorem)** *A bounded and entire function is constant.*

We have already observed through examples that this result is quite different from any that could possibly hold for real-valued functions of a real variable. We remark that Liouville's theorem[10] can be viewed as a statement about the range of a non-constant entire function. Therefore,

[10] Liouville (1809-1882), a French mathematician, is known for his work on analysis and differential geometry.

an equivalent formulation of Liouville's theorem is that *"the range of a bounded and entire function is a singleton set"*.

As an immediate consequence of Liouville's theorem, we have the following simple applications:

- *Non-constant entire functions must be unbounded.* For example, as we have already seen, the entire functions $\sin z$ and $\cos z$ are unbounded unlike their real counterparts.
- *Every analytic function in the extended complex plane is necessarily constant.* In fact if f is analytic at $z = \infty$, then $\lim_{|z|\to\infty} f(z)$ is finite. Let this limit be L. This means that, given $\epsilon > 0$ there exists an $R > 0$ such that

  $$|f(z)| - |L| \leq |f(z) - L| < \epsilon \quad \text{whenever} \quad |z| > R;$$

  and so, in particular, f is bounded for $|z| > R$ and thus, by the continuity of f on the compact set $\{z : |z| \leq R\}$, f is bounded on the whole of $\mathbb{C}$. Hence, by Liouville's theorem, f is constant. Equivalently, we say that *the only function which is analytic on the Riemann sphere is the constant function.*
- *If f is entire and if there exists an $M > 0$ with $|f(z)| > M$ for all $z \in \mathbb{C}$, then f is constant.* This is because the given conditions imply that $f'(z)$ exists and $f(z) \neq 0$ in $\mathbb{C}$ so that $1/f(z)$ is analytic on $\mathbb{C}$ and $|1/f(z)| < 1/M$ for all $z \in \mathbb{C}$. Now applying Liouville's theorem to $1/f(z)$, we conclude that f is constant.

6.56. Problem. Do we really need the boundedness condition in the hypotheses of Liouville's theorem?

For instance, can we replace the 'boundedness' condition in Liouville's theorem by a condition such as the following?

- $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded on $\mathbb{C}$
- $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ lies in a half-plane.

The answer is 'yes' under each of the above conditions. For example, if f is entire and $\operatorname{Re} f(z) \leq M$ for some fixed $M \in \mathbb{R}$, then f is bounded. Indeed,

$$\begin{aligned}
f(z) \text{ is entire} &\implies \phi(z) = e^{f(z)} \text{ is entire} \\
&\implies |\phi(z)| = |e^{f(z)}| = e^{\operatorname{Re} f(z)} \leq e^M \text{ for all } z \in \mathbb{C} \\
&\implies \phi(z) \text{ is constant, by Liouville's theorem,} \\
&\implies \phi'(z) = e^{f(z)} f'(z) = 0 \text{ for all } z \in \mathbb{C} \\
&\implies f'(z) = 0 \text{ in } \mathbb{C}, \text{ since } e^{f(z)} \neq 0 \text{ in } \mathbb{C} \\
&\implies f(z) \text{ is constant.}
\end{aligned}$$

Alternatively, it suffices to observe that if f is entire and $\operatorname{Re} f(z) \leq M$ then $g(z) = 1/[1+(M-f(z))]$ is entire and bounded by 1 so that g (and, hence, f) is constant.

The other cases may be handled in a similar fashion. However, it is important to remark that all these examples follow as a simple consequence of Picard's theorem which we shall discuss in Section 12.7. Note also that the above discussion (together with the fact that a harmonic function in $\mathbb{C}$ possesses a harmonic conjugate in $\mathbb{C}$) clearly shows the following:

6.57. Theorem. *A function which is harmonic and bounded in* $\mathbb{C}$ *must be constant.*

This statement can be viewed as the harmonic analog of Liouville's theorem.

6.58. First proof of Liouville's theorem. Since f is bounded on $\mathbb{C}$, there is a finite M such that $|f(z)| \leq M$ for $z \in \mathbb{C}$. Let a be an arbitrary complex number and let $\Gamma = \{z : z = Re^{i\theta},\, 0 \leq \theta < 2\pi\}$, where $|a| < R < \infty$. Then, according to the Cauchy integral formula, we have

$$f(a) - f(0) = \frac{1}{2\pi i}\int_\Gamma \left\{\frac{1}{z-a} - \frac{1}{z}\right\} f(z)\,dz = \frac{a}{2\pi i}\int_\Gamma \frac{f(z)}{z(z-a)}\,dz$$

so that for each fixed a, we have

$$|f(a) - f(0)| \leq \frac{|a|}{2\pi}\left\{\max_{|z|=R}\left|\frac{f(z)}{z(z-a)}\right|\right\} 2\pi R \leq \frac{M|a|}{R-|a|}$$

which approaches zero as $R \to \infty$. Thus, $f(a) = f(0)$ for each $a \in \mathbb{C}$ and hence, f is constant. ∎

6.59. Second proof of Liouville's theorem. By hypothesis, there exists a finite $M > 0$ such that $|f(z)| \leq M$ for $|z| < R$ and for any $R > 0$. Equivalently, $|f(Rz)| \leq M$ for $|z| < 1$. In particular, $|f(0)| \leq M$. Set

$$g(z) = \frac{f(Rz) - f(0)}{2M}, \quad |z| < 1.$$

Then g satisfies the hypotheses of Schwarz' lemma for each $R > 0$, since $g(0) = 0$ and

$$|g(z)| \leq \frac{|f(Rz)| + |f(0)|}{2M} \leq \frac{M+M}{2M} = 1.$$

Hence, we have $|g(z)| \leq |z|$ for $|z| < 1$. In other words,

$$|f(Rz) - f(0)| \leq 2M|z| \quad \text{for} \quad |z| < 1,$$

or equivalently,

$$|f(z) - f(0)| \leq \frac{2M}{R}|z| \quad \text{for} \quad |z| < R.$$

Again remember that M is a fixed constant, whereas R is at our disposal, and can be chosen as large as we please. Thus, restricting z in the unit disk $|z| < 1$ and letting R tend to infinity in the inequality, we find that $f(z) = f(0)$ for $|z| < 1$, which, by the Uniqueness theorem, gives $f(z) = f(0)$ for all $z \in \mathbb{C}$) and so f is constant. ∎

Third proof of Liouville's theorem is actually an immediate consequence of *Cauchy estimate*. So we leave this as a simple exercise.

Our fourth proof is contained in the following theorem which shows that if f is an entire function such that $|f(z)|$ increases slower than some power of $|z|$ as $|z| \to \infty$, then the function must be a polynomial.

6.60. Theorem. (A Generalized Version of Liouville's Theorem). *An entire function f, which satisfies the inequality $|f(z)| \leq M|z|^\alpha$ for some $\alpha \geq 0$, $M > 0$ and all sufficiently large $|z|$, reduces to a polynomial of degree n where n is the largest integer such that $n \leq \alpha$.*

Proof. Note that the case $\alpha = 0$ is Liouville's theorem. Let f be entire. Then, f admits a Taylor expansion around 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz \quad \text{for any} \quad R > 0.$$

Hence, taking $R \geq M$ and using the standard estimate for integrals and the given growth condition on f, we have

$$|a_k| \leq \frac{1}{2\pi} \left(\frac{MR^\alpha}{R^{k+1}} \right) \cdot 2\pi R = MR^{\alpha - k}.$$

Since R can be made arbitrarily large, it follows that for $k > \alpha$ we must have $a_k = 0$. We conclude that f can only be a polynomial of degree not greater than α. ∎

For example if $f(z)$ is entire such that $f(z)/z^n$ is bounded in $\mathbb{C}$, then $f(z) = cz^n$ for some constant c. For a generalization of Theorem 6.60, we refer to Exercise 6.89.

6.61. Remark. We remark that it is possible that a real differentiable function f can map $\mathbb{C}$ onto the unit disk $\Delta = \{z : |z| < 1\}$, as the example

$$z \mapsto \frac{z}{\sqrt{1 + |z|^2}}, \quad \text{i.e.} \quad f(x, y) = \left(\frac{x}{\sqrt{1 + x^2 + y^2}}, \frac{y}{\sqrt{1 + x^2 + y^2}} \right),$$

shows. Note that this has an inverse given by

$$w \mapsto \frac{w}{\sqrt{1-|w|^2}}, \quad \text{i.e.} \quad f^{-1}(u,v) = \left(\frac{u}{\sqrt{1-(u^2+v^2)}}, \frac{v}{\sqrt{1-(u^2+v^2)}} \right).$$

However, an immediate application of Liouville's theorem shows that this is not the case with analytic functions as "every analytic function $f : \mathbb{C} \to \Delta$ is constant." In other words, there can be no *bianalytic* (i.e. a bijection which is analytic together with its inverse) mapping of the unit disk Δ onto the whole complex plane $\mathbb{C}$ or of the upper half-plane onto $\mathbb{C}$. ●

Moreover, as a consequence of Theorem 6.60, we conclude that if f is entire and $|f(z)| \leq A + B|z|^\alpha$ for all sufficiently large $|z|$ and for some fixed constants A, B and $0 \leq \alpha < 1$, then f is constant. This observation reminds us that we need not assume that $|f(z)|$ is bounded in Liouville's theorem, only that its growth is sufficiently slow.

6.62. Example. Suppose that $f = u + iv$ is an entire function which is blessed with the property that $u_y - v_x = -2$ for all $z \in \mathbb{C}$. What can you say about the function f? Can it be a constant function? Clearly not! Can this be a polynomial of higher degree? The given condition shows that this is not the case (how?). Let us now try to find the precise form of this function. By the C-R equation $u_y = -v_x$, the given condition is the same as $v_x = 1$ which, by the fact that $f'(z) = u_x + iv_x$, is equivalent to

$$\operatorname{Im} f'(z) = 1 \quad \text{for } z \in \mathbb{C}.$$

This observation implies that $f'(z)$ is a constant, say a, so that f has the form $f(z) = az + b$ with $\operatorname{Im} a = 1$. Alternatively, as $\operatorname{Im} f'(z) = 1$, $h(z)$ defined by $h(z) = e^{if'(z)}$ is entire and $|h(z)| = e^{-1}$ shows that h (and hence $f'(z)$) is constant, by Liouville's theorem. ●

6.5 Doubly Periodic Entire Functions

Recall that a function f has a period ω if $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$. To obtain another application of Liouville's theorem, we first recall two familiar periodic functions of a real variable x:

$$\begin{cases} \sin(x + 2\pi) = \sin x \\ \cos(x + 2\pi) = \cos x \end{cases}, \quad \text{for all } x \in \mathbb{R}.$$

Does the property hold when we allow x to be complex rather than just real, for each of these functions? By the Uniqueness theorem, yes it is. Thus, $\sin z$ and $\cos z$ are periodic functions with period 2π. Is e^x periodic if $x \in \mathbb{R}$? No. Indeed, this function is a strictly increasing one-to-one function on $\mathbb{R}$. However, in view of the periodicity property of $\sin z$ and $\cos z$, we have

$$\cos(z+2\pi)+i\sin(z+2\pi)=\cos z+i\sin z \quad \text{for all } z\in\mathbb{C};$$

that is $e^{z+2\pi i}=e^z$ for all $z\in\mathbb{C}$ showing that the complex exponential function e^z is periodic with period $2\pi i$, which is a purely imaginary number. So, 2π is a period for the entire functions e^{iz}, $\cos z$ and $\sin z$. It is therefore natural to ask whether there exists a non-trivial *doubly periodic* entire function f (i.e. f with two independent periods ω_1 and ω_2 such that $f(z)=f(z+\omega_1)=f(z+\omega_2)$ for all $z\in\mathbb{C}$). Equivalently, we raise the following

6.63. Problem. Does there exist an entire function f having both a real and an imaginary period? More precisely, are there two non-zero real numbers α, β such that

$$\begin{cases} f(z+\alpha)=f(z) \\ f(z+i\beta)=f(z) \end{cases} \quad \text{for all } z\in\mathbb{C}?$$

Let us first show that every entire function f such that

$$f(z)=f(z+1)=f(z+i) \ \text{ on } \mathbb{C}$$

is necessarily a constant. To do this, we consider the solid square

$$S=\{x+iy:\ 0\le x,y\le 1\}.$$

Since f is continuous on S, there exists an M such that $|f(z)|\le M$ on S. Now, let $z\in\mathbb{C}$ be arbitrary. Clearly, we can find two integers m and n such that $z+m+in\in S$ and therefore, $|f(z+m+in)|\le M$ for $z\in\mathbb{C}$. Further, the hypotheses imply that

$$f(z)=f(z\pm 1)=\cdots=f(z+m)=f(\underbrace{z+m}+i)=\cdots=f(\underbrace{z+m}+in)$$

showing that f is bounded in $\mathbb{C}$ as the behavior of $f(z)$ over $\mathbb{C}$ is completely characterized by its behavior on the compact set S. Therefore, by Liouville's theorem, f is a constant. More generally, a simple modification of this discussion gives the following result which answers Problem 6.63.

6.64. Theorem. *If $f:\mathbb{C}\to\mathbb{C}$ is analytic and*

$$f(z)=f(z+z_1)=f(z+z_2) \ \text{ for all } z\in\mathbb{C},$$

where z_1 and z_2 are the two non-zero complex numbers such that $z_1/z_2\notin\mathbb{R}$, then f is constant.

Proof. Since z_1/z_2 is not real, each $z \in \mathbb{C}$ can be written in the form

$$z = \lambda_1 z_1 + \lambda_2 z_2$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. But λ_1 and λ_2 may be written as $\lambda_1 = t_1 + n_1$ and $\lambda_2 = t_2 + n_2$, that is

$$z = (t_1 + n_1)z_1 + (t_2 + n_2)z_2 = (t_1 z_1 + t_2 z_2) + (n_1 z_1 + n_2 z_2)$$

for some integers n_1, n_2 and some $0 \leq t_1, t_2 < 1$. Obviously if z_1 and z_2 are the periods of f, so is $m_1 z_1 + m_2 z_2$ for any integers m_1 and m_2 and hence, we must have $f(z) = f(t_1 z_1 + t_2 z_2)$. Thus, the behavior of f is now entirely characterized by its behavior on the parallelogram

$$\{t_1 z_1 + t_2 z_2 : 0 \leq t_1, t_2 < 1\}.$$

From the analyticity of f, it follows that $|f|$ is continuous on the closed parallelogram $D = \{t_1 z_1 + t_2 z_2 : 0 \leq t_1, t_2 \leq 1\}$ and so, f is bounded for all $z \in D$. Consequently, f is bounded on $\mathbb{C}$. We then conclude, by Liouville's theorem, that f is constant. ∎

6.6 Fundamental Theorem of Algebra

Does every analytic function have a zero in $\mathbb{C}$? Clearly, the answer is no as the familiar exponential function e^z shows. Thus, a transcendental entire function may have no zeros in $\mathbb{C}$. Does every polynomial in $x \in \mathbb{R}$ with real coefficients have a zero in $\mathbb{R}$? Again the answer is no as the equation $x^2 + 1 = 0$ has no zeros in $\mathbb{R}$. Does every polynomial in $x \in \mathbb{R}$ with rational coefficients have a rational zero? Observe that the equation $x^2 - 3 = 0$ has no rational zeros. How about a non-constant polynomial with complex coefficients? The answer to this question is given by the fundamental theorem of algebra.

Liouville's theorem can be used to provide a natural and short proof for the fundamental theorem of algebra which asserts that *every non-constant polynomial with complex coefficients has at least one complex zero and hence has exactly n zeros.* In fact, from the observations made below Liouville's theorem, it is straightforward to see that if p is a polynomial of degree $n \geq 1$, then $p(z) = 0$ has a zero in $\mathbb{C}$; because if it did not have, then the function $1/p(z)$ would be bounded (how?) and analytic in $\mathbb{C}$ and would therefore be constant, which is a contradiction to the hypothesis. It remains to prove that if p does not have any zeros, then it is bounded in $\mathbb{C}$. To do this, without loss of generality we can suppose that the polynomial (with complex coefficients) has the form

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \quad (a_0 \neq 0,\, n \geq 1).$$

Then intuitively, for large z, we can expect that $p(z)$ should behave like z^n, since the largest power dominates the other ones. Indeed, for $|z| \geq 1$ (so that $|z|^n \geq |z|^{n-1} \geq \cdots \geq |z|$), we have

$$\begin{aligned} |p(z)| &= \left| z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + 1 \right) \right| \\ &\geq |z|^n \left\{ 1 - \left| \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right| \right\}, \quad \text{by the triangle inequality,} \\ &\geq |z|^n \left\{ 1 - \left(\frac{|a_0|}{|z|^n} + \cdots + \frac{|a_{n-1}|}{|z|} \right) \right\} \\ &\geq |z|^n \left(1 - \frac{1}{|z|}(|a_0| + \cdots + |a_{n-1}|) \right). \end{aligned}$$

Hence, for sufficiently large $|z|$ (e.g. $|z| = R \geq R_0 = \max\{1, 2(|a_0| + \cdots + |a_{n-1}|)\}$), we have

$$|p(z)| \geq \frac{|z|^n}{2}. \tag{6.65}$$

Then, for $|z| \geq R$,

$$\left| \frac{1}{p(z)} \right| \leq \frac{2}{|z|^n} \leq \frac{2}{R^n}.$$

On the compact set $\overline{\Delta}_R = \{z \in \mathbb{C} : |z| \leq R\}$, the function $1/p(z)$, being continuous on $\overline{\Delta}_R$, is bounded in the disk by some $M = \max_{|z|=R} |1/p(z)|$. Therefore, $|1/p(z)|$ is bounded above for all $z \in \mathbb{C}$ by $\max\{M, 2R^{-n}\}$. Thus, $1/p(z)$ is a bounded entire function and hence, must be constant which is absurd. Therefore, $p(z) = 0$ has a zero.

Here is a precise formulation of the fundamental theorem of algebra.

6.66. Theorem. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a non-constant polynomial of degree $n \geq 1$ with complex coefficients. Then, p has n zeros in $\mathbb{C}$. That is, there exist n complex numbers $z_1, z_2, \ldots, z_n$, not necessarily distinct, such that $p(z) = a_n \prod_{k=1}^{n} (z - z_k)$.*

Proof. If $a \in \mathbb{C}$, then by the 'division algorithm' there is a polynomial q of degree $n-1$ such that $p(z) = (z-a)q(z) + R$, where R is a constant. Clearly,

$$R = 0 \iff p(a) = 0 \iff z - a \text{ is a factor of } p(z).$$

Since there exists a z_1 such that $p(z_1) = 0$, $z - z_1$ is a factor of $p(z)$ with no remainder term. By the 'division algorithm' there is a polynomial p_{n-1} of degree $n-1$ such that $p(z) = (z - z_1)p_{n-1}(z)$, because

$$\begin{aligned} p(z) - p(z_1) &= a_1(z - z_1) + \cdots + a_{n-1}(z^{n-1} - z_1^{n-1}) + (z^n - z_1^n) \\ &= (z - z_1)p_{n-1}(z). \end{aligned}$$

This shows that p has a linear factor $z-z_1$. Thus, if $n>1$, then by applying the same principle we conclude that there is another complex number, say z_2, such that $p_{n-1}(z_2)=0$ and so p_{n-1} has a linear factor $z-z_2$. Proceeding in this manner, we can express p uniquely as a product of linear factors:

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

where $z_1, z_2, \ldots, z_n$ are (not necessarily distinct) the zeros of $p(z)$. ∎

Observe that there is no analogue of the fundamental theorem of algebra in the case of real numbers. This can be easily seen by considering the quadratic polynomial $p(x) = 1 + x^2$ which has no real zeros. The following result is referred to as an abbreviated statement of the fundamental theorem of algebra.

6.67. Corollary. *Every polynomial $p(z)$ of positive degree omits no values, i.e. $p(\mathbb{C}) = \mathbb{C}$. Each $w \in \mathbb{C}$ is the image of exactly n points in $\mathbb{C}$.*

Proof. If $p(z)$ is a polynomial of degree n, then $q(z) = p(z) - a$ is also a polynomial of degree n for each fixed $a \in \mathbb{C}$. By Theorem 6.66, $p(z)$ has n-zeros. In other words, for each $a \in \mathbb{C}$ there are n points $z_1, z_2, \ldots, z_n$ such that $p(z_j) - a = 0$ for $j = 1, 2, \ldots, n$. Thus, $p(\mathbb{C}) = \mathbb{C}$. ∎

6.7 Zeros of certain Polynomials

We start with a simple result before we move on to a discussion on the location of the zeros of certain polynomials.

6.68. Theorem. *If a polynomial $p(z)$ with real coefficients has a zero at α such that $\mathrm{Im}\,(\alpha) \neq 0$, then the complex conjugate $\overline{\alpha}$ is also a zero of $p(z)$. Indeed, if α is a zero of order k then $\overline{\alpha}$ is also a zero with order k.*

Proof. Set $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n$ are all real and $a_n \neq 0$. Since α is a zero of $p(z)$, we have $p(\alpha) = 0$. Taking conjugation on both sides, we have $p(\overline{\alpha}) = 0$. Hence, $\overline{\alpha}$ is also a zero of the polynomial $p(z)$. To prove the second assertion we note that if $p(z)$ has a zero at α of order k, then

$$p(\alpha) = p'(\alpha) = \cdots = p^{(k-1)}(\alpha) = 0 \quad \text{and} \quad p^{(k)}(\alpha) \neq 0$$

which would then imply that

$$p(\overline{\alpha}) = p'(\overline{\alpha}) = \cdots = p^{(k-1)}(\overline{\alpha}) = 0 \quad \text{and} \quad p^{(k)}(\overline{\alpha}) \neq 0.$$

Thus, complex zeros occur as conjugate pairs with the same multiplicity. ∎

The *convex hull* of a set $D \subset \mathbb{C}$ is the intersection of all the convex sets containing D. The *closed convex hull* of D is the intersection of all the closed convex sets containing D. In fact, it can be seen that the convex hull of points $z_1, z_2, \dots, z_n \in \mathbb{C}$ is the set of all linear combinations $z = \sum_{j=1}^n \lambda_j z_j$ with $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^n \lambda_j = 1$. The concept of convex hull helps us to discuss the location of the zeros of certain polynomials.

6.69. Theorem. **(Gauss's Theorem)** *Suppose that $p(z)$ is a polynomial of degree $n \geq 1$. Then, every zero of $p'(z)$ lies in the convex hull of the set of zeros of $p(z)$.*

Proof. Let $p(z)$ be a polynomial of the form

$$p(z) = \prod_{k=1}^n (z - z_k),$$

where $z_1, z_2, \dots, z_n$ are (not necessarily distinct) the zeros of $p(z)$. Thus, by logarithmic differentiation, it follows from the above representation that for $z \neq z_k$

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^n \frac{1}{z - z_k} = \sum_{k=1}^n \frac{\overline{(z - z_k)}}{|z - z_k|^2} \tag{6.70}$$

so that

$$\overline{\left(\frac{p'(z)}{p(z)}\right)} = \sum_{k=1}^n \left(\frac{1}{|z - z_k|^2}\right)(z - z_k).$$

If $c \in \mathbb{C}$ is such that $p(c) \neq 0$ and $p'(c) = 0$, then the above equation becomes

$$c = \frac{\sum_{k=1}^n z_k |c - z_k|^{-2}}{\sum_{k=1}^n |c - z_k|^{-2}}.$$

Hence c is of the form $c = \sum_{k=1}^n \lambda_k z_k$, where

$$\lambda_j = \frac{|c - z_j|^{-2}}{\sum_{k=1}^n |c - z_k|^{-2}}, \quad j = 1, 2, \dots, n.$$

This shows that if $z_1, z_2, \dots, z_n$ are the zeros of $p(z)$, then for every zero c of $p'(z)$ there are non-negative numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$c = \sum_{k=1}^n \lambda_k z_k, \quad \text{with} \quad \sum_{k=1}^n \lambda_k = 1.$$

The above construction uses the fact that $p(c) \neq 0$. If $p(c) = 0 = p'(c)$, then we simply take $\lambda_1 = 1$ and $c = 1 \cdot c$. ∎

As an example to Theorem 6.69, we conclude that if all the zeros of a polynomial $p(z)$ have negative real parts, then all the zeros of $p'(z)$ have negative real parts.

6.71. Alternate proof of Gauss's Theorem. Recall that every half-plane H can be defined by the inequality (see 5.42)

$$\operatorname{Im}\left(\frac{z-a}{b}\right) > 0$$

for some complex constants a and b ($b \neq 0$). Consider the polynomial $p(z) = \prod_{k=1}^{n}(z - z_k)$. Assume that $z_k \in H$ for each $k = 1, 2, \ldots, n$. Then,

$$\operatorname{Im}\left(\frac{z_k-a}{b}\right) > 0.$$

If $c \in \mathbb{C}\backslash H$, then $p(c) \neq 0$. Therefore, Im $((c-a)/b) \leq 0$ and

$$\operatorname{Im}\left(\frac{c-z_k}{b}\right) = \operatorname{Im}\left(\frac{c-a}{b}\right) - \operatorname{Im}\left(\frac{z_k-a}{b}\right) < 0$$

which implies that Im $(b/(c-z_k)) > 0$ for each $k = 1, 2, \ldots, n$. Finally, by (6.70), we see that

$$\operatorname{Im}\left(b\frac{p'(c)}{p(c)}\right) = \sum_{k=1}^{n} \operatorname{Im}\left(\frac{b}{c-z_k}\right) > 0 \quad \text{for } c \notin H$$

and consequently,

$$\frac{bp'(c)}{p(c)} \neq 0 \ \text{ or } p'(c) \neq 0 \text{ whenever } c \notin H.$$

We complete the proof of Gauss's theorem.

As an immediate application of Gauss's theorem, we have

6.72. Theorem. (**Lucas's Theorem**) *If all the zeros of a polynomial lie in a half-plane, then the zeros of its derivative also lie in the same half-plane.*

If we apply Lucas theorem with reference to half-planes determined by each side of a convex polygon, we obtain the following

6.73. Corollary. *If all the zeros of a polynomial lie in the smallest convex polygon, then the zeros of its derivative also lie in the same polygon.*

The first equality in (6.70) immediately yields the following result. Later we shall see that this is a consequence of a general result.

6.74. Theorem. *If N is the number of zeros (counted according to multiplicity) of the polynomial $p(z)$ in $\Delta(a; R)$, then*

$$\int_{|z-a|=R} \frac{p'(z)}{p(z)}\, dz = 2\pi i N.$$

6.8 Exercises

6.75. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) There exists an analytic function f on $\overline{\Delta}$, $f(0) = 1 + i$, $|f(z)| > 2$ for $|z| = 1$, and having a zero in Δ.

(b) For each $n \in \mathbb{N}$, one has $\max_{|z|\leq r} |z^n + b| = r^n + |b|$ and the maximum is attained at $re^{i(\operatorname{Arg} b + 2k\pi)/n}$, $k \in \mathbb{Z}$.

(c) Let $f(z) = z^m/(z^n + 2p)$, where $m, n \in \mathbb{N}$ are fixed, and p is real such that $p > 1/2$. Then, $\max_{|z|\leq 1} |f(z)| = 1/(2p - 1)$.

(d) If $\Omega = \{z = x + iy : 0 \leq x, y \leq 1\}$, then $\max_{z\in\Omega} |z^2 - 2z| = \sqrt{5}$.

(e) Suppose that $f \in \mathcal{H}(\overline{\Delta}_R)$ and satisfies the conditions $|f(z)| \leq 2$ on $\partial\Delta_R$ and $f(0) = \sqrt{3} + i$. Then, f is a constant on $\overline{\Delta}_R$.

(f) Suppose that f is an entire function and has zeros at $z = \pm 2i$. If $M = \max_{|z|=3} |f(z)|$, then $|f(z)| \leq (M/5)|z^2 + 4|$ for $|z| < 3$.

(g) Let f be an entire function and has n zeros at $\omega_k = e^{2k\pi i/n}$ $(k = 0, 1, 2, \dots, n-1)$, the n-th roots of unity. If $M = \max_{|z|=3} |f(z)|$, then $|f(z)| \leq M(3^n - 1)^{-1}|z^n - 1|$ for $|z| < 3$.

(h) If $f \in \mathcal{H}(\overline{\Delta})$ such that f has a zero of order n at $z = 0$, and $M = \max_{|z|=1} |f(z)|$, then $|f(z)| \leq M|z|^n$ for $|z| \leq 1$.

(i) If $w = \varphi_a(z)$ is a Möbius map of Δ conformally onto itself, $f \in \mathcal{H}(\Delta)$ and $g(w) = f(z)$, then $(1 - |w|^2)\,|g'(w)| = (1 - |z|^2)\,|f'(z)|$.
Note: What happens when $g(w) = w$?

(j) There exists an analytic function f of Δ onto itself such that $f(0) = 1/2$, $f(1/2) = 1/3$, and $f(1/3) = 1/4$.

(k) There exists an analytic function $f : \Delta \to \Delta$ such that $f(1/2) = 0$ and $|f'(1/2)| \leq 4/3$.

(l) There exists an analytic function $f : \Delta \to \Delta$ such that $f(0) = 1/2$ and $f'(0) = 3/4$.

(m) There exists no analytic function $f : \Delta \to \overline{\Delta}$ such that $f(1/2) = 3/4$ and $f'(1/2) = 3/4$.

(n) If f is entire and $|f'(z)| \leq |z|$ for all z, then f is of the form $a + bz^2$ with $|b| \leq 1/2$.

(o) If $|a_i| \leq 1$, $i = 1, 2, \dots, n-1$, then each zero of the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_2z^2 + a_1z + 1$ lies in the annulus $D = \{z : 1/2 < |z| < 2\}$.

(p) Each polynomial of the form $p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n$ satisfies the inequality $\sup\{|p(z)| : |z| \leq 1\} \geq 1$.

(q) If f is entire and $\operatorname{Re} f(z)$ is bounded as $|z| \to \infty$, then f is constant.

(r) Suppose f is entire which takes real z into real and purely imaginary into purely imaginary. Then, f is an odd function.

(s) If f is entire such that f omits a non-empty disk, then f is constant.

(t) An entire function f whose imaginary part is the square of the real part is constant in $\mathbb{C}$.

(u) If $k \neq 1$ is a fixed constant and f is entire such that $f(z) = f(kz)$ for all $z \in \mathbb{C}$, then f is constant on $\mathbb{C}$.

(v) An entire function f such that $f(z) \neq 0$ in $\mathbb{C}$ and $\lim_{z\to\infty} f(z) \neq 0$ is necessarily a constant.

(w) An entire function f such that $|f'(z)| \leq 2|f(z)|$ must be of the form be^{az} for some constants a and b with $|a| \leq 2$.

(x) If $f = u + iv$ is entire and $u^2 \leq v^2 + 2004$ on $\mathbb{C}$, then f is a constant.

(y) If $u(z) = u(x, y)$ is harmonic in $\mathbb{C}$ satisfying the condition $u(z) \leq a|\ln|z|\,| + b$ for some positive constants a, b and for all $z \in \mathbb{C}$, then show that u is a constant.

(z) If $u(z) = u(x, y)$ is harmonic in $\mathbb{C}$ such that $u(z) \leq |z|^n$ for some $n \in \mathbb{N}$ and for all $z \in \mathbb{C}$, then $u(z)$ is a polynomial in x and y.

6.76. Define $f(z) = 1 - z$ for $|z| \leq 1$. Show that $|f(z)|$ attains its maximum value when $z = -1$. If $f(z)$ is replaced by $g(z) = (1 - z)^2$ or $1 - z^2$, does the same conclusion hold?

6.77. If $f \in \mathcal{H}(\Delta)$ is such that $|f(z)| < 1$ for $z \in \Delta$ and f fixes two distinct points of Δ, then show that f is the identity function.

6.78. Does there exist an analytic function $f : \Delta \to \overline{\Delta}$ with $f(0) = -1/2$ and $f'(0) = 3/4$? Either find such an f or state why there does not exist such a function f. Answer the same question when $f(0) = \pm 1/2$ and $f'(0) = 4/5$.

6.79. Suppose that f is analytic and $\operatorname{Re} f(z) < 0$ in Δ. Find an estimate for $|f'(0)|$.

6.80. If $f \in \mathcal{H}(\Delta)$, $f(0) = 1$ and $|\operatorname{Re} f(z)| < 1$ for $z \in \Delta$, then show that $|f'(0)| \leq 4/\pi$.

6.81. If $f \in \mathcal{H}(\Delta)$, $f(0) = 0$ and if there exists a constant $\alpha > 0$ such that $\operatorname{Re} f(z) < \alpha$ for all $z \in \Delta$, then show that $|f(z)| \leq 2\alpha|z|/(1 - |z|)$ for $z \in \Delta$ and $|f'(0)| \leq 2\alpha$.

6.82. Suppose that f is analytic on Δ_R with $|f(z)| \le M < \infty$ for all $z \in \Delta_R$, and $f(z_0) = w_0$ for some z_0 with $|z_0| < R$. Show that

$$M\left|\frac{f(z) - w_0}{M^2 - \overline{w}_0 f(z)}\right| \le R\frac{|z - z_0|}{|R^2 - \overline{z}_0 z|} \quad \text{for} \quad |z| < R.$$

Interpret the case $z_0 = 0 = w_0$ geometrically and show that in this case the equality is achieved for some $\zeta_0 \in \Delta_R \setminus \{0\}$ iff f is of the form $f(z) = Me^{i\theta}z/R$, where θ is real.

6.83. Suppose f is analytic and bounded by M for $z \in \Delta$ and has zeros at the points $z_1, \dots, z_N \in \Delta$. Prove that $|f(z)| \le M\prod_{j=1}^{N}\left|\frac{z-z_j}{1-\overline{z}_j z}\right|$ for all $z \in \Delta$. Is this an improvement over the hypothesized inequality $|f(z)| \le M$? What can you say about f if equality holds for some $z \notin \{z_1, z_2, \dots, z_N\}$?

6.84. If $f : \Delta \to \Delta$ is analytic such that $f(0) = f(1/3) = f(-1/3)$, then show that $|f(1/4)| \le 7/572$. Show also that the bound $7/572$ cannot be made smaller.

6.85. Prove or disprove the following: there is no bianalytic mapping of the right half-plane $U = \{z : \operatorname{Re} z > 0\}$ onto the whole complex plane.

6.86. Let f be entire such that $|f(z)| \le e^{\operatorname{Re} z}$ for $z \in \mathbb{C}$. What can you say about f?

6.87. Find the set of all entire functions f such that $|f(z)| \le |z|^{5/2} + |z|^{9/2}$ for $z \in \Delta$. Justify your answer with a proof.

6.88. If $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ $(n \ge 1)$, then show that there exists a real $R > 0$ such that $2^{-1}|z|^n \le |p(z)| \le 2|z|^n$ for $|z| \ge R$.

6.89. If f is an entire function which for some real numbers α and β satisfies $\operatorname{Re} f(z) \le \beta|z|^\alpha$ for all z with sufficiently large $|z|$, then f is a polynomial of degree not greater than α (see also Exercise 10.35).

6.90. In each case given below, determine whether or not there exists a non-constant entire function $f(z)$ satisfying the following conditions. If there is, give an example. If not explain why not.

(i) $f(0) = e^{i\alpha}$ and $|f(z)| = 1/2$ for all $z \in \partial\Delta$
(ii) $f(e^{i\alpha}) = 3$ and $|f(z)| = 1$ for all z with $|z| = 3$
(iii) $f(0) = 1$, $f(i) = 0$, and $|f(z)| \le 10$ for all $z \in \mathbb{C}$
(iv) $f(0) = 4 - 3i$ and $|f(z)| \le 5$ for all $z \in \Delta$
(v) $f(z) = 0$ for all $z = n\pi$, $n \in \mathbb{Z}$.

Chapter 7

Classification of Singularities

Consider the functions

$$\frac{1}{x^2},\ \ x\sin\frac{1}{x},\ \ \exp(-1/x^3),\ \ \frac{1}{x(x^2+2)}.$$

Then we see that the point $x = 0$ is a singular point for each of these functions in the sense that the function is defined in a deleted neighborhood of 0. The problem of classifying singularities is not satisfactory for functions defined only on $\mathbb{R}$. On the other hand the situation is quite different for functions defined on domains in $\mathbb{C}$. In Section 7.1, we start our discussion on isolated and non-isolated singularities and classify an isolated singularity as a removable singularity, or a pole, or an essential singularity. In Section 7.2, we discuss removable singularities and present Riemann's removable singularity theorem for characterizing whether an isolated singularity is removable. Section 7.3 is devoted to a discussion on poles. In Section 7.4, we show that isolated singularities can be classified in a simple way using Laurent's series. In Section 7.5, we introduce the notion of an isolated singularity at ∞. Analytic functions with only poles as singularities play a prominent role in function theory with a special name, meromorphic functions. In Section 7.6, we discuss some aspects of meromorphic functions through a number of examples and characterizations. As a consequence, we present an analog of Liouville's theorem for meromorphic functions. In Section 7.7, we discuss the erratic behavior of functions near an essential singularity via Casorati-Weierstrass theorem which provides a basis for understanding the importance of Picard's little theorem.

7.1 Isolated and Non-isolated Singularities

A point $z = a$ is called a *regular point* for a complex-valued function f if f is analytic at a. Point a is called *a singular point* or *a singularity*, of f, if f is not analytic at a but every neighborhood of a contains at

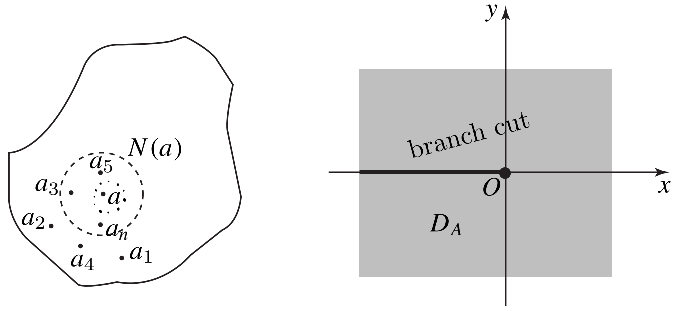


Figure 7.1: Description for a non-isolated singularity at a.

least one point at which f is analytic. A singular point a is said to be *an isolated singular point* or *an isolated singularity* of f if f is analytic in some deleted neighborhood of a. Otherwise, we say a is a *non-isolated singular point* of f (see Figures 7.1 and 7.2). Equivalently, we say that a point a is a non-isolated singularity of f iff a is a singularity and every deleted neighborhood of a contains at least one singularity of f. For example, z^{-1} has an isolated singular point at $z = 0$ while the function $1/\sin(\pi z)$ has isolated singularities at every integer point n, $n \in \mathbb{Z}$. On the other hand, every point on the negative real axis (including the point $z = 0$) is a non-isolated singularity of $\operatorname{Log} z$, the principal branch of the logarithm function. How about the singularities of $\sqrt{z}$, the principal square root function?

Note that the concept of singularities of a function f very much depends upon the domain of the function f. For instance, $f(z) = z^{-2}$ on $\mathbb{C}\backslash\{0\}$ has an isolated singularity at $z = 0$ whereas the restriction $g(z) = z^{-2}$ on $\mathbb{C}\backslash\Delta$ does not have a singularity at $z = 0$. However, this ambiguity will be removed during the discussion on analytic continuation.

Entire functions have no singular points but can have zeros, for example $f(z) = z^n e^z$, $n \in \mathbb{N}$. Rational functions $p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, have isolated singularities at points where $q(z) = 0$. For example, the rational function

$$\frac{z+1}{z^2 - 3z} \quad \text{on } \mathbb{C}\backslash\{0, 3\}$$

has isolated singularities at $0, 3$.

7.1. Example. What are the singular points of each of the functions

$$\overline{z},\ \operatorname{Im} z,\ \operatorname{Re} z,\ z\operatorname{Im} z,\ z\operatorname{Re} z,\ |z|^2?$$

Observe that each of the functions listed here is nowhere analytic. This does not mean that every point of $\mathbb{C}$ is a singularity. In fact, as there exists no neighborhood (about any point of $\mathbb{C}$) which contains a point at which

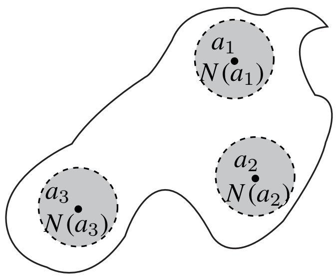


Figure 7.2: Description for isolated Singularities at a_1, a_2 and a_3.

the function is analytic, none of the functions listed above has singularities in $\mathbb{C}$. ●

7.2. Example. Let us discuss the singularities of $g(z) = 1/f(z)$, where $f(z) = \sin(1/z)$. So the discussion of singularities of g follows immediately from the zeros of $\sin(1/z)$. Thus, the singularities of g are at points $z_n = 1/n\pi$ for $n \in \mathbb{Z}$ and at their limit point $z = 0$. This shows g has isolated singularities at $z_n = 1/n\pi$ for $n \in \mathbb{Z}$ and a non-isolated singularity at the limit point $z = 0$.

Similar discussion shows that $\phi(z) = 1/\cos(1/z)$ has isolated singularities at the points $z_n = 2((2n+1)\pi)^{-1}, n \in \mathbb{Z}$. Note that, every neighborhood N of $z = 0$ contains singular points different from $z = 0$. So, $z = 0$ is a non-isolated singularity of $\phi(z)$. In both the cases, for each z_n, there are neighborhoods (say circles of sufficiently small radius δ) which contain no other singularity. This means that these singularities are isolated. ●

Let us now look at the following functions defined for $z \in \mathbb{C}\setminus\{0\}$:

$$f_1(z) = \frac{\sin z}{z}, \quad f_2(z) = \frac{1}{z^n} \ (n \in \mathbb{N}), \quad \text{and} \quad f_3(z) = e^{1/z}.$$

Clearly, each of these functions is analytic in $\mathbb{C}\setminus\{0\}$. Is it possible to define each of these functions at the origin so that the resulting function in each case becomes continuous at the origin? We observe the following:

(1) As $\lim_{z\to 0} f_1(z) = 1$, there exists an entire function $F_1(z)$ such that $F_1(0) = 1$ and $F_1(z) = f_1(z)$ for $z \in \mathbb{C}\setminus\{0\}$.

(2) Since $|f_2(z)|$ is large when $|z|$ is small, $f_2(z)$ is *unbounded* near 0. So, there is no way we can define $f_2(z)$ at 0 so that the resulting function becomes continuous at the origin. Also, $\lim_{z\to 0} f_2(z) = \infty$,

$$\lim_{z\to 0} z^n f_2(z) = 1 \quad \text{and} \quad \lim_{z\to 0} z^m f_2(z) = 0 \quad \text{for all } m > n.$$

(3) Finally,

(a) if $z = x > 0$, then for x near 0, $|f_3(z)| = e^{1/x}$ is large
(b) if $z = -x > 0$, then for x near 0, $|f_3(z)| = e^{-1/x}$ is small
(c) if $z = iy \neq 0$, then $|f_3(z)| = |e^{-i/y}| = 1$ for each $y \neq 0$.

Thus, $\lim_{z\to 0} f_3(z)$ fails to exist (including the possibility of the limit being infinity). In particular, there exists no entire function $F_3(z)$ such that $F_3(z) = f_3(z)$ for $z \in \mathbb{C}\backslash\{0\}$. Moreover, there exists no $n \in \mathbb{N}$ such that the limit $\lim_{z\to 0} z^n f_3(z)$ exists. Indeed, with $x > 0$ and a fixed $n \in \mathbb{N}$, we have

$$\lim_{z=x\to 0} z^n e^{1/z} = \lim_{1/x=t\to\infty} \frac{e^t}{t^n} = \infty$$

and

$$\lim_{z=-x\to 0} z^n e^{1/z} = \lim_{\substack{t\to\infty \\ t>0}} \frac{e^{-t}}{(-t)^n} = (-1)^n \lim_{\substack{t\to\infty \\ t>0}} \frac{e^{-t}}{t^n} = 0$$

so that neither $f_3(z)$ nor $z^n f_3(z)$ is bounded near 0 for any $n \in \mathbb{N}$.

The examples illustrated above motivates one to classify isolated singularities of f into three categories. Suppose that f has an isolated singularity at a point a. Then exactly one of the following holds:

(a) $f(z)$ is bounded near a
(b) $f(z)$ is unbounded near a but $(z-a)^n f(z)$ is bounded near a for some $n \in \mathbb{N}$
(c) there exists no $n \in \mathbb{N}$ such that $(z-a)^n f(z)$ is bounded near a, i.e. neither (a) nor (b) holds.

Thus, the above three situations are classified respectively as follows:

(1) *Removable singularity*, which upon closer examination reveals that this is not actually considered to be a singular point at all. More precisely, an isolated singularity $z = a$ of f is said to be a removable singularity for f if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$.
(2) *Pole* arises from the reciprocal of an analytic function with zero. More precisely, an isolated singularity $z = a$ of f is said to be a pole for f if $\lim_{z\to a} f(z) = \infty$. (Note that $f(z)$ is defined if z is near enough to a and $z \neq a$.)
(3) *Essential singularity*, which is neither a removable singularity nor a pole. More precisely, an isolated singularity $z = a$ of f is said to be an essential singularity for f if $\lim_{z\to a} f(z)$ does not exists in $\mathbb{C}_\infty$. The behavior of a function in a neighborhood of an essential singularity is described by the *Casorati-Weierstrass theorem* (Theorem 7.40).

We discuss each of these cases in detail, and prove a few relevant theorems in each case.

7.2 Removable Singularities

An isolated singularity z_0 of f is called *removable* or that f has a *removable singularity* at z_0 if f can be defined at z_0 so that it becomes analytic at z_0.

7.3. Example. Consider the following functions:

$$f_1(z) = \sin z \ \ (z \neq 1), \qquad f_4(z) = \left(\frac{e^z - 1}{z}\right)^2 \ \ (z \neq 0)$$

$$f_2(z) = \frac{z}{e^z - 1} \ \ (z \neq 0), \qquad f_5(z) = \frac{2(1 - \cos z)}{z^2} \ \ (z \neq 0)$$

$$f_3(z) = \frac{\sin z}{z} \ \ (z \neq 0), \qquad f_6(z) = \frac{-\operatorname{Log}(1 - z)}{z} \ \ (z \neq 0).$$

All of these functions except f_1 have removable singularities at 0; and f_1 has a removable singularity at $z = 1$. These singularities can be removed by letting $f_1(1) = \sin 1$ and $f_j(0) = 1$ $(2 \leq j \leq 6)$, respectively. ●

If f is analytic on an open set D and $z_0 \in D$, then the function F defined by

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0}, \ \ z \in D \backslash \{z_0\}$$

is analytic on $D \backslash \{z_0\}$ and $\lim_{z \to z_0} F(z) = f'(z_0)$. Thus F has a removable singularity at z_0, which can be removed by letting $F(z_0) = f'(z_0)$.

7.4. Example. Let f be defined by

$$f(z) = \frac{z^2 + a^2}{z + ia}, \ \ z \neq -ia.$$

Clearly, $f \in \mathcal{H}(\mathbb{C} \backslash \{-ia\})$. For $z \neq -ia$, we have $f(z) = z - ia$ and therefore, $\lim\limits_{z \to -ia} f(z) = -2ai$. Thus, f has removable singularity at $z = -ia$. Now we set

$$g(z) = \begin{cases} \dfrac{z^2 + a^2}{z + ia} & \text{for } z \neq -ia \\ -2ai & \text{for } z = -ia. \end{cases}$$

Then, g becomes analytic everywhere including at $z = -ia$. ●

Suppose that f has a removable singularity at a point z_0, say. Then there is a function g analytic at z_0 such that $f(z) = g(z)$ for all z in a deleted neighborhood of z_0. In particular, f is bounded near z_0. The converse of this statement provides a useful criterion for determining whether an isolated singularity is removable.

7.5. Theorem. (Riemann's Removable Singularity Theorem) *If f has an isolated singularity at z_0, then $z = z_0$ is a removable singularity iff one of the following conditions holds:*

(i) *f is bounded in a deleted neighborhood of z_0,*

(ii) $\lim_{z\to z_0} f(z)$ *exists,*

(iii) $\lim_{z\to z_0} (z-z_0)f(z) = 0.$

Proof. Suppose that f has an isolated singularity at z_0. Then f is analytic in some deleted neighborhood of z_0, say on $0 < |z-z_0| < \delta$. For proving '(iii) implies z_0 is a removable singularity', we introduce

$$h(z) = \begin{cases} (z-z_0)f(z) & \text{for } 0 < |z-z_0| < \delta \\ 0 & \text{for } z = z_0. \end{cases}$$

By the hypothesis (iii), h is continuous at z_0. Since h, like f, is analytic in the deleted neighborhood $0 < |z-z_0| < \delta$, it follows that h is analytic at $z = z_0$ (see Corollary 4.88). This means h is analytic throughout the neighborhood $|z-z_0| < \delta$. For $z \neq z_0$, let g be defined by

$$g(z) = \frac{h(z)-h(z_0)}{z-z_0} = \frac{h(z)}{z-z_0}.$$

As $\lim_{z\to z_0} g(z)$ exists and equals $h'(z_0)$ and $g = f$ for $z \neq z_0$, we define $f(z_0) = h'(z_0)$. Thus, when (iii) holds, f can be extended to be analytic in $|z-z_0| < \delta$. This observation, by definition, shows that the singularity at $z = z_0$ is removable.

The remaining parts are deducible from (iii), as $\lim_{z\to z_0}(z-z_0)f(z) = 0$ holds under (i) or (ii). Indeed, (ii) implies (iii) is trivial.

Finally, it remains to show that (i) implies that f has a removable singularity. Now, we suppose that (i) holds. Then, $|f(z)| \leq M$ for $0 < |z-z_0| \leq r$ and for a small $r > 0$. The Laurent coefficient a_k gives

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^{k+1}} \right| \leq \frac{M}{r^k}$$

which approaches zero as $r \to 0$, when $k < 0$. We deduce that $a_k = 0$ when $-k \in \mathbb{N}$. Consequently, $\lim_{z\to z_0} f(z)$ exists and therefore, z_0 is a removable singularity of f. ∎

From the proof of Theorem 7.5, it follows that if f has an isolated singularity at $z = z_0$ and satisfies the condition

$$|f(z)| \leq \frac{M}{|z-z_0|^{5/6}}$$

in a neighborhood of $z = z_0$, then $z = z_0$ must be a removable singularity of f. Is it the case if we replace $5/6$ by an α with $\alpha < 1$? Also, Theorem 7.5

infers that there exists no function analytic at z_0 with $f(z) \sim (z - z_0)^{-\alpha}$ whenever $\alpha \in (0, 1)$.

Next we illustrate by an example that a result similar to Theorem 7.5 is not available for functions of a real variable. Consider $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by $f(x) = x^{-1/3}$. We see that $|xf(x)| = |x|^{2/3} \to 0$ as $x \to 0$ but $f(x)$ cannot be extended to be (real) differentiable near 0. Here is another example. Consider $g(x) = |x|$ for $x \neq 0$. Then $\lim_{x\to 0} g(x) = 0$ but $g(x)$ cannot be extended so that it is (real) differentiable at $x = 0$.

7.6. Example. Consider the function

$$f(z) = \cos(1/|z|) \quad \text{or} \quad \sin(1/|z|), \quad z \in \mathbb{C} \setminus \{0\}.$$

Then f is continuous (in fact real differentiable infinitely often) and bounded on $\mathbb{C}\setminus\{0\}$, yet f cannot be extended to be continuous on any neighborhood of the origin. Thus, the situation for (real-valued) C^∞-functions is different from that of the case of analytic functions. ●

7.7. Remark. Let us look at the statement (ii) of Theorem 7.5. Then the following situations occur:

(a) f may not be defined at z_0

(b) f may be defined at z_0 but $f(z_0)$ may not be equal to $\lim_{z\to z_0} f(z)$

(c) f may be defined at z_0 and $f(z_0)$ is equal to $\lim_{z\to z_0} f(z)$.

In the last case, f is not singular at z_0. Thus if f has a removable singularity at z_0 then either (a) or (b) holds, as discussed in the introduction. ●

7.8. Remark. If f and g are analytic and if both have a zero of order n at z_0, then we may write

$$f(z) = (z - z_0)^n f_0(z), \quad \text{and} \quad g(z) = (z - z_0)^n g_0(z)$$

where f_0 and g_0 are both analytic and non-zero at z_0, and hence

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f_0(z_0)}{g_0(z_0)}$$

exists. This means that $f(z)/g(z)$ has a removable singularity at z_0. For instance each of the functions f_j $(j = 1, 2)$, where

$$f_1(z) = \frac{e^z - 1}{z} \quad \text{and} \quad f_2(z) = \frac{z - \sin z}{z \sin z},$$

has a removable singularity at $z = 0$. ●

7.9. Example. Consider the function

$$f(z) = (1 - z^2)\csc(\pi z).$$

Then f is analytic on $\mathbb{C} \setminus \mathbb{Z}$, and so f can be expressed as a Laurent series about 0. Considering the location of the singularities of $f(z)$, we see that the Laurent series about 0 is valid for $0 < |z| < 1$. Clearly, f extends to be analytic both at $z = 1$ and $z = -1$. This observation shows that the largest open set on which the Laurent expansion for $f(z)$ about 0 converges is $0 < |z| < 2$.

Moreover, for functions such as $g(z) = \csc(\pi z)$, there exist infinitely many Laurent series about the origin. For the function g, the Laurent series about 0 is valid in each of the annuli

$$D_k = \{z \in \mathbb{C} : k < |z| < k + 1\} \quad (k = 0, 1, 2, \ldots). \qquad \bullet$$

7.3 Poles

We have seen that if f is bounded in a deleted neighborhood of an isolated singularity at z_0, then z_0 is a removable singularity of f. Thus if z_0 is not a removable singularity of f then, by Theorem 7.5, f is not bounded near z_0. We might then ask whether $(z - z_0)^n f(z)$ is bounded near z_0 for some $n \in \mathbb{N}$. In this case, we say that the point z_0 is a pole of f and the smallest positive integer n such that $(z - z_0)^n f(z)$ is bounded near z_0 is called the *order of the pole* at z_0. A pole of order one is called a *simple pole*. For example, the function $f(z) = z^8/(z - 1)^2$ has a double pole at $z = 1$.

7.10. Example. The function $f(z) = 1/(1 + e^{i\pi z})$ has simple poles at $z = 1 + 2k$ $(k \in \mathbb{Z})$ and $z = \infty$ is the limit point of these poles, since the condition $1 + e^{i\pi z} = 0$ implies that $e^{i\pi z} = e^{i\pi} e^{2ki\pi}$, $k \in \mathbb{Z}$. •

7.11. Example. Consider the function

$$f(z) = \frac{z\cos(\pi z/2a)}{(z - a)(z^2 + b^2)^7 \sin^5 z},$$

where a and b are distinct non-zero real numbers. Then we see that f has poles of orders: 7 at $\pm ib$, 4 at $z = 0$, 5 at $\pm k\pi$ $(k \in \mathbb{N})$ and a removable singularity at $z = a$. •

Observe that the condition $\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$ is equivalent to the condition that n is the smallest positive integer such that $(z - z_0)^n f(z)$ has a removable singularity at z_0. This expresses the following

7.12. Theorem. *If f is analytic in a deleted neighborhood of z_0, then f has a pole at z_0 iff there exists an n such that $(z - z_0)^n f(z)$ is* *bounded near z_0. More precisely, f has a pole of order n iff* $\lim_{z\to z_0}(z - z_0)^n f(z) \neq 0$, *and $(z - z_0)^n f(z)$ has a removable singularity at z_0, i.e.* $\lim_{z\to z_0}(z - z_0)^{n+1} f(z) = 0$.

7.13. Remark. Let f be analytic in a domain D and $z_0 \in D$. Recall that the function f has a *zero of order n* at z_0 iff $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \dots, n-1$ and $f^{(n)}(z_0) \neq 0$. From Taylor's expansion, it follows that f has a zero of order n iff

$$f(z) = (z - z_0)^n g(z)$$

where g is analytic at z_0 and $g(z_0) = f^{(n)}(z_0)/n! \neq 0$. Thus, we see that $f(z)$ has a zero of order n at z_0 iff $1/f(z)$ has a pole of order n at z_0. For instance let $f(z) = \sin(z^2)$. Then we see that $f(z)$ has a double zero at $z_0 = 0$ and simple zeros at $z_k = \sqrt{\pi} *\sqrt{k}$, $k = \pm 1, \pm 2, \dots$, because $f(z_k) = 0$ and $f'(z_k) = 2z_k \cos(z_k^2) \neq 0$ for $k \neq 0$, and $f(z_0) = f'(z_0) = 0$, $f''(z_0) \neq 0$ (Here the notation $*\sqrt{k}$ is defined as in item 1.3). Thus, $1/f(z)$ has a double pole at z_0 and simple poles at z_k, $k = \pm 1, \pm 2, \dots$. Note that Laurent's expansion for $1/f(z)$ in $0 < |z| < \sqrt{\pi}$ contains even powers of z only. ●

In general, the following theorem tells us how we may determine poles by their behavior in a deleted neighborhood.

7.14. Theorem. *Let N be a deleted neighborhood of z_0 such that f is analytic in N. If f has an isolated singularity at z_0, then z_0 is a pole of order n iff there are positive constants C_1 and C_2 such that*

$$C_2 \leq |(z - z_0)^n f(z)| \leq C_1$$

for some deleted neighborhood N_0 of z_0 such that $N_0 \subset N$.

Proof. Suppose that the inequalities hold. Then the right hand side of the above inequality shows that $(z - z_0)^n f(z)$ is bounded for points near z_0 so that

$$(z - z_0)^{n+1} f(z) \to 0 \quad \text{as} \quad z \to z_0$$

whereas the left inequality gives $|(z - z_0)^n f(z)| \geq C_2$ as $z \to z_0$. So, by Theorem 7.12, z_0 is a pole of order n for f.

Assume that f has a pole of order n at z_0. Then, $(z - z_0)^n f(z)$ is bounded for points near z_0. By the removability theorem, there exists a function g which is analytic at z_0 such that $(z - z_0)^n f(z) = g(z)$ in some deleted neighborhood N_0 of z_0. If g were such that $g(z) = (z - z_0)\widetilde{g}(z)$ with $\widetilde{g}$ analytic at z_0, then

$$\widetilde{g}(z) = (z - z_0)^{n-1} f(z)$$

in N_0 and is bounded near z_0. This contradicts the fact that z_0 is a pole of order n. This fact shows that $(z-z_0)^{n-1}f(z)$ is unbounded near z_0. That there are constants C_1 and C_2 satisfying the required inequalities is then a consequence of boundedness or unboundedness as the case may be. ∎

Note that Theorem 7.14 characterizes poles of f through the behavior of the values of f near z_0 in the following form:

7.15. Theorem. *If $f(z)$ has an isolated singularity at z_0, then $f(z)$ has a pole at z_0 if and only if* $\lim_{z\to z_0} f(z) = \infty$.

Proof. Suppose that f has a pole of order n at z_0. Then

$$f(z) = (z-z_0)^{-n}g(z),$$

where g is analytic and is non-zero at z_0. So, by the continuity of g at z_0 (see for example the proof of Theorem 2.10(c)), there is a neighborhood N of z_0 on which f is defined (except at z_0) and

$$|g(z)| \geq \frac{|g(z_0)|}{2} \quad \text{for} \quad z \in N.$$

Hence, $|f(z)| \geq |g(z_0)|\,|z-z_0|^{-n}/2$ for $z \in N\setminus\{z_0\}$; i.e. $\lim_{z\to z_0}|f(z)| = \infty$.

Conversely, suppose that f has an isolated singularity at z_0 such that $\lim_{z\to z_0} f(z) = \infty$. Then, for a given $\epsilon = 1$, there exists a $\delta > 0$ such that

$$|f(z)| > \epsilon = 1 \quad \text{for } 0 < |z-z_0| < \delta.$$

Consequently, the function $1/f(z)$ is analytic and bounded on the punctured disk $0 < |z-z_0| < \delta$ with $\lim_{z\to z_0} 1/f(z) = 0$. By Riemann's removability theorem, it follows that $1/f(z)$ has a removable singularity at z_0. Define

$$g(z) = \begin{cases} 1/f(z) & \text{for } 0 < |z-z_0| < \delta \\ 0 & \text{for } z = z_0. \end{cases}$$

Then, $g \in \mathcal{H}(\Delta(z_0;\delta))$. Clearly, g is not identically zero on $\Delta(z_0;\delta)$ but $g(z)$ has a zero at z_0. It follows that there exists an $n \in \mathbb{N}$ such that

$$g(z) = (z-z_0)^n \psi(z),$$

where $\psi \in \mathcal{H}(\Delta(z_0;\delta))$ and $\psi(z_0) \neq 0$. As $\psi(z_0) \neq 0$ and $g(z) \neq 0$ on $0 < |z-z_0| < \delta$, we have

$$f(z) = \frac{1}{g(z)} = \frac{\phi(z)}{(z-z_0)^n} \qquad \left(\phi(z) = \frac{1}{\psi(z)}\right)$$

where $\phi(z)$ is analytic at z_0. Hence, $\lim\limits_{z\to z_0} (z-z_0)^n f(z) = \phi(z_0) \neq 0$ showing that f has pole of order n at z_0. ∎

The following theorem is an equivalent formulation of Theorem 7.14 in the language of zeros—a useful characterization of zeros.

7.16. Theorem. *If f is analytic in a deleted neighborhood N of z_0, then z_0 is a zero of order n iff there are finite positive constants C_1 and C_2 such that $C_2 \le |(z-z_0)^{-n}f(z)| \le C_1$ for some deleted neighborhood N_0 of z_0 such that $N_0 \subset N$.*

7.17. Example. Let us now discuss the singularities of

$$f(z) = \frac{z-1-i}{z^2-(4+3i)z+(1+5i)}.$$

First we obtain that the poles of f, if any, are determined by

$$z^2-(4+3i)z+(1+5i)=0,$$

that is (with $t_j^2 = 3+4i$ for $j=1,2$), we have $2z = 4+3i+t_j = 4+3i\pm(2+i)$. Therefore, we write

$$f(z) = \frac{z-\alpha}{(z-\alpha)(z-\beta)}$$

with $\alpha = 1+i$ and $\beta = 3+2i$. Thus, f has a simple pole at $z=\beta$ and a removable singularity at $z=\alpha$. ●

In view of Theorems 7.15 and 7.5, we arrive at a characterization of isolated essential singularities.

7.18. Theorem. *If f has an isolated singularities at z_0, then $f(z)$ has an essential singularity at z_0 iff $\lim_{z\to z_0} f(z)$ fails to exist either as a finite value or as an infinite limit.*

7.4 Further Illustrations through Laurent's Series

An obvious tool in characterizing singularities is the Laurent series expansion of a given function about its isolated singularities. If f has an isolated singularity at z_0, then we have the unique representation

$$f(z) = \sum_{k=-\infty}^{-1} a_k(z-z_0)^k + \sum_{k=0}^{\infty} a_k(z-z_0)^k,\ \ 0<|z-z_0|<r, \tag{7.19}$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{k+1}} \tag{7.20}$$

and C can be any circle with center at z_0 and radius less than r. If z_0 is the only singularity of $f(z)$ in $\mathbb{C}$, then (7.19) is valid with $r=\infty$. Note also that the integral over C is taken in the positive direction and has the same value on any positively oriented curve which encloses z_0 but no other singularity of f (see the Cauchy deformation theorem). Then, according to (7.20),

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz. \tag{7.21}$$

The coefficient a_{-1} of $(z-z_0)^{-1}$ in the Laurent expansion (7.19) of f about z_0, which is of special significance, is called the *residue* of f at z_0. We use the notation

$$a_{-1} = \text{Res}\,[f(z); z_0]$$

to denote the residue of f at z_0. Equation (7.21) provides a convenient way of evaluating certain integrals (see Chapters 8 and 9) of a function f around an isolated singularity, since its value is simply the product of $2\pi i$ and the coefficient of $(z-z_0)^{-1}$ in its Laurent expansion (7.19).

Note that, throughout the above discussion, we have supposed that f is defined for all z near z_0, but not necessarily at z_0 itself. Thus, the classification of the isolated singularity of f at z_0 depends only on the local behavior of f in a deleted neighborhood of z_0.

The first part in (7.19), i.e. the series with negative powers of $(z-z_0)$, is called *principal part* whereas the second part in (7.19), i.e. the series with non-negative powers of $(z-z_0)$, is called the *regular/holomorphic/analytic part*. It is the first part which plays an important role in deriving the character of the singularities of f at the isolated singularity z_0. The following three cases are mutually exclusive.

Case 1. No principal part. In this case (7.19) takes the form

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k, \quad 0 < |z-z_0| < r.$$

Thus if we define $f(z_0) = a_0 = \lim_{z\to z_0} f(z)$, then f becomes analytic at z_0 and so, on the entire disk $|z-z_0| < r$. This observation shows that z_0 is a removable singularity. Similarly if f has a removable singularity at z_0, then $a_k = 0$ for $k \leq -1$; for, since we may write $f(z) = g(z)$ for $z \neq z_0$, the Laurent series expansion for f must coincide with the Taylor series expansion for g near z_0. In other words, "*f has a removable singularity iff, in* (7.19), $a_{-k} = 0$ *for* $k \geq 1$." For example, consider the following functions:

$$\begin{aligned} f_1(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad (|z| > 0) \\ f_2(z) &= 1 - \frac{z^2}{2!} + \frac{z^3}{3!} - \cdots \quad (|z| > 0) \\ f_3(z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad (0 < |z| < 1). \end{aligned}$$

Then each of these functions have a removable singularity at $z = 0$, because f_j $(j = 1, 2, 3)$ can be defined at $z = 0$ in such a way that f_j $(j = 1, 2, 3)$ becomes analytic at $z = 0$.

7.22. Example. Let $f \in \mathcal{H}(\Delta \setminus \{0\})$ such that $|f(z)| \le \ln(1/|z|)$ for $z \in \Delta \setminus \{0\}$. Then, $z = 0$ is a removable singularity of f. Indeed, the Laurent coefficients a_n $(n \in \mathbb{Z})$ of f are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz$$

so that $|a_n| \le r^{-n} \ln(1/r)$. When $n < 0$, letting $r \to 0$ gives that $a_n = 0$, and when $n = 0$, letting $r \to 1^-$ implies that $a_0 = 0$. This observation shows that the Laurent series expansion of f about the origin does not have the principal part, and so $z = 0$ is a removable singularity of f. ●

Case 2. Finite principal part. In this case, according to Theorem 7.14, (7.19) becomes $f(z) = (z - z_0)^{-n} g(z)$ where g is analytic and non-zero at z_0. Hence if $g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$, then we may write

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k, \;\; b_{n+k} = a_k,$$

$(a_{-n} = b_0 = g(z_0) \ne 0)$. The uniqueness of $f(z)$ is obvious from the uniqueness of $g(z)$. The converse assertion is clear. In other words, "*f has a pole of order n iff, in* (7.19), *$a_{-n} \ne 0$, but $a_{-k} = 0$ for $k \ge n + 1$.*" Moreover, the discussion here is equivalent to

7.23. Theorem. *An isolated singularity at z_0 of $f(z)$ is a pole of order n iff $f(z) = (z - z_0)^{-n} g(z)$, where g is analytic at z_0 and $g(z_0) \ne 0$.*

Consider the following functions:

$$\begin{aligned} f_1(z) &= \frac{e^z}{(z-1)^4} \\ &= \frac{e}{(z-1)^4} + \frac{e}{(z-1)^3} + \frac{e}{2!(z-1)^2} + \cdots, \quad |z-1| > 0, \\ f_2(z) &= \frac{(1 - z^n)e^z}{z^m} = \frac{1}{z^m} + \cdots, \quad |z| > 0 \ (m > n). \end{aligned}$$

Then, it is clear that f_1 has a pole of order 4 at $z = 1$ and f_2 has a pole of order m at $z = 0$.

Case 3. Infinite principal part. The function f has an essential singularity at $z = z_0$ exactly when neither Case 1 nor Case 2 prevails. In other words, "*f has an essential singularity iff, in* (7.19), *$a_{-k} \ne 0$ for infinitely many $k \ge 1$.*"

In this case, $\lim_{z\to z_0} f(z)$ fails to exist (including the possibility of the limit being infinity) as we have seen with the function $f(z) = e^{1/z}$. Here $e^{1/z}$ has an essential singularity at $z = 0$. Similarly, it is easy to see that the function $f(z) = e^{z/(z-b)} = ee^{b/(z-b)}$ $(b \neq 0)$ has an essential singularity at $z = b$.

Consider the Laurent series

$$\sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}. \tag{7.24}$$

Observe that the first series converges for $|z| > 1$, while the second for $|z| < 2$. Then the combined series converges to $f(z)$ for $1 < |z| < 2$, where

$$f(z) = \frac{1/z}{1-1/z} + \frac{1/2}{1-z/2} = \frac{-1}{(z-2)(z-1)}.$$

Observe that the Laurent series (7.24) has an infinite number of negative powers of z. But, since the region of convergence of the given Laurent series does not include a deleted neighborhood of the origin, it would not be correct to conclude that the origin is an essential singularity of f. In fact, the limit function f has simple poles at $z = 1$ and $z = 2$. This remark is to caution the reader when dealing with a series of the form (7.24).

7.25. Example. Let $f(z) = (z-a)^{2004} \sin(1/(z-b))$. Then, we may rewrite $f(z)$ as

$$f(z) = (z-b+b-a)^{2004} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)!} \frac{1}{(z-b)^{2k-1}}$$

which implies that $z = b$ is clearly an essential singularity of f. Similarly, it is easy to see that each of the functions ze^{1/z^2}, $e^{-1/z}$, $\sin(1/z)$, and $\cos(1/z)$ has an essential singularity at $z = 0$. ●

7.5 Isolated Singularities at Infinity

In the earlier sections, we have discussed singularities of a function which lie in $\mathbb{C}$. In this section we shall be concerned with functions that have singularities at the point at infinity $-$ as it is sometimes useful to think about ∞ just like any other point in $\mathbb{C}$.

The role of the point at infinity is understood through the inversion $w = z^{-1}$ as it allows us to pass back and forth between the neighborhoods of ∞ and the neighborhoods of 0. This function is defined for every $z \neq 0$ and maps each point z in $\mathbb{C}$ except $z \neq 0$, into a point in the w-plane. For instance, the circle $|z| = R$ is mapped onto $|w| = 1/R$. If we assign the point at infinity in the extended w-plane to $z = 0$ and $w = 0$ to the point at infinity in the extended z-plane, then the inversion is one-to-one from the extended z-plane to the extended w-plane, i.e. $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$, a fact which was discussed in Chapter 5.

Let $f(z)$ be analytic for $|z| > R$, for some R with $0 \le R < \infty$. By putting $z = 1/w$ in $f(z)$, we obtain

$$F(w) = f(1/w). \tag{7.26}$$

Then, $F(w)$ is analytic in the deleted neighborhood $\{w : 0 < |w| < 1/R\}$ of zero. The nature of the singularity of $f(z)$ at $z = \infty$ (the point at infinity) is defined to be the same as that of $F(w)$ at $w = 0$. That is, $f(z)$ has an isolated singularity at ∞ iff $F(w) = f(1/w)$ has an isolated singularity at $w = 0$. In particular, "*an analytic function $f(z)$ on $|z| > R$ is said to have a removable singularity, a pole of order n, or an essential singularity at ∞ iff the function $f(1/z)$ has a removable singularity, a pole of order n, or an essential singularity at 0, respectively*".

For instance, the function f defined by $f(z) = z^n$ ($n \in \mathbb{N}$) has a pole of order n at $z = \infty$, since the corresponding F defined by

$$F(w) = f(1/w) = w^{-n}$$

has a pole of order n at $w = 0$. More generally, every polynomial of degree n has a pole of order n at ∞. Similarly, the function f defined by $f(z) = z^{-2} + z^m$ has a pole of order two at $z = 0$ and a pole of order m at infinity. On the other hand, the function f defined by $f(z) = e^z$ has an essential singularity at $z = \infty$, since F defined by $F(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$. In the same way, it is easy to verify that each of the functions e^{iz}, e^{-z}, $\sinh z$, $\cosh z$, $\sin z$, $\cos z$ has an essential singularity at $z = \infty$. Also, we see that every nonconstant and nonvanishing entire function on $\mathbb{C}$ necessarily has an isolated essential singularity at ∞.

Suppose that $f(z)$ has an isolated singularity at $z = \infty$ and R is the distance from the origin to the farthest singular point of $f(z)$. (If point at infinity is the only singularity, then R may be chosen as an arbitrary large positive number). Since we may write

$$f(z) = f(1/w) = F(w) \quad (w = 1/z,\ |z| > R),$$

$F(w)$ has an isolated singularity at $w = 0$ and so the nearest singularity of $F(w)$ from $w = 0$ is at a distance $1/R$. Therefore, $F(w)$ has a Laurent expansion about $w = 0$:

$$f(1/w) = F(w) = \sum_{n=-\infty}^{\infty} a_n w^n, \quad 0 < |w| < 1/R. \tag{7.27}$$

Thus,

$$f(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n = \sum_{n=-\infty}^{0} a_{-n} z^n + \sum_{n=1}^{\infty} a_{-n} z^n, \quad |z| > R.$$

Then the following cases arise.

Case 1. Suppose that, in (7.27), $a_{-n} = 0$ for all $n \geq 1$. Then, we define $F(0) = a_0$ so that the corresponding $F(w)$ is analytic at $w = 0$. Hence, $F(w)$ has a removable singularity at $w = 0$ iff, in (7.27), $a_{-n} = 0$ for all $n \geq 1$. That is, $f(z)$ has a removable singularity at $z = \infty$ iff its Laurent expansion about $z = \infty$ has the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad |z| > R;$$

that is, iff the Laurent expansion of $f(z)$ on $|z| > R$ has no positive power of z with nonzero coefficients.

Case 2. Recall that $F(w)$ has a pole of order k at $w = 0$ iff (7.27) takes the form

$$F(w) = \frac{a_{-k}}{w^k} + \cdots + \frac{a_{-1}}{w} + \sum_{n=0}^{\infty} a_n w^n \quad (a_{-k} \neq 0,\ 0 < |w| < 1/R).$$

Hence, $f(z)$ has a pole of order k at $z = \infty$ iff its Laurent expansion has the form

$$f(z) = a_{-k}z^k + \cdots + a_{-1}z + \sum_{n=0}^{\infty} a_n z^{-n} \quad (a_{-k} \neq 0,\ |z| > R);$$

that is iff the Laurent series of $f(z)$ for $|z| > R$ has only a finite number of positive powers of z with nonzero coefficients. Here we define the principal part of $f(z)$ at $z = \infty$ to be the polynomial

$$a_{-k}z^k + a_{-k+1}z^{k-1} + \cdots + a_{-1}z.$$

Case 3. If an infinite number of a_{-n} for $n \geq 1$ in (7.27) do not vanish, then $F(w)$ has an essential singularity at $w = 0$. In terms of $f(z)$, we say that $f(z)$ has an essential singularity at $z = \infty$ and we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}, \quad |z| > R,$$

where an infinite number of a_{-n}'s are non-zero for $n \geq 1$.

Case 3 gives rise to the following interesting observation. "*An entire function $f(z)$ is transcendental iff $f(1/z)$ has an essential singularity at* 0." To see this we suppose that $f(z)$ is an entire transcendental. Then, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all} \quad z \in \mathbb{C}.$$

If 0 is not an essential singular point of $f(1/z)$, then there exists a k (possibly for a sufficiently large k) such that

$$z^k f(1/z) = \sum_{n=0}^{\infty} a_n z^{k-n}, \quad |z| > 0,$$

has a removable singularity at 0. This means that $a_n = 0$ for all $n \geq k+1$, that is $f(z)$ is a polynomial of degree at most k, contrary to our assumption that $f(z)$ is entire transcendental.

To prove the converse part, suppose $f(z)$ is not transcendental. Then, by the definition of transcendental function, $f(z)$ is a polynomial and so we write

$$g(z) = f(1/z) = a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}.$$

But then g has either a pole (if $n > 0$) or a removable singularity (if $n = 0$) at $z = 0$. The proof of the observation is complete.

The above discussion can be equivalently stated as follows: "*an entire transcendental function has necessarily an essential singular point at infinity*", for, otherwise it would have no singularities at all and would, by Liouville's theorem, reduce to a constant.

Let f be defined in some open set Ω of $\mathbb{C}_\infty$, and let f have isolated singularities at $a_1, a_2, a_3, \ldots$. Also, let $a \in \mathbb{C}_\infty$ be a limit point of the set of these singularities. Then, for any neighborhood N of a, there is a singularity (different from a) inside N, i.e. f cannot be analytic in any deleted neighborhood ($\Delta(a;\delta)\backslash\{a\}$ if $a \in \mathbb{C}$ and $|z| > \delta$ if $a \in \infty$) of a. So f has no Laurent expansion about a. In other words, a is neither a regular point nor an isolated singularity. Thus, a is a non-isolated (essential) singularity of f.

For instance, $\cot(1/z)$ has poles at $z = 1/k\pi$, $k = \pm 1, \pm 2, \ldots$. Hence the limit point $z = 0$ of these poles is a non-isolated (essential) singularity of this function.

7.28. Example. Consider $f(z) = 1/\sin z$ for $z \neq k\pi$, $k \in \mathbb{Z}$. Let us discuss the singularity of f at ∞. For this we define

$$g(z) = f(1/z) = 1/\sin(1/z)$$

and discuss the corresponding singularity of g at 0. Note that g is not analytic at the origin, since every neighborhood of the origin contains a singularity. It follows that $g(z)$ has poles at the points $z = 1/(k\pi)$ $(k \in \mathbb{Z})$. Note that $z = 0$ is not an isolated essential singularity for $g(z)$ as $z = 0$ is a limit point for its poles at $1/(k\pi)$ $(k \in \mathbb{Z})$. This observation shows that $f(z)$ is not analytic in $\{z : |z| > R\}$ for any $R > 0$ and ∞ is a non-isolated (essential) singularity for $f(z)$.

Similarly, it is easy to see that ∞ is a non-isolated (essential) singularity for $f(z) = (e^z - 1)^{-1}$. ●

7.29. Example. Consider the function $g(z) = z^{-1}(1+z)^{-1}$. Then $g(1/w) = w^2/(1+w)$ which is analytic at $w = 0$ and hence, $g(z)$ is analytic at infinity. Similarly, we may easily verify that each of the functions

$$\frac{1}{1+z}, \ \frac{z}{1+z}, \ \exp(1/z^2)$$

is analytic at the point at infinity.

On the other hand, functions $\sin z$ and $\cos z$ are not analytic at the point at infinity, because both $\sin(1/z)$ and $\cos(1/z)$ have an isolated essential singularity at $z = 0$. ●

7.30. Example. The function g defined by $g(z) = z^4 \sin(1/(z+1))$ has a pole of order three at the point at infinity. On the other hand, the function $1/(z^2 \sin(1/z))$ has a removable singularity at infinity. ●

Suppose that f is entire. Then f has a power series expansion of the form $f(z) = \sum_{n\geq 0} a_n z^n$ which converges absolutely for all $z \in \mathbb{C}$. Thus,

$$F(z) = f(1/z) = \sum_{n\geq 0} a_n z^{-n} \quad \text{for } |z| > 0$$

and so $z = \infty$ is an isolated singularity of $f(z)$, since 0 is an isolated singularity of $F(z)$. There are now three possibilities:

(i) $f(z)$ has a removable singularity at $z = \infty$

(ii) $f(z)$ has a pole of order k at $z = \infty$

(iii) $f(z)$ has an essential singularity at $z = \infty$.

In case (i), the expansion contains no powers of z and so $f(z) = a_0$. Alternatively, by Liouville's theorem, $f(z)$ reduces to a constant. Thus, "*an entire function* $f : \mathbb{C} \to \mathbb{C}$ *has a removable singularity at* $z = \infty$ *iff* $f(z)$ *is constant*".

In case (ii), as discussed above, $f(z)$ has a pole of order $k\,(\geq 1)$ at ∞ iff the expansion of $f(z)$ contains only a finite number of positive powers and $a_n = 0$ for $n > k$. Consequently, "*an entire function* $f : \mathbb{C} \to \mathbb{C}$ *has a pole at* $z = \infty$ (*of order* k) *iff* $f(z)$ *is a nonconstant polynomial* (*of degree* k)".

Case (iii) occurs iff $a_n \neq 0$ for infinitely many n's. Consequently, $f(z)$ is a transcendental entire function-a fact which has been confirmed earlier too.

7.6 Meromorphic Functions

A function analytic in a domain $D \subseteq \mathbb{C}$ except possibly for poles is called *meromorphic* in D.

Suppose that f is meromorphic in D such that D contains $\{z : |z| > r\}$ for some $r > 0$. Then, the function f is said to be *meromorphic at* $z = \infty$ if $g(z) = f(1/z)$ is meromorphic in a neighborhood of $z = 0$, i.e. in the usual sense on $\Delta_{1/r}$. The definition as stated is equivalent to requiring that f has a pole at ∞ but has no poles in $\{z : |z| > R\}$ for some $R > r$.

Here are some basic examples of meromorphic functions in $\mathbb{C}$.

(i) $f(z) = e^z/z^2$ is meromorphic in $\mathbb{C}$, since f is analytic in $\mathbb{C}\backslash\{0\}$ and the singularity at 0 is a double pole.

(ii) every analytic function in $D \subseteq \mathbb{C}$ is obviously meromorphic. Consequently, sums and products of meromorphic functions are meromorphic. It is important to remark that the functions $f(z) = e^z/z^2$ and $g(z) = z - e^z/z^2$ are meromorphic in $\mathbb{C}$, but $f + g$ has a removable singularity at $z = 0$. Here we regard $f + g$ as an extended analytic function in $\mathbb{C}$ (by defining its value at the origin as its limiting value, namely $(f + g)(0) = 0$) and hence, we treat $f + g$ as a meromorphic function.

(iii) $f(z) = 1/\sinh z$ is meromorphic in $\mathbb{C}$ since the singularities of f are poles at $z = k\pi i$, $k \in \mathbb{Z}$. Further, functions $1/\sin z$, $1/\cos z$, $\cos z/\sin z$, $\sin z/\cos z$, $1/\cosh z$ and $1/(e^z - 1)$ are all meromorphic in $\mathbb{C}$. Note also that ∞ is the limit point of poles for these functions.

(iv) any rational function $R(z)$, where the numerator and the denominator of $R(z)$ have no common factor, is meromorphic in $\mathbb{C}$ since it is analytic in $\mathbb{C}$ except at the zeros of the denominator where $R(z)$ has poles. (Note that each of the functions involved in (iii) is not rational since the set of poles for each of these functions is countably infinite). We can express $R(z)$ as a quotient of polynomials of the form

$$R(z) = A\frac{\prod_{i=1}^{k}(z - a_i)^{n_i}}{\prod_{j=1}^{l}(z - b_j)^{m_j}},$$

where a_i's and b_j's are all distinct, and A is some constant.

(v) the quotient of a meromorphic function is meromorphic, provided that the denominator term is not identically zero.

(vi) the functions $(z^2 + 4)^{-1}e^{1/z}$ and $(z^2 - 4)^{-1}\sin(1/z)$ are meromorphic in $\mathbb{C}\backslash\{0\}$. They are not, however, meromorphic in $\mathbb{C}$, because each of them has an essential singularity at the origin.

(vii) every meromorphic function in an open set Ω admits a representation as the quotient of two analytic functions in Ω, which will be confirmed in Chapter 11.

(viii) the only singularities of $f(z) = \cot z$ are the simple poles at $n\pi$ $(n \in \mathbb{Z})$ and hence, $f(z)$ is meromorphic in $\mathbb{C}$. It is, however, not meromorphic in $\mathbb{C}_\infty$ because ∞ is a limit point of poles of $f(z)$.

7.31. Remark. A meromorphic function in $\mathbb{C}$ can have only a finite number of poles in any bounded subset D of $\mathbb{C}$. For instance, neither $f(z) = 1/\sin(\pi/z)$ nor $g(z) = 1/\cos(1/\pi z)$ is meromorphic in $\mathbb{C}$. Note that the set of poles of $f(z)$ is a bounded set $S = \{1/n : n \in \mathbb{Z}\}$ and the limit point of S is 0, which is a non-isolated (essential) singularity of $f(z)$. A similar argument shows that $g(z)$ is not meromorphic in $\mathbb{C}$. On the other hand, both $f(z)$ and $g(z)$ are meromorphic in $\mathbb{C}\backslash\{0\}$. However, the functions such as $1/(e^z + 1)$ and $1/(1 + \cos z)$ are all meromorphic in $\mathbb{C}$. ●

When we speak of a meromorphic function f without mentioning the domain of definition for f, it is understood that f is meromorphic in $\mathbb{C}$. We observe that analytic functions are, in some sense, a generalization of polynomials while the meromorphic functions are then a generalization of rational functions.

The next two theorems which characterize rational functions are simple and elegant. From Liouville's theorem it follows that an entire function on $\mathbb{C}_\infty$ is constant. An analog of this result for meromorphic functions in $\mathbb{C}_\infty$ follows.

7.32. Theorem. *A function $f(z)$ is rational iff it is meromorphic in the extended complex plane $\mathbb{C}_\infty$.*

Proof. As noticed above a rational function is meromorphic in $\mathbb{C}_\infty$. Suppose conversely f is meromorphic in $\mathbb{C}_\infty$. Note that $f(z)$ can have only finitely many poles in $\mathbb{C}_\infty$ since otherwise the limit point of the poles would be a non-isolated (essential) singularity which is not a pole at all. Let the poles of $f(z)$ be $z_1, z_2, \dots, z_m \in \mathbb{C}$, of orders $k_1, k_2, \dots, k_m$, respectively. Then, g defined by

$$g(z) = f(z) \cdot \prod_{j=1}^{m} (z - z_j)^{k_j}$$

is an entire function. Thus g is a polynomial, since otherwise g would necessarily have an essential singularity at ∞. The result follows. ■

For instance, $f(z) = (z^5 + 3z^2 + 1)(z^2 - 2z - 1)^{-1}$ has a triple pole at ∞ because the function $f(1/z)$ has a triple pole at the origin.

7.33. Theorem. *Let $f(z)$ be meromorphic in $\mathbb{C}$ and there exist a natural number n, $M > 0$, and $R > 0$ such that*

$$|f(z)| \le M|z|^n \quad \text{for } |z| > R. \tag{7.34}$$

Then, f is a rational function.

Proof. Proceeding exactly as in Theorem 7.32, we obtain an entire function g, where $g(z) = f(z) \cdot p(z)$ with $p(z) = \prod_{j=1}^{m} (z - z_j)^{k_j}$. Therefore, for sufficiently large $|z|$, we easily have (see (6.65))

$$|p(z)| \leq 2|z|^N \tag{7.35}$$

where $N = k_1 + k_2 + \cdots + k_m$. Thus, by (7.34) and (7.35),

$$|g(z)| = |f(z)|\,|p(z)| \leq 2M|z|^{N+n}$$

for sufficiently large $|z|$. It follows from Theorem 6.60 that g is a polynomial, and hence, $f(z)$ is a rational function. ∎

Note that if, in Theorem 7.33, f is assumed to be analytic in $\mathbb{C}$, then f would be a polynomial. Thus, Theorem 7.33 is clearly a natural generalization of Liouville's theorem (see Theorem 6.60).

7.7 Essential Singularities and Picard's Theorem

We have already proved that $\sin(\mathbb{C})$, $\cos(\mathbb{C})$ and $\exp(\mathbb{C})$ are all unbounded subsets in $\mathbb{C}$. Therefore, it is natural to ask: *Can we say something more about entire unbounded functions?* In fact, such questions lead to certain beautiful conclusion about the image domains of non-constant entire unbounded functions. For certain familiar entire functions much more is true:

- each non-constant polynomial $p(z)$ assumes every complex number as a value; that is $p(\mathbb{C}) = \mathbb{C}$.
- each of the trigonometric functions $\sin z$, $\cos z$, and of the hyperbolic functions $\sinh z$, $\cosh z$ assumes every complex number as a value; that is, $\sin(\mathbb{C}) = \cos(\mathbb{C}) = \sinh(\mathbb{C}) = \cosh(\mathbb{C}) = \mathbb{C}$.
- the exponential function e^z never assumes zero, as $\exp(\mathbb{C}) = \mathbb{C}\setminus\{0\}$.

The first assertion has been proved in Theorem 6.66 while the other two assertions can be proved by looking at the solutions of the equation $f(z) = c$ for a given $c \in \mathbb{C}$, where $f(z)$ is one of the functions given by

$$\sin z, \quad \cos z, \quad \sinh z, \quad \cosh z, \quad \text{and} \quad e^z.$$

For example, as $\sin z = (e^{iz} - e^{-iz})/(2i)$, the solution of $\sin z = c$ is obtained from

$$e^{iz} = ic \pm \sqrt{1 - c^2}.$$

Note that there exists no $c \in \mathbb{C}$ such that $ic \pm \sqrt{1-c^2} = 0$. Consequently, for each c, the equation $\sin z = c$ has infinitely many solutions given by

$$z = -i \log(ic \pm \sqrt{1-c^2}).$$

These observations probably led to "the Picard little theorem" which we shall discuss soon. Let us start with the following preliminary result which may be called a weaker form of the Casorati-Weierstrass theorem (see Theorem 7.40).

7.36. Theorem. *The range of a non-constant entire function is a dense subset of* $\mathbb{C}$.

Proof. Let f be a non-constant entire function. Suppose on the contrary that $f(\mathbb{C})$ is not dense. Then, there would exist a point $w_0 \in \mathbb{C}$ and a neighborhood $\Delta(w_0;\epsilon)$ such that $\Delta(w_0;\epsilon) \cap f(\mathbb{C}) = \emptyset$. Then, for all $z \in \mathbb{C}$, we have $|f(z) - w_0| > \epsilon$ so that $|g(z)| < 1$ for $z \in \mathbb{C}$, where

$$g(z) = \frac{\epsilon}{f(z) - w_0}.$$

But then, g being a bounded entire function, would be constant, and hence f would be constant, which is a contradiction. ∎

Note that a subset D of D' is said to be *dense* in D', if for each $z_0 \in D'$ and any $\delta > 0$, $\Delta(z_0;\delta) \cap D \neq \emptyset$.

According to Theorem 7.36, it is almost trivial to show the following: Suppose that f is an entire function satisfying any one of the following conditions for all $z \in \mathbb{C}$ and for some fixed $M > 0$:

(a) $|f(z)| \geq M$;

(b) $\operatorname{Re} f(z) \leq M$ or $\operatorname{Im} f(z) \leq M$ or $|\operatorname{Re} f(z)| > M$ or $|\operatorname{Im} f(z)| > M$.

Then, f is necessarily a constant. We have already shown these as a consequence of Liouville's theorem. In particular, this result demonstrates a simple fact that *"the range of a non-constant entire function is never contained in a half-plane or in bounded domain or in the complement of any bounded simply connected domain in* $\mathbb{C}$*"*. This piece of information clearly improves Liouville's theorem but obviously falls far short of Picard's little and big theorems. To be a little more precise, much more than Theorem 7.36 holds: *if f is a non-constant entire function then $\mathbb{C}\setminus f(\mathbb{C})$ contains at most one point.* This result is known as Picard's little theorem.[11] We may restate it in the following form. For a proof of this result, we refer to p. 540.

7.37. Theorem. **(Picard's Little Theorem)** *Every non-constant entire function omits at most one complex number as its value.*[12] *In other words, if an entire function omits two values, then it is constant.*

[11] Emile Picard (1856-1941) was a French mathematician who published the proofs of the two famous theorems at the age of 22. He has made a number of contributions in many other areas.

[12] A number $c \in \mathbb{C}$ is said to be an exceptional value of a complex-valued function f if c does not belong to the range of f.

Now we need some preparations for the proof of Picard's little theorem. Picard's original proof of his little theorem is of entirely different character. The proof which we are going to present is due to Landau-König. Let us start with a remark that Picard's little theorem is an astonishing generalization of the theorem of Liouville as well as the theorem of Casorati-Weierstrass. In 1879, with the aid of elliptic modular functions, Picard proved a more general and deep result−the so called Picard's great theorem: *"if f has an essential singularity at a, then the range under f of any deleted neighborhood of a is the whole complex plane with at most one exception"*. Equivalently, we may state it in the following form.

7.38. Theorem. (Picard's Great Theorem) *Suppose that f is analytic in $\Delta(z_0;r)\backslash\{z_0\}$ and $z = z_0$ is an essential singularity of f. Then $\mathbb{C}\backslash f(\Delta(z_0;r)\backslash\{z_0\})$ is a singleton set.*

Examples 7.46 and 7.48 below are special cases of Picard's theorem. We shall present the proof of Picard's little theorem in Section 12.7 as it requires Bloch's theorem. However, let us first restrict ourselves to the following much weaker and simpler result−also called the Casorati-Weierstrass theorem.

7.39. Theorem. (Weaker Form of Picard's Great Theorem) *Suppose that $f \in \mathcal{H}(\Delta(z_0;R)\backslash\{z_0\})$ and z_0 is an essential singularity of f. Then for each $\delta > 0$ with $\delta \leq R$, $f(\Delta(z_0;\delta)\backslash\{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Suppose on the contrarary that the range $f(\{0 < |z - z_0| < \delta\})$ is not dense. Then there is a point $w_0 \in \mathbb{C}$ and a disk $\Delta(w_0;\epsilon) = \{w \in \mathbb{C} : |w - w_0| < \epsilon\}$ such that $\Delta(w_0;\epsilon) \cap f(\{0 < |z - z_0| < \delta\}) = \emptyset$.

For all z with $0 < |z-z_0| < \delta$, we have $|f(z)-w_0| > \epsilon > 0$ and therefore, the function g defined by

$$g(z) = \frac{\epsilon}{f(z) - w_0}$$

is bounded and analytic in the deleted neighborhood $\Delta(z_0;\delta)\backslash\{z_0\}$, since $f(z) - w_0$ is analytic and non-zero there. From the removability theorem, it follows that g has a removable singularity at z_0. Thus, by defining $g(z_0)$ properly, g becomes analytic in $\Delta(z_0;\delta)$. Clearly, $g(z) \not\equiv 0$ in $\Delta(z_0;\delta)$. If $g(z_0) \neq 0$,

$$f(z) = \frac{\epsilon}{g(z)} + w_0$$

is analytic at z_0. If $g(z)$ has a zero at z_0, then z_0 is a pole of $1/g(z)$ and hence, the same holds for $f(z)$. In either case this contradicts the hypothesis that z_0 is an essential singularity of $f(z)$. ∎

7.40. Theorem. (Casorati-Weierstrass Theorem) *If f has an essential singularity at z_0 and if w_0 is a given finite complex number, then* *there exists a sequence* $\{z_n\}$ *with* $z_n \to z_0$ *such that* $f(z_n) \to w_0$. *In other words,* f *takes values arbitrarily close to every complex number in every neighborhood of an essential singularity.*

Proof. By hypothesis, $f(z)$ is analytic throughout a deleted neighborhood of z_0. Suppose that there exists a complex number w_0 and a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) \not\to w_0$. Then there exist an $\epsilon > 0$ and a $\delta > 0$ such that $|f(z) - w_0| \geq \epsilon$ for $0 < |z - z_0| < \delta$. Define

$$g(z) = \frac{1}{f(z) - w_0}.$$

Now, proceeding exactly as in Theorem 7.39, we would get that $f(z)$ has either a removable singularity or a pole at $z = z_0$. In either case this contradicts the hypothesis that z_0 is an essential singularity of $f(z)$. ■

7.41. Remark. Note that Theorem 7.40 and Theorem 7.39 are equivalent. ●

If ∞ is regarded as an isolated essential singularity of $g(z)$, then 0 is an isolated essential singularity of $f(z)$, where $f(z) = g(1/z)$. In view of this observation, Theorem 7.39 may be rephrased as follows:

7.42. Theorem. *If* $g(z)$ *is an entire transcendental function, then near* ∞, *the values assumed by* $g(z)$ *are dense in* $\mathbb{C}$. *In other words,* $g(z)$ *takes values arbitrarily close to every complex number in every neighborhood of* ∞.

Next we record a very useful application of Theorem 7.40.

7.43. Theorem. *An entire function* $f(z)$ *is univalent in* $\mathbb{C}$ *iff* $f(z) = a_0 + a_1 z$ $(z \in \mathbb{C})$, *where* a_0, a_1 *are constants with* $a_1 \neq 0$*; that is,*

$$\mathrm{Aut}\,(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) : f(z) = a_0 + a_1 z\}.$$

Proof. If $f(z)$ is entire, then, by Theorem 4.93 (see also Corollary 3.73), $f(z)$ has a power series expansion of the form $f(z) = \sum_{n \geq 0} a_n z^n$ which converges absolutely for all $z \in \mathbb{C}$. There are now three possibilities:

(i) $f(z) = a_0$, i.e. constant

(ii) $f(z) = \sum_{n=0}^{k} a_n z^n$, $a_k \neq 0$ $(k \geq 1)$

(iii) $f(z) = \sum_{n \geq 0} a_n z^n$, $a_n \neq 0$ for infinitely many n.

Suppose further that $f(z)$ is univalent in $\mathbb{C}$. Then case (i) cannot occur.
In case (ii), $f(z)$ is a polynomial of degree $k \geq 1$. But, by the fundamental theorem of algebra, $f(z)$ has k zeros which lie inside some circle

$|z| = R$, R large enough. Therefore, f is univalent only when $k \not> 1$. Hence, $k = 1$ and so, $f(z) = a_0 + a_1 z$, where $a_1 \neq 0$.

In case (iii), $f(z)$ is transcendental and hence $f(z)$ has an essential singularity at ∞. Then, by the Casorati-Weierstrass theorem (see also Theorem 7.42), for any given complex number w, there exists a sequence $\{z_n\}$ such that $z_n \to \infty$ and $f(z_n) \to w$. In particular, for $w = 0$, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} f^{-1}(f(z_n)) = f^{-1}(0) \neq \infty.$$

This contradiction shows that $f(z)$ is a polynomial which by case (ii) yields that f is linear. The converse part is obvious. This proves that an entire function which assumes every complex value exactly once is precisely the linear function. ∎

Picard's results mark the beginning of a development which eventually culminated in the value distribution theory of R. Nevanlinna. In 1896, E. Borel derived Picard's little theorem with elementary function-theoretic tools. The theory then took a surprising turn in 1924, when A. Bloch discovered the theorem named after him. We shall prove Picard's theorem using Bloch's theorem (see Sections 12.6 and 12.7).

Now, we observe that an entire function h omits two distinct values, namely a and b with $a \neq b$, iff the function f defined by

$$f(z) = \frac{h(z) - a}{b - a}$$

is entire and omits 0 and 1. Therefore, h is constant if and only if f is constant. In view of this observation, Theorem 7.37 is equivalent to the following

7.44. Theorem. *If f is an entire function such that $0 \notin f(\mathbb{C})$ and $1 \notin f(\mathbb{C})$, then f is constant.*

It is natural to see what happens when we replace "$f(z)$ is entire" by "$f(z)$ is meromorphic in $\mathbb{C}$." To see this, we consider

$$\phi(z) = \frac{1}{1 + e^z}$$

which never assumes 0 or 1 as its value. Clearly, ϕ is meromorphic in $\mathbb{C}$, ϕ omits 0 and 1, but ϕ is not a constant function. This example shows that meromorphic functions in $\mathbb{C}$ can omit two complex numbers. A value that a meromorphic function does not assume is known as Picard's exceptional value. For instance, the function $\phi(z)$ has two exceptional values, namely $0, 1$. Similarly, it is easy to see that $\tan z$ is meromorphic in $\mathbb{C}$ and has an essential singularity at $z = \infty$ which is the limit point of poles of $\tan z$.

Moreover, $\tan z$ assumes every complex value with Picard's exceptional values $\pm i$. Meromorphic functions of this type are significant in view of the following result.

7.45. Theorem. (Picard's Little Theorem for Meromorphic Functions) *Every meromorphic function ϕ in $\mathbb{C}$ that omits three distinct values $a, b, c \in \mathbb{C}$ is necessarily constant.*

Proof. The desired conclusion follows from Theorem 7.44 as the function f defined by $f(z) = (\phi(z) - a)^{-1}$ is entire and omits two distinct values $1/(b-a)$ and $1/(c-a)$. ∎

7.46. Example. We show that $e^{1/z}$ assumes every value infinitely many times with an exception of zero (Picard's exceptional value). To do this we recall that $\exp(1/z)$ has an essential singularity at the origin and omits the value zero. By Picard's theorem, it therefore assumes all the other values in each punctured neighborhood $\Delta(0; r)\setminus\{0\}$. Let us now verify this by a direct computation. Consider $e^{1/z} = c = e^{\log c}$ $(c \neq 0)$, or equivalently,

$$e^{1/z} = e^{\ln |c| + i \arg c} \tag{7.47}$$

where c may be taken as a complex number. If c is real and non-negative, then (7.47) implies

$$\frac{1}{z} = \ln |c| + i \arg c = \ln |c| + 2k\pi i; \quad \text{i.e.} \quad z_k = \frac{1}{\ln |c| + 2k\pi i}, \quad k \in \mathbb{Z}.$$

Thus, we have a sequence $\{z_n\}$ such that $z_n \to 0$ and $e^{1/z_n} = c$ for each $n \in \mathbb{N}$. The general case for $c \neq 0$ can be handled similarly. Observe that Picard's exceptional value 'zero' is a limit point. ●

7.48. Example. Let us discuss one more function which has an essential singularity at $z = 0$. Consider $\sin(1/z)$. Suppose for c real and $c \neq 0$, $\sin(1/z) = c$. Then the solution to this equation is given by

$$\frac{1}{z} = \arcsin c = \frac{1}{i} \log(ic + \sqrt{1-c^2}), \quad \text{i.e.} \quad z = \frac{i}{\log(ic + \sqrt{1-c^2})}.$$

For $k \in \mathbb{Z}$, let

$$z_k = \frac{i}{\ln |ic + \sqrt{1-c^2}| + \operatorname{Arg}(ic + \sqrt{1-c^2}) + 2k\pi i}.$$

Then, we obtain a sequence $\{z_n\}$ such that $z_n \to 0$ and $\sin(1/z_n) = c$, $n \in \mathbb{N}$.

In particular, if $c = 1$ then $\sin(1/z) = 1$ is satisfied by infinitely many values, namely, $z_k = 2/(4k+1)\pi$, $k \in \mathbb{Z}$.

However, $z = 0$ is also an essential singularity of $\cos(1/z)$ and there are no exceptional values (except ∞). ●

7.8 Exercises

7.49. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) An entire function $f(z)$ having $z = \infty$ as a removable singularity is constant.
Note: See the discussion at the end of Section 7.5.

(b) An entire function $f(z)$ has a pole of order n at infinity iff $f(z)$ is a polynomial of degree n.

(c) If $f(z)$ has a pole at z_0, then $\exp\{f(z)\}$ has an essential singularity at this point.

(d) If $f(z)$ is a nonconstant entire function, then $\exp\{f(z)\}$ has an essential singularity at $z = \infty$.

(e) If a is an isolated singularity of f which is not removable, then a is an essential singularity of $\exp\{f(z)\}$.

(f) If f is analytic in $\mathbb{C}_\infty$ except for a finite number of poles, then the number poles and zeros of f (counted according to multiplicity) are equal.

(g) If f is non-constant and analytic at z_0, then $f^{(n)}(z_0) \neq 0$ for some $n \geq 1$.

(h) Poles are isolated. That is, f has a pole of order m at a iff $f(z) = (z-a)^{-m}g(z)$, where g is analytic at a and $g(a) \neq 0$.

(i) Suppose f has an essential singularity at $z = a$ and g has a pole at $z = a$. Then the product fg has an essential singularity at $z = a$.

(j) The function $f(z) = \dfrac{z^4+1}{(\alpha z - |z|^2)(\beta z - |z|^2)}$, $\alpha,\ \beta \in \mathbb{C}\backslash\{0\}$ $(\alpha \neq \beta)$ has two simple poles at $z = \overline{\alpha},\ \overline{\beta}$ and a double pole at $z = 0$.

(k) The function f defined by the Laurent series

$$f(z) = \sum_{k=-\infty}^{-1} \frac{z^{2k}}{(-k)!} + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}$$

has an essential singularity at the origin.

(l) If f is meromorphic, then f and f' have the same poles and the order of the poles of f' increases by one.

(m) The function $f(z) = 1/\sin(1/z)$ has a singularity at the origin and the Laurent series expansion about the origin does not exist.

(n) If $z = a$ is an isolated essential singularity for $f(z)$, then $z = a$ is neither a regular point nor a pole for $g(z) = 1/f(z)$. Further, $z = a$ is not necessarily an isolated essential singularity for $g(z)$.

(o) Suppose f has an essential singularity at z_0. Then there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $\lim_{n\to\infty} |f(z_n)| = \infty$.

(p) If f is a transcendental entire function, then there exists a sequence $z_n \to \infty$ for which $f(z_n) \to 0$.

(q) If $f(z)$ is analytic at ∞, then $f'(\infty) = 0$.

(r) The function $f(z) = \dfrac{e^{1/(z-1)}}{e^z - 1}$ has a simple pole at $z = 2k\pi i$ ($k \in \mathbb{Z}$) and an essential singularity at $z = 1$.

(s) Every $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ such that $|f(z)| \leq a|z|^{1/2} + b|z|^{-1/2}$ for some $a, b > 0$ is necessarily constant.

(t) Every $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ such that $|f(z)| \leq |z|^2 + |z|^{-1/2}$ for all $z \in \mathbb{C}\backslash\{0\}$ is necessarily a polynomial of degree at most two.

(u) There does not exist a function $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ that satisfies $|f(z)| \geq |z|^{-\alpha}$ for all $z \in \mathbb{C}\backslash\{0\}$, and for a fixed $\alpha \in (0, 1)$.

(v) The meromorphic function $\cot z$ never assumes i and $-i$ and so, $\pm i$ are the Picard exceptional values for $\cot z$.

(w) The meromorphic function $e^{iz}/\cos z$ never assumes 0 and 2.

(x) If f and g are two entire functions such that $e^f + e^g = 1$, then f and g are constant functions.

(y) If $f(z) = \dfrac{e^{\sqrt{z}} - e^{\sqrt{-z}}}{\sin\sqrt{z}}$, then $z = 0$ is not a branch point of $f(z)$.

(z) If $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$, then $z = 0$ is a removable singularity of f.

7.50. Suppose that $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ and satisfies $|f(z)| \leq a|z|^2 + b|z|^{-2}$ for all $z \in \mathbb{C}\backslash\{0\}$, and for some $a, b > 0$. If f is an odd function of z, what form must the Laurent series of f have? How about when f is an even function of z?

7.51. Classify [type (and order where applicable)] the isolated singular points (including at the point at ∞) of the following functions:

(a) $f(z) = \dfrac{\sin^4 z}{z^8}$

(b) $f(z) = \dfrac{(1+z)\cos z}{z}$

(c) $f(z) = \dfrac{1 - 8z^3}{1 - 4z^2}$

(d) $f(z) = \dfrac{\sin(\pi z)\, e^{1/z^3}}{(z-1)z^3}$

(e) $f(z) = \sin\left(\dfrac{1}{z}\right) + \dfrac{1}{z^2(2-z)}$

(f) $f(z) = \csc z - \dfrac{1}{z}$

(g) $f(z) = \dfrac{z^2}{(z-ip)^4(z+q)^3}$, $p,\ q \in \mathbb{R}\backslash\{0\}$

(h) $f(z) = \dfrac{z^{10}(z^2+1)e^{1/(z-4)^5}}{(z-\pi)^2\sin^{10} z}$

7.52. Prove or disprove that $f(z) = 1/\sin z$ is not meromorphic on the Riemann sphere. Does $f(\mathbb{C}) = \mathbb{C}\backslash\{0\}$? Does $\sec(\mathbb{C}) = \mathbb{C}\backslash\{0\}$?

7.53. Prove or disprove the following: The point $z = 0$ is the only singularity of the function $f(z) = \sin(1 - 1/z)$ and $z = 0$ is a simple pole.

7.54. Give an example of a function which is meromorphic in $\mathbb{C}$ without being meromorphic on the Riemann sphere.

7.55. Suppose that f is any one of $(z^3 - z)e^{1/z}$, $z\sin(1/z)$, $e^{1/z} - e^{-1/z}$ and $e^{1/z} + e^{-1/z}$. Assuming the validity of Picard's theorem, decide whether $f(0 < |z| < 1) = \mathbb{C}$ or not.

7.56. Let f_j $(j = 1, 2, \ldots, 6)$ be defined as in Example 7.3. Classify the singularity of each of these functions at ∞.

7.57. Find the general form of a function in $\mathbb{C}_\infty$ having the following singularities:

(i) only simple pole at z_0,
(ii) one pole of order n at z_0,
(iii) one pole of order n at infinity and a pole of order m at z_0.

7.58. Suppose $z = a$ is a singularity of $f(z)$. Can $z = a$ be a singularity for $1/f(z)$? If so what will be the nature of the singularity. Discuss in details with an example for each case.

7.59. Suppose f and g are analytic in a neighborhood of z_0, $f(z_0) = 0$ with multiplicity m, $g(z_0) = 0$ with multiplicity n. What is the multiplicity of z_0 as a zero of the composite function $f \circ g$?

7.60. Construct a function ϕ which is analytic except at the four distinct points z_j, $j = 1, 2, 3, 4$, where it has the following properties:

(i) simple pole at z_1,
(ii) simple zeros at z_2, z_3, z_4,
(iii) simple pole at infinity,
(iv) $\lim_{|z|\to\infty} z^{-1}\phi(z) = 2$.

7.61. Find the constant c such that

$$f(z) = \frac{1}{z^n + z^{n-1} + \cdots + z^2 + z - n} + \frac{c}{z-1}$$

can be extended to be analytic at $z = 1$, where $n \in \mathbb{N}$ is fixed.

7.62. Prove that the function f defined by $f(z) = (e^z - 1)/(z(z-2))$ has a removable singularity at $z = 0$ whereas it has a simple pole at $z = 2$. Prove also that it has an essential singularity at $z = \infty$.

7.63. Prove that the function f defined by $f(z) = \exp(z/\sin z)$ has a removable singularity at $z = 0$ whereas it has essential singularities at $z = k\pi$, $k \in \mathbb{Z}\setminus\{0\}$.

7.64. Suppose that $f(z)$ is analytic on $\mathbb{C}$ except for a double pole at $z = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Is $\{a_n\}$ bounded? Does $\{a_n\}$ converge? Justify your answer. How about if we replace the word 'double pole' by 'simple pole'?

7.65. Let $f \in \mathcal{H}(\Delta_3 \setminus \{0\})$. Suppose that f has a simple pole at $z = 2i$ and $f(z) = \sum_{n\geq 0} a_n z^n$ for $|z| < 2$. Is $\{a_n\}$ bounded? Does $\{a_n\}$ converge? Justify your answer.

7.66. Let f be a meromorphic function in a domain Ω. Then prove that neither the set of zeros nor the set of poles of f can have a limit point in Ω unless f is identically zero in Ω.

7.67. Assume that $\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in the annulus $0 < |z| < 1$. Find a_{-n} for $n \geq 1$.

Chapter 8

Calculus of Residues

If f is analytic at a point $z = a$, then there is a neighborhood N of a inside which f is analytic. Let C be a positively oriented closed contour contained in N. Then the celebrated Cauchy theorem tells us that $\int_C f(z)dz = 0$. If, however, f fails to be analytic at finitely many isolated singularities inside C, then the above argument fails; which means each of these singularities contribute a specified value to the value of the integral. This motivates us to generalize the Cauchy theorem to functions which have *isolated singularities*. This generalization results in the *Residue theorem*. This result is one of the most important and often used, tools that applied scientists need, from the theory of complex functions.

In Section 8.1, we are concerned with the notion of what we call "*the residue at an isolated singularity*" and discuss the concept of residue in detail and illustrate it with a number of examples for finding the residue of a given function at an isolated singularity. In Section 8.2, we discuss the notion of residue at the point at ∞. The main result in Section 8.3 is Cauchy's residue theorem which states that the integral of an analytic function f around a simple closed contour C equals $2\pi i$ times the sum of the residues of f at the isolated singular points inside C. Using this important theorem, we shall then develop and illustrate some of the basic methods employed in complex integration for evaluating complex line integrals. The residue theorem extends Cauchy's theorem by allowing for a finite number of isolated singularities inside the contour of integration. Formulae enabling us to do this include an alternate proof of the so-called generalized Cauchy integral formula. When there are no singularities, the residue theorem simply reduces to Cauchy's theorem. The materials presented in Section 8.1-8.3, provide a good training ground for the evaluation of complex integration in the next chapter. In Section 8.4, we derive the argument principle which is another of the most important applications of the Cauchy residue theorem. Also, we discuss several of its consequences, for example, locating the zeros of an analytic function. The argument principle provide a tool in the form of Rouché's theorem to see how the number of zeros of analytic functions remains constant under small perturbations. We discuss a version of Rouché's theorem in Section 8.5.

8.1 Residue at a Finite Point

We recall that if f has an isolated singularity at z_0, then the residue of $f(z)$ at z_0 is

$$\text{Res}\,[f(z); z_0] := a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz, \tag{8.1}$$

where C is any circle centered at z_0 and lying inside a disk about z_0. For instance, consider $f(z) = \cot z$. Since $\sin z = 0 \Longleftrightarrow z = k\pi$ $(k \in \mathbb{Z})$, and

$$\lim_{z\to k\pi}(z-k\pi)\cot z = \lim_{z\to k\pi}\frac{z-k\pi}{\sin z}\cdot\lim_{z\to k\pi}\cos z = \lim_{z\to k\pi}\frac{1}{\cos z}\cdot\lim_{z\to k\pi}\cos z = 1,$$

f has simple poles at $z = k\pi$, $k \in \mathbb{Z}$. Suppose we choose $z_0 = 0$. Then

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (0 < |z| < \pi)$$

and, in view of (8.1), this gives,

$$\int_C \cot z\,dz = 2\pi i.$$

Here C could be any circle around zero in $0 < |z| < \pi$. If $z_0 = k\pi$, then

$$f(z) = \frac{1}{z-k\pi} + \sum_{n=0}^{\infty} a_n (z-k\pi)^n, \quad |k|\pi < |z-k\pi| < (|k|+1)\pi.$$

8.2. Example. We know that the coefficient of z^k in $(1+z)^n$ is $\binom{n}{k}$. So, we may write

$$\begin{aligned}\binom{n}{k} &= \text{coefficient of } z^{-1} \text{ in } (1+z)^n/z^{k+1}\\ &= \text{Res}\,[f(z);0],\quad f(z) = (1+z)^n/z^{k+1},\\ &= \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz,\end{aligned}$$

where C is any simple closed contour enclosing the origin (note that f is analytic for all $z \in \mathbb{C}\setminus\{0\}$). Similarly, we also see that

$$\begin{aligned}\binom{n}{k} &= \text{coefficient of } z^{-k} \text{ in } (1+1/z)^n\\ &= \text{constant term in } z^k(1+1/z)^n\end{aligned}$$

and therefore, we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}^2 &= \text{coefficient of } z^{-1} \text{ in } \left(\frac{(1+z)^n}{z^{k+1}}\right)[z^k(1+1/z)^n] \\
&= \frac{1}{2\pi i}\int_C F(z)\,dz, \quad F(z) = \frac{(1+z)^{2n}}{z^{n+1}}, \\
&= \text{Res}\,[F(z);0] \\
&= \text{coefficient of } z^n \text{ in } (1+z)^{2n} \\
&= \binom{2n}{n}.
\end{aligned}$$

●

We wish to point out that the most convenient way to find the residue is directly from the Laurent expansion (if it were already available). We want actually to develop techniques for calculating the residue of a function f without having to find its Laurent expansion. However if z_0 is an essential singularity of f, then, in most of cases, the Laurent expansion of f about z_0 will be needed in order to find the residue at z_0. For instance, at the essential singular point $z = 0$ of $\sin(1/z)$ we have

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \cdots, \quad |z| > 0.$$

Thus, we see that

$$\text{Res}\,[\sin(1/z);0] = 1, \quad \text{i.e.} \quad \int_C \sin(1/z)\,dz = 2\pi i.$$

Similarly, we obtain

$$\text{Res}\,[z^3\sin(1/z^2);0] = 0, \quad \text{i.e.} \quad \int_C z^3\sin(1/z^2)\,dz = 0,$$

where C is any circle around the origin in the punctured plane, $\mathbb{C}\backslash\{0\}$. The same idea may be used to derive examples of this type (using Taylor's series expansion of the exponential function):

(a) $\text{Res}\,[z^{-4}e^{iaz};0] = -\dfrac{ia^3}{3!}$, $a \in \mathbb{C}$.

(b) If $a \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$$\text{Res}\,[e^{az}/z^{n+1};0] = \frac{a^n}{n!} \quad \text{and} \quad \int_{|z|=r} \frac{e^{az}}{z^{n+1}}\,dz = \frac{2\pi i a^n}{n!}.$$

If a is real and $r = 1$, then we let $z = e^{i\theta}$ and easily obtain

$$\int_0^{2\pi} e^{a\cos\theta}e^{i(a\sin\theta - n\theta)}\,d\theta = \frac{2\pi a^n}{n!}$$

which, by equating real and imaginary parts, gives

$$\int_0^{2\pi} e^{a\cos\theta}\sin(n\theta - a\sin\theta)\,d\theta = 0$$

and

$$\int_0^{2\pi} e^{a\cos\theta}\cos(n\theta - a\sin\theta)\,d\theta = \frac{2\pi a^n}{n!}.$$

(c) $\displaystyle\frac{1}{2\pi i}\int_C e^{1/z^n}\,dz = \text{Res}\,[f(z);0] = \begin{cases} 0 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}$, where C is any circle around zero in the punctured plane $\mathbb{C}\backslash\{0\}$.

(d) $\text{Res}\,[(z-2)^{-4}\sin(z-2);2] = -1/6!$ and $\text{Res}\,[\sin(1/(z-2));2] = 1$.

8.3. Example. Define

$$f(z) = \frac{\sin(z^2)}{z^2(z-a)}, \quad a \neq 0.$$

Then the singularities of f are at $z = 0$ and $z = a$. Since $\lim_{z\to 0} f(z) = -1/a$, $z = 0$ is a removable singularity of f. Thus, $\text{Res}\,[f(z);0] = 0$. As f admits a Laurent series expansion about zero in $0 < |z| < |a|$, we conclude that, for any r with $r < |a|$,

$$\int_{|z|=r} \frac{\sin(z^2)}{z^2(z-a)}\,dz = 0. \qquad \bullet$$

8.4. Remark. If f is analytic at z_0, then $\text{Res}\,[f(z);z_0] = 0$. However, this simple fact holds for other situations as well. For instance if

$$f_n(z) = b_n(z-z_0)^{-n}, \; n = 1, 2, \ldots, \text{ and } b_1 \neq 0,$$

then $\text{Res}\,[f_n(z);z_0] = 0$ for $n \geq 2$. Also, we observe that $\text{Res}\,[f_1(z);z_0] = b_1$ but $\text{Res}\,[f_1^2(z);z_0] \neq b_1^2$. $\bullet$

In view of Laurent's expansion and Cauchy's principle of deformation of contour, the following result is almost trivial.

8.5. Theorem. *If f has a removable singularity at z_0, then we have $\text{Res}\,[f(z);z_0] = 0$. In particular, if C is a simple closed contour containing only removable singularities at z_k $(k = 1, 2, \ldots, n)$ inside C, then $\int_C f(z)\,dz = 0$.*

For instance, using Theorem 8.5, we have

$$\text{Res}\,[(\cos z - 1)^2/z^2;0] = 0 \quad \text{and} \quad \text{Res}\,[z^2/\sin^2 z;0] = 0.$$

Note that the functions involved here are even. More generally, we have

8.6. Theorem. *If f has an isolated singularity at z_0 and if f is even in $z - z_0$, i.e. $f(z - z_0) = f(-(z - z_0))$, then* $\operatorname{Res}[f(z); z_0] = 0$.

Proof. Suppose that f is even in $z - z_0$. Then the Laurent series expansion around z_0 cannot have odd powers of $z - z_0$. Hence, the assertion follows. ∎

Consider the function f defined by

$$f(z) = \frac{1}{z^2(z^2 - a^2)} \quad (a \in \mathbb{R}\setminus\{0\}).$$

Then the point $z = 0$ is a double pole and $z = \pm a$ is a simple pole. Observe that $f(z) = f(-z)$ and hence, by Theorem 8.6, we have $\operatorname{Res}[f(z); 0] = 0$. This shows that the residue can be zero, even though f has a non-removable singularity at $z = 0$ (see Remark 8.4).

Similarly, $f(z) = e^{1/z^2}$ has an essential singularity at $z = 0$. Again, as f is even, we have $\operatorname{Res}[f(z); 0] = 0$. By Theorem 8.6, we easily obtain the following:

(i) $\operatorname{Res}[(\sin z)^{-2}; k\pi] = 0, \ k = 0, \pm 1, \pm 2, \dots \ (\sin^2(z - k\pi) = \sin^2 z)$
(ii) $\operatorname{Res}[1/\sin(z^2); 0] = 0$
(iii) $\operatorname{Res}[z^3 \sin(1/z); 0] = 0$
(iv) $\operatorname{Res}[(\sin z - z)/z^3; 0] = 0$
(v) $\operatorname{Res}[e^{-1/z^2} \cos(1/z); 0] = 0$
(vi) $\operatorname{Res}[1/[2 + z^2 - 2\cosh z]; 0] = 0$.

For evaluating residues in concrete examples, the following theorem is very useful.

8.7. Theorem. *If f has a pole of order n at z_0, then*

$$\operatorname{Res}[f(z); z_0] = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z)). \tag{8.8}$$

The coefficient of $(z - z_0)^{-k}$ in the Laurent expansion is

$$a_{-k} = \frac{1}{(n-k)!} \lim_{z \to z_0} \frac{d^{n-k}}{dz^{n-k}}((z - z_0)^n f(z)), \quad k = 1, 2, \dots, n.$$

(*Case $k = n$ means that we have only to look at $(z - z_0)^n f(z)$.*)

Proof. Suppose that f has a pole of order n at z_0. Then, we have

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (0 < |z - z_0| < \delta)$$

for some $\delta > 0$, where $a_{-n} \neq 0$. Further, $\lim_{z\to z_0}(z-z_0)^n f(z)$ exists and is non-zero. Note that for $z \neq z_0$,

$$(z-z_0)^n f(z) = a_{-n}+a_{-n+1}(z-z_0)+\cdots+a_{-1}(z-z_0)^{n-1}+\sum_{k=0}^{\infty} a_k(z-z_0)^{k+n}$$

and the formula (8.8) follows at once if we differentiate the last equation $(n-1)$-times and then allow $z \to z_0$. Alternatively, define g by

$$g(z) = \begin{cases} (z-z_0)^n f(z) & \text{for } 0 < |z-z_0| < \delta \\ \lim\limits_{z\to z_0}(z-z_0)^n f(z) & \text{for } z = z_0 \end{cases}$$

so that $f(z) = (z-z_0)^{-n} g(z)$, $0 < |z-z_0| < \delta$, where g is analytic in $|z-z_0| < \delta$ with $g(z_0) = a_{-n} \neq 0$. This observation, according to the Cauchy integral formula applied to $C = \{z : |z-z_0| = r\}$ for $0 < r < \delta$, implies

$$\begin{aligned} \text{Res}\,[f(z); z_0] &= \frac{1}{2\pi i}\int_C f(z)\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{g(z)}{(z-z_0)^n}\,dz \\ &= \frac{1}{(n-1)!} g^{(n-1)}(z_0) \end{aligned}$$

and the result follows at once. By a similar argument, we have the second part. ■

Theorem 8.7 is useful in the case of rational functions. For instance, consider the function

$$f(z) = \frac{z(z-2)}{(z+4)^2(z-1)^2},$$

which has a double pole at $z = 1$. Using (8.8), we obtain

$$\text{Res}\,[f(z); 1] = \frac{d}{dz}\left(\frac{z(z-2)}{(z+4)^2}\right)\bigg|_{z=1} = \frac{2}{125}.$$

At the double pole $z = -4$, we have

$$\text{Res}\,[f(z); -4] = \frac{d}{dz}\left(\frac{z(z-2)}{(z-1)^2}\right)\bigg|_{z=-4} = \frac{d}{dz}\left(\frac{-1}{(z-1)^2}\right)\bigg|_{z=-4} = \frac{2}{125}.$$

Similarly, we see that

$$\text{Res}\,[(z^3+7)(z-2)^{-3}; 2] = \frac{1}{2!}\frac{d^2}{dz^2}(z^3+7)\bigg|_{z=2} = 6.$$

8.9. Example. Consider the function $f(z) = e^{2z}/\cosh \pi z$. Since $\cos z = 0$ iff $z = (2k+1)\pi/2,\ k \in \mathbb{Z}$, and $\cos(iz) = \cosh z$, we have

$$\cosh \pi z = 0 \iff z = -\frac{(2k+1)i}{2}.$$

Thus, f has a simple pole at $z = -(2k+1)i/2$ ($k \in \mathbb{Z}$). From this we see that if $C = \{z : |z - i/2| = 1/2\}$, then

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}\left[\frac{e^{2z}}{\cosh \pi z}; \frac{i}{2}\right] = 2\pi i \left[\frac{e^i}{\pi \sinh(\pi i/2)}\right] = \frac{2\pi i e^i}{\pi i} = 2e^i.$$

Inside the contour $\gamma : |z - i/2| = 1$, f has a singularity only at $z = i/2$. Therefore, if C is any circle around $i/2$ lying completely inside γ then we have $\int_C f(z)\,dz = 2e^i$. ●

8.10. Example. Define

$$f(z) = \frac{(1+z^2)^{n+k}}{z^{2n+1}}.$$

Then f has a pole of order $2n+1$ at $z = 0$ and

$$\begin{aligned} \operatorname{Res}[f(z); 0] &= \text{coefficient of } z^{-1} \text{ in } f(z) \\ &= \text{coefficient of } z^{2n} \text{ in } (1+z^2)^{n+k} \\ &= \text{coefficient of } z^{n} \text{ in } (1+z)^{n+k} \\ &= \binom{n+k}{n}. \end{aligned}$$

On the other hand, by Theorem 8.6, we have $\operatorname{Res}[zf(z); 0] = 0$. ●

8.11. Example. Consider

$$f(z) = \frac{1}{(z^3-1)(z+1)^2}.$$

Then f has a double pole at -1 and simple poles at $1, \omega, \omega^2$, where ω is a cube root of unity. We easily see that

$$\operatorname{Res}[f(z); -1] = \lim_{z\to -1} \frac{d}{dz}\left(\frac{1}{z^3-1}\right) = \lim_{z\to -1} \frac{-3z^2}{(z^3-1)^2} = -\frac{3}{4}.$$

If a is any one of the cube roots of unity, then we have

$$\operatorname{Res}[f(z); a] = \lim_{z\to a} \frac{(z-a)}{z^3-1}(z+1)^{-2} = \frac{1}{3a^2(1+a)^2} = \frac{1}{3(2+a+a^2)}$$

so that $\text{Res}\,[f(z);1] = 1/12$, $\text{Res}\,[f(z);\omega] = \text{Res}\,[f(z);\omega^2] = 1/3$. •

From Theorem 8.7, we have a simple result for computing residues: "*if f has a simple pole at $z = z_0$, then we have* $\text{Res}\,[f(z);z_0] = \lim_{z\to z_0}(z-z_0)f(z)$." From this observation, we also obtain another result which is also extremely useful in practice.

8.12. Theorem. *If f has a simple pole at $z = z_0$ and if h is analytic at z_0 with $h(z_0) \neq 0$, then* $\text{Res}\,[f(z)h(z);z_0] = h(z_0)\text{Res}\,[f(z);z_0]$.

Proof. Observe that

$$\lim_{z\to z_0}(z-z_0)f(z)h(z) = h(z_0)\lim_{z\to z_0}(z-z_0)f(z)$$

and the result follows immediately. ■

Recall that, f has simple pole at z_0 iff $g(z) = 1/f(z)$ has simple zero at z_0. Thus, in this case (since $g(z_0) = 0$ and $g'(z_0) \neq 0$),

$$\text{Res}\,[f(z);z_0] = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{z-z_0}{g(z)} = \frac{1}{g'(z_0)}.$$

Hence we have

8.13. Theorem. *Suppose ϕ is analytic at z_0 with $\phi(z_0) \neq 0$ and g has a simple zero at z_0. Then* $\text{Res}\,[\phi(z)/g(z);z_0] = \phi(z_0)/g'(z_0)$. *In particular,* $\text{Res}\,[1/g(z);z_0] = 1/g'(z_0)$.

Consider the function

$$f(z) = \frac{\phi(z)}{a^n + z^n} \quad (a \neq 0,\ n \geq 1),$$

where ϕ is analytic at z_0 such that $\phi(z_0) \neq 0$ for each $z_0 \in \mathbb{C}$ satisfying $z_0^n + a^n = 0$. Then, using Theorem 8.13 (since f has simple poles), we have

$$\text{Res}\,[f(z);z_0] = \frac{\phi(z_0)}{nz_0^{n-1}} = \frac{z_0\phi(z_0)}{nz_0^n} = -\frac{z_0\phi(z_0)}{na^n}.$$

In particular if $\phi(z) = 1$, then

$$\text{Res}\,[(z^n + a^n)^{-1};z_k] = -\frac{z_k}{na^n}$$

where z_k $(k = 1, 2, \ldots, n)$ is the simple pole of $1/(z^n + a^n)$. If $\phi(z) = z^{n-1}$, then

$$\text{Res}\left[\frac{z^{n-1}}{z^n + a^n};z_k\right] = \frac{1}{n}, \quad a \neq 0.$$

As the order of poles increases, the formula for the residue becomes more complicated. However, at a double pole, we have the following result which can be proved easily.

8.14. Theorem. *Suppose ϕ is analytic at z_0 with $\phi(z_0) \neq 0$, g has a pole of order two at z_0 and h has a zero of order two at z_0. Then we have*

(i) $\text{Res}\,[\phi(z)g(z); z_0] = \phi'(z_0)\text{Res}\,[(z-z_0)g(z); z_0] + \phi(z_0)\text{Res}\,[g(z); z_0]$

(ii) $\text{Res}\left[\dfrac{\phi(z)}{h(z)}; z_0\right] = \left.\dfrac{6\phi'(z)h''(z) - 2\phi(z)h'''(z)}{3[h''(z)]^2}\right|_{z=z_0}.$

Proof. By hypothesis, there exist r_1, r_2 such that

$$\begin{aligned} \phi(z) &= a_0 + a_1(z-z_0) + \cdots \quad \text{for } |z-z_0| < r_1, \\ g(z) &= \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{z-z_0} + b_0 + b_1(z-z_0) + \cdots \end{aligned}$$

for $0 < |z-z_0| < r_2$, respectively. The coefficient of $(z-z_0)^{-1}$ in $\phi(z)g(z)$ is $a_0 b_{-1} + a_1 b_{-2}$ and therefore

$$\begin{aligned} \text{Res}\,[\phi(z)g(z); z_0] &= a_0 b_{-1} + a_1 b_{-2} \\ &= \phi(z_0)\text{Res}\,[g(z); z_0] + \phi'(z_0)\text{Res}\,[(z-z_0)g(z); z_0] \end{aligned}$$

and the second part follows similarly. ∎

8.15. Example. Consider $f(z) = \phi(z)/g(z)$, with $\phi(z) = 1 + z$ and $g(z) = \cos z - 1$. Then the singularities of f are given by $g(z) = 0$. Since $\cos z = 1 \iff z = 2k\pi$ $(k \in \mathbb{Z})$, the singularities occur at $z_0 = 2k\pi$, $k \in \mathbb{Z}$. Further, we note that

$$g'(z) = -\sin z \quad \text{and} \quad g''(z) = -\cos z$$

and, since $\sin z = 0 \iff z = n\pi$ $(n \in \mathbb{Z})$, we see that

$$g'(2\pi k) = 0 \quad \text{and} \quad g''(2\pi k) \neq 0 \text{ for } k \in \mathbb{Z}.$$

Consequently, g has a double zero at $z_0 = 2\pi k$ for $k \in \mathbb{Z}$. In other words, for each such k, f has a double pole at $z_0 = 2\pi k$. Since ϕ and g considered above are analytic at z_0 with $g(z_0) = 0 = g'(z_0)$ and $g''(z_0) \neq 0$, we compute that

$$\text{Res}\,[f(z); z_0] = \left.\frac{6\phi'(z)g''(z) - 2\phi(z)g'''(z)}{3[g''(z)]^2}\right|_{z=z_0} = -\frac{2}{\cos z_0} = -2.$$

Similarly, $\text{Res}\,\left[e^z(z-z_0)^{-2}; z_0\right] = e^{z_0}$. ●

8.16. Example. As an immediate application of Theorem 8.13, we obtain (a) to (g) in the following:

(a) Since $\sin z$ has simple zeros at $z = k\pi$, $k \in \mathbb{Z}$,

$$\mathrm{Res}\left[\frac{\cos z}{\sin z}; k\pi\right] = \mathrm{Res}\left[\frac{\pi \cos \pi z}{\sin \pi z}; k\right] = \frac{\cos k\pi}{\cos k\pi} = 1.$$

Similarly, we have $\mathrm{Res}\,[\cosh z \cot z; k\pi] = \cosh k\pi$.

(b) $\mathrm{Res}\,[\csc z; k\pi] = \mathrm{Res}\,[\pi \csc \pi z; k] = (-1)^k,\ k \in \mathbb{Z}$.

(c) $\mathrm{Res}\,[e^z/\sin z; k\pi] = \dfrac{e^{k\pi}}{\cos k\pi},\ k \in \mathbb{Z}$.

(d) For $k \in \mathbb{Z}$, $\mathrm{Res}\left[\dfrac{\sinh z}{\cosh z}; (2k+1)\pi i/2\right] = \dfrac{\sinh((2k+1)\pi i/2)}{\sinh((2k+1)\pi i/2)} = 1$, where $\cosh z$ has simple zeros at $z = (2k+1)\pi i/2$.

(e) For $z_0 \neq z_0'$, $\mathrm{Res}\left[\dfrac{e^z}{(z-z_0)(z-z_0')}; z_0\right] = \dfrac{e^{z_0}}{z_0 - z_0'}$.

(f) $\mathrm{Res}\,[f(z); 1] = -e$ and $\mathrm{Res}\,[f(z); 0] = e - 1$, where $f(z) = \dfrac{e^{1/z}}{1-z}$.

(g) For the function $f(z) = \pi \cot \pi z/z^2$, we have (using Theorem 8.7) $\mathrm{Res}\,[f(z); k] = 1/k^2$ for $k \in \mathbb{Z}\setminus\{0\}$ and $\mathrm{Res}\,[f(z); 0] = -\pi^2/3$, because f has a pole of order 3 at $z = 0$ and a simple pole at k, $k \in \mathbb{Z}\setminus\{0\}$. ●

8.17. Example. Consider $f(z) = (1 + z + z^2 + z^3)^{-1}$. As

$$1 - z^4 = (1-z)(1 + z + z^2 + z^3),$$

f has a simple pole at $c_k = e^{k\pi i/2},\ k = 1, 2, 3$. So (see Theorem 8.13),

$$\mathrm{Res}\,[f(z); c_k] = \left.\frac{1-z}{-4z^3}\right|_{z=c_k} = -\left.\frac{z(1-z)}{4}\right|_{z=c_k} = -\frac{c_k(1-c_k)}{4}, \quad k = 1, 2, 3.$$

A simplification gives

$$\mathrm{Res}\,[f(z); e^{k\pi i/2}] = -\frac{e^{k\pi i/2}(1 - e^{k\pi i/2})}{4} = i\frac{e^{3k\pi i/4}}{2}\sin\left(\frac{k\pi}{4}\right).$$ ●

8.18. Example. Consider the function $f(z) = (z \sin z)^{-1}$. Then f has a double pole at $z = 0$ and a simple pole at $z = k\pi$, $k \in \mathbb{Z}\setminus\{0\}$. Note that f is even. By Theorem 8.6, we immediately have $\mathrm{Res}\,[f(z); 0] = 0$. Since the nearest singularity of f to zero is at $\pm\pi$, we obtain

$$\int_C \frac{dz}{z \sin z} = 0$$

for any circle C centered at zero and radius less than π.

Similarly, we see that the function $f(z) = (z^2 \sin z)^{-1}$ has a triple pole at $z = 0$ and simple pole at $z = k\pi$, $k \in \mathbb{Z} \setminus \{0\}$. Since

$$f(z) = \frac{1}{z^3}\left[1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)\right]^{-1} = \frac{1}{z^3} + \frac{1}{3!}\frac{1}{z} + \cdots \quad \text{for } |z| > 0,$$

we obtain

$$\operatorname{Res}\left[\frac{1}{z^2 \sin z}; 0\right] = \frac{1}{6}; \quad \text{i.e.} \int_{|z|=r<\pi} \frac{dz}{z^2 \sin z} = \frac{\pi i}{3}. \qquad \bullet$$

8.19. Theorem. *If ϕ and g are analytic at z_0, where ϕ has a zero of order m at z_0 and g has a zero of order n at z_0. Then*

(a) *ϕ/g has a removable singularity at z_0 if $m \geq n$*
(b) *ϕ/g has a pole of order $n - m$ at z_0 if $m < n$.*

Proof. By hypotheses, there exists a disk $|z - z_0| < R$ such that

$$\phi(z) = \sum_{k=m}^{\infty} \frac{\phi^{(k)}(z_0)}{k!}(z - z_0)^k, \quad \text{and} \quad g(z) = \sum_{k=n}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Since zeros are isolated, we have $g(z) \neq 0$ in $0 < |z - z_0| < \delta$ for some $\delta < R$. This observation shows that

(i) $m > n \Longrightarrow \lim_{z \to z_0} \frac{\phi(z)}{g(z)} = 0$
(ii) $m = n \Longrightarrow \lim_{z \to z_0} \frac{\phi(z)}{g(z)} = \frac{\phi^{(m)}(z_0)}{g^{(n)}(z_0)}$
(iii) $m < n \Longrightarrow \lim_{z \to z_0} \frac{\phi(z)}{g(z)} = \infty$

from which the required conclusions follow. ∎

8.20. Example. Take $\phi(z) = \sin z$ and $g(z) = (1 - e^z)^2$. Then $\phi(0) = 0$, $\phi'(0) = 1$, $g(0) = g'(0) = 0$ and $g''(0) = -2$. This implies that the function f defined by $f(z) = \phi(z)/g(z)$ has a simple pole at the origin. Therefore, $\operatorname{Res}[f(z); 0] = \lim_{z \to 0} \frac{z \sin z}{(1 - e^z)^2} = 1$. •

8.21. Example. We wish to construct a function $f(z)$ which has the following properties:

(i) The only singularities of $f(z)$ in $\mathbb{C}_\infty$ are poles of order 1 and 2 at $z = 1$ and $z = -1$, respectively.
(ii) $f(0) = 0 = f(-1/2)$ and $\operatorname{Res}[f(z); 1] = 1 = \operatorname{Res}[f(z); -1]$.

To do this, by hypotheses, we observe that the principal part of $f(z)$ has the form

$$P(z) = \frac{1}{z-1} + \frac{1}{z+1} + \frac{b}{(z+1)^2}$$

so that $F(z) = f(z) - P(z)$ extends to be analytic in $\mathbb{C}$. As f has no other singularities in $\mathbb{C}_\infty$, f is analytic at ∞ and so, $F(z)$ is analytic at ∞. Thus, by Liouville's theorem, $F(z)$ is constant. Hence, $f(z) = P(z) + a$ for some constant a. Finally, as $f(0) = 0 = f(-1/2)$, it follows that

$$a + b = 0 \quad \text{and} \quad 4b + a = -4/3.$$

Solving these equations imply that $b = -a = -4/9$. •

The final example of this section relates to finding the residue of a branch of a multi-valued function.

8.22. Example. Consider the function

$$f(z) = (1+z^2)^{-2}\,\mathrm{Log}\,(1+z).$$

Note that $\mathrm{Log}\,(1+z)$ is analytic in $\mathbb{C}\backslash(-\infty,-1]$ and f has a pole of order 2 at $z = i$ and $-i$. Hence,

$$\begin{aligned}
\mathrm{Res}\,[f(z);-i] &= \lim_{z\to -i} \frac{d}{dz}[(z+i)^2 f(z)] \\
&= \lim_{z\to -i} \frac{d}{dz}\left[\frac{\mathrm{Log}\,(z+1)}{(z-i)^2}\right] \\
&= \lim_{z\to -i} \left[\frac{1}{1+z}\cdot\frac{1}{(z-i)^2} + \mathrm{Log}\,(1+z)\cdot\left(\frac{-2}{(z-i)^3}\right)\right] \\
&= \frac{1}{1-i}\cdot\frac{1}{(-2i)^2} + \mathrm{Log}\,(1-i)\cdot\left(\frac{-2}{(-2i)^3}\right) \\
&= \frac{1+i}{2}\cdot\left(-\frac{1}{4}\right) + \left[\ln\sqrt{2} - i\frac{\pi}{4}\right]\cdot\left(\frac{-2}{8i}\right) \\
&= \left[-\frac{1}{8} + \frac{\pi}{16}\right] + i\left[-\frac{1}{8} + \frac{\ln\sqrt{2}}{4}\right].
\end{aligned}$$

$\mathrm{Res}\,[f(z);i]$ may similarly be computed. •

8.2 Residue at the Point at Infinity

First consider $z = 1/w$ and recall the geometric aspect of this transformation. If we set $z = Me^{-i\theta}$, then we have $w = M^{-1}e^{i\theta}$. This shows that as z describes the circle $|z| = M$ in the z-plane in the clockwise direction, w describes the circle $|w| = 1/M$ in the w-plane in the anti-clockwise direction.

Thus, the point $z_0 = \rho e^{-i\theta}$ $(\rho > M)$ outside the circle $|z| = M$ corresponds to a point $w_0 = \rho^{-1}e^{i\theta}$ inside the circle $|w| = 1/M$.

Let f be analytic in a deleted neighborhood of the point at infinity. Then f admits a Laurent series expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad (R < |z| < \infty), \quad a_k = \frac{1}{2\pi i}\int_{C^+} \frac{f(z)}{z^{k+1}}\,dz,$$

with $R > 0$ sufficiently large. Here C^+ denotes the positively oriented circle $|z| = M$, where $M > R$ and M is sufficiently large so that a finite number of singularities in $\mathbb{C}$ will be inside C_0, where $\operatorname{int} C_0 \subset \operatorname{int} C_R = \{z : |z| < R\}$. Define

$$C^- = \{z : |z| = M > R,\ M \ \text{ is sufficiently large}\}$$

where C^- is traversed in the clockwise direction (so that the point at infinity is to the left of C^- as in the case of a finite point). Put $z = Me^{-i\theta}$. Then it follows that (since the termwise integration is permissible)

$$\begin{aligned}
\int_{C^-} f(z)\,dz &= \sum_{k=-\infty}^{\infty} a_k \int_{C^-} z^k\,dz \\
&= -\sum_{k=-\infty}^{\infty} i a_k M^{k+1} \int_0^{2\pi} e^{-i(k+1)\theta} d\theta \\
&= -i a_{-1} \int_0^{2\pi} d\theta = -2\pi i a_{-1}.
\end{aligned}$$

In view of this reasoning, it is natural to define

$$\operatorname{Res}[f(z); \infty] = \frac{1}{2\pi i}\int_{C^-} f(z)\,dz = -a_{-1}.$$

In other words, $\operatorname{Res}[f(z); \infty]$ is the negative of the coefficient of $1/z$ in the Laurent series expansion of $f(z)$ with center at the point at infinity. One should be alerted that a_{-1} here is neither the residue of f at infinity nor the residue of f at $z = 0$. Further, we observe that for $z = Me^{-i\theta}$ with $M = 1/R'$,

$$\begin{aligned}
\operatorname{Res}[f(z); \infty] &= \frac{1}{2\pi i}\int_{C^-} f(z)\,dz \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} f(Me^{-i\theta}) i M e^{-i\theta} d\theta \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} f\left(\frac{1}{R'e^{i\theta}}\right) \frac{d(R'e^{i\theta})}{(R'e^{i\theta})^2} \\
&= -\frac{1}{2\pi i}\int_C f\left(\frac{1}{w}\right)\frac{dw}{w^2} \\
&= -\operatorname{Res}\left[\frac{f(1/w)}{w^2}; 0\right],
\end{aligned}$$

where $C = \{w : |w| = 1/R'\}$ is described in the anti-clockwise (positive) direction. Alternatively, if we replace z by $1/z$, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{k=-\infty}^{\infty} \frac{a_k}{z^{k+2}} = \sum_{k=-\infty}^{\infty} \frac{a_{k-2}}{z^k} \quad (0 < |z| < 1/R)$$

so that

$$\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right] = a_{-1} = \text{coefficient of } z^{-1} \text{ in } \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

and hence, in both methods, we quickly have

$$\text{Res}\,[f(z); \infty] = -\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right]. \tag{8.23}$$

Here is an alternate proof of Liouville's Theorem (Theorem 6.55). Let $f(z)$ be a bounded entire function. Then $f(z)$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$, so that g defined by

$$g(z) = f(1/z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

is analytic for all $|z| > 0$. Since $f(z)$ is bounded in $\mathbb{C}$, $g(z)$ is bounded in a deleted neighborhood of 0, and so $g(z)$ has a removable singularity at 0. Therefore, $a_n = 0$ for all $n > 0$. Thus, $f(z) = a_0$ is a constant.

8.24. Example. Consider the function $f(z) = 1 + z^{-1}$, $z \neq 0$. Then

$$F(w) = f(1/w) = 1 + w \ (w \neq 0), \quad \text{and} \quad \lim_{w \to 0} F(w) = 1.$$

Thus, $F(w)$ has a removable singularity at $w = 0$ and therefore, the point at infinity is a removable singularity of $f(z)$. Further, $\text{Res}\,[f(z); \infty] = -1$. From this we also observe that if f has a removable singularity at the point at infinity, then the residue of f at ∞ may prove to be non-zero in contrast to the case when f has a removable singularity at a finite point. ●

8.3 Residue Theorem

The effectiveness of the residue theorem depends, of course, on how effectively we can evaluate residues at various singularities. However, caution must be exercised to avoid reaching a hasty conclusion based on appearances. Having identified the type of singularities, we have to choose a proper contour. Most often the following theorem will be applied in the next chapter to evaluate different types of line integrals.

8.25. Theorem. (Cauchy's Residue Theorem) *If f is analytic in a domain D except for isolated singularities at $a_1, a_2, \dots, a_n$, then, for any closed contour γ in D on which none of the points a_k lie, we have*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} n(\gamma; a_k)\mathrm{Res}\,[f(z); a_k].$$

Proof. Since γ does not pass through any of a_j's, we can choose numbers $\delta_1, \delta_2, \dots, \delta_n$ so small that

(i) for every $j = 1, 2, \dots, n$ no two circles $\gamma_j : |z - a_j| = \delta_j$ intersect,
(ii) every circle γ_j $(j = 1, 2, \dots, n)$ lies inside γ.

Since a_j is an isolated singularity of f, f admits a Laurent series expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(j)}(z - a_j)^n, \quad 0 < |z - a_j| \le \delta_j, \tag{8.26}$$

for each $j = 1, 2, \dots, n$. We denote the principal part of $f(z)$ at each of these isolated singularities by

$$p_j(z) = \sum_{n=-\infty}^{-1} a_n^{(j)}(z - a_j)^n.$$

Then for each j, function p_j is analytic on and outside the circle γ_j (see Theorems 4.117 and 4.139).

Since a_j lies inside γ and p_j converges uniformly on γ_j, we have

$$\begin{aligned}
\int_\gamma p_j(z)\,dz &= a_{-1}^{(j)} \int_\gamma \frac{dz}{z - a_j} \quad (a_j \text{ lies inside } \gamma) \\
&= 2\pi i a_{-1}^{(j)} n(a_j; \gamma) \\
&= 2\pi i n(a_j; \gamma)\ \mathrm{Res}\,[f(z); a_j], \quad j = 1, 2, \dots, n.
\end{aligned}$$

If we subtract the principal parts $p_1(z), p_2(z), \dots, p_n(z)$ from f, it follows that the function g defined by the difference

$$g(z) = f(z) - \sum_{j=1}^{n} p_j(z) \tag{8.27}$$

is analytic on $D \setminus \{a_1, a_2, \dots, a_n\}$. It follows that all the a_j's are removable singularities of g and, by Cauchy's theorem (see also Theorem 8.5), we have $\int_\gamma g(z)\,dz = 0$. Consequently, by (8.27),

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_\gamma p_k(z)\,dz = 2\pi i \sum_{k=1}^{n} n(a_k; \gamma)\ \mathrm{Res}\,[f(z); a_k]. \qquad \blacksquare$$

In most of the applications γ will be a simple closed contour with positive orientation and hence, in these cases (see Section 4.5),

$$n(\gamma; a_k) = \begin{cases} 0 & \text{if } a_k \text{ is in the unbounded component of } \mathbb{C}\backslash\{\gamma\} \\ 1 & \text{if } a_k \text{ is inside } \gamma. \end{cases}$$

Thus, the residue formula becomes elegant for simple closed contours. Thus if $\gamma \subset D$ is a simple closed contour with positive orientation, then under the hypotheses of Theorem 8.25 we have the following simple form.

8.28. Theorem. (Residue Formula)

$$\int_\gamma f(z)\,dz = 2\pi i \sum \operatorname{Res}\,[f(z); a_k].$$

Here the sum is taken over all a_k's inside γ.

The proof of this special case is trivial because for the simple closed contour γ in D, by Cauchy's principle of deformation of contour, we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\,[f(z); a_k].$$

The general case can also be proved in the same spirit.

The Cauchy integral formula (see Theorem 4.63) can be considered as a special case of the residue theorem. Indeed, if f is analytic in D and $a \in D$, then g defined by

$$g(z) = \frac{f(z)}{z-a}$$

is analytic in $D\backslash\{a\}$ and has the residue $f(a)$ at the simple pole a, by Theorem 8.13. In fact, the Cauchy integral formula for higher order derivatives can also be deduced as a special case of Theorem 8.25.

8.29. Remark. In Theorem 8.25, $f(z)$ can have only a finite number of singularities, because otherwise singularities of $f(z)$ would have a limit point ζ (possibly at the point at infinity), and so ζ would not be an isolated singularity of $f(z)$, contrary to our assumption. ●

8.30. Example. Let us evaluate

$$I_\alpha = \int_{|z|=1} \frac{\operatorname{Re} z}{z-\alpha}\,dz, \quad |\alpha| \neq 1.$$

For $|z| = 1$, we have $\operatorname{Re} z = \left(z + z^{-1}\right)/2$ so that

$$I_\alpha = \frac{1}{2} \int_{|z|=1} f(z)\,dz,$$

where

$$f(z) = \frac{z^2+1}{z(z-\alpha)} = \begin{cases} \dfrac{z^2+1}{z^2} & \text{if } \alpha = 0 \\ \dfrac{z^2+1}{\alpha}\left(\dfrac{1}{z-\alpha} - \dfrac{1}{z}\right) & \text{if } \alpha \neq 0. \end{cases}$$

To compute the integral, we may either use the Cauchy integral formula or the residue theorem. Note that $\text{Res}\,[f(z);0] = 0$ if $\alpha = 0$.Thus, $I_0 = 0$.

When $|\alpha| > 1$, the Cauchy integral formula gives that

$$I_\alpha = \frac{1}{2}\int_{|z|=1} \frac{(z^2+1)/(z-\alpha)}{z}\,dz = \frac{2\pi i}{2}\left(-\frac{1}{\alpha}\right) = -\frac{\pi i}{\alpha}.$$

When $0 < |\alpha| < 1$, we see that

$$\text{Res}\,[f(z);0] = -\frac{1}{\alpha} \quad\text{and}\quad \text{Res}\,[f(z);\alpha] = \frac{\alpha^2+1}{\alpha},$$

and so, if $0 < |\alpha| < 1$ then the Cauchy residue theorem gives

$$I_\alpha = \frac{2\pi i}{2}\{\text{Res}\,[f(z);0] + \text{Res}\,[f(z);\alpha]\} = \pi\alpha i.$$

Similarly, for $|\alpha| \neq 1$, we can easily see that

$$\int_{|z|=1} \frac{\text{Im}\, z}{z-\alpha}\,dz = \begin{cases} 0 & \text{if } \alpha = 0 \\ \pi/\alpha & \text{if } |\alpha| > 1 \\ \pi\alpha & \text{if } 0 < |\alpha| < 1. \end{cases}$$ ●

8.31. Example. For $|a| \neq R$ and $C = \{z : |z| = R\}$, we wish to show that

$$I = \int_C \frac{|dz|}{|z-a|^2} = \frac{2\pi R}{|R^2 - |a|^2|}.$$

For a proof, we let $z = Re^{i\theta}$. Then $dz = iz\,d\theta$, $|dz| = Rd\theta = R\,dz/iz$ and

$$|z-a|^2 = (z-a)(\overline{z}-\overline{a}) = z^{-1}(z-a)(R^2 - \overline{a}z)$$

so that I takes the form

$$I = \frac{R}{i}\int_C f(z)\,dz, \quad f(z) = \frac{1}{(z-a)(R^2-\overline{a}z)}.$$

For $a = 0$, the result is trivial. For $a \neq 0$, f has two simple poles at $z = a$ and $z = R^2/\overline{a}$. If one of them lies inside the circle $|z| = R$, then the other lies outside. Finally, the result follows from the residue theorem. ●

Combining the residue at the point at infinity and Theorem 8.25, we have the "*Residue Formula for the Extended Complex Plane*" as follows:

8.32. Theorem. (Extended Residue Formula) *Let f be analytic in $\mathbb{C}$ except for isolated singularities at $a_1, a_2, \ldots, a_n$. Then we have*

(i) *the sum of all residues (including the residue at infinity) of f is zero. Equivalently (by Theorem 8.28 and equation (8.23)), we write*

$$\mathrm{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right] = \sum_{k=1}^{n}\mathrm{Res}\,[f(z);a_k].$$

(ii) *if γ is a simple closed contour in $\mathbb{C}$ such that all a_k's are interior to γ, then*

$$\int_\gamma f(z)\,dz = 2\pi i\mathrm{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right].$$

The extended residue formula can be used to give another simple proof of Liouville's theorem (see Theorem 6.55).

8.33. Proof of Liouville's Theorem (see Theorem 6.55). Suppose that f is entire and bounded in $\mathbb{C}$. Choose two distinct points, say 0 and a in $\mathbb{C}$, and consider the function

$$F(z) = \frac{f(z)}{z(z-a)}.$$

Then $F(z)$ has singularities at $z = 0, a$ and possibly at the point at infinity. Since $\lim_{|z|\to\infty} zF(z) = 0$ (as f is bounded in $\mathbb{C}$), $\mathrm{Res}\,[F(z);\infty] = 0$. Note also that

$$\mathrm{Res}\,[F(z);0] = -\frac{f(0)}{a}, \quad \text{and} \quad \mathrm{Res}\,[F(z);a] = \frac{f(a)}{a}.$$

In view of Theorem 8.32, we have

$$\mathrm{Res}\,[F(z);0] + \mathrm{Res}\,[F(z);a] + \mathrm{Res}\,[F(z);\infty] = -\frac{f(0)}{a} + \frac{f(a)}{a} = 0$$

which proves $f(a) = f(0)$ for each $a \in \mathbb{C}$. Hence, f must be constant. ■

On most occasions, calculation of the residues at many isolated singularities of the integrand is quite difficult. In this situation, we can use Theorem 8.32 to evaluate certain contour integrals. Now, we demonstrate this advantageous situation by a number of examples. Consider

$$f(z) = \frac{z^{21}}{(z^2-1)^4(z^4-2)^3}.$$

Then we see that all the singularities of f lie inside the circle $|z| = 3$. Note that f has a simple pole at infinity with $\mathrm{Res}\,[f(z);\infty] = -4$. Consequently,

$$\int_{|z|=3} f(z)\,dz = -2\pi i\,\mathrm{Res}\,[f(z);\infty] = 8\pi i.$$

8.34. Example. Let us evaluate the integral

$$I = \frac{1}{2\pi i}\int_{|z|=R} f(z)\,dz, \quad f(z) = \frac{z^{2n+3m-1}}{(z^2+a)^n(z^3+b)^m},$$

where $a, b \in \mathbb{C}\backslash\{0\}$, $R > \max\{\sqrt{|a|}, |b|^{1/3}\}$ and m and n are fixed positive integers. First we note that f has poles of order n at the zeros of z^2+a, say z_1 and z_2, and poles of order m at the zeros of z^3+b, say z_3, z_4 and z_5. By the conditions on a, b and R, these poles lie inside the circle $|z| = R$. Therefore, by the residue theorem,

$$I = \sum_{j=1}^{5} \text{Res}\,[f(z); z_j].$$

As the calculation of residues at these poles is quite difficult, to complete the solution, we make use of Theorem 8.32. According to this, $I + \text{Res}\,[f(z); \infty] = 0$. Since

$$-\text{Res}\,[f(z); \infty] = \text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right] = \text{Res}\left[\frac{1/z}{(1+az^2)^n(1+bz^3)^m}; 0\right] = 1,$$

this gives $I = 1$. •

The occasional short cut method used in the above two examples should not be missed.

8.4 Number of Zeros and Poles

Before we consider some useful consequences of Theorem 8.32, let us define the change in $\arg f(z)$ as z goes around C. This is denoted by $\Delta_C \arg f(z)$. Let f be analytic inside and on a simple closed contour C except possibly for poles inside C and $f(z) \neq 0$ on C. As z describes C once in the positive direction in the z-plane, the image point $w = f(z)$ describes a closed curve $\Gamma = f(C)$ in the w-plane in a particular direction which determines the orientation of the image curve Γ. Since $f(z) \neq 0$ on C, Γ never passes through the origin in the w-plane. Let w_0 be an arbitrary fixed point on Γ and let ϕ_0 be a value of the argument of w_0. Then, let $\arg w$ run continuously from ϕ_0, as the point begins at w_0 and traverses Γ once in the direction of orientation assigned to it by $w = f(z)$. If w returns to the starting point w_0, then $\arg w$ assumes a particular value of $\arg w_0$ which we denote by ϕ_1. We define

$$\Delta_C \arg f(z) = \phi_1 - \phi_0.$$

Note that the difference $\phi_1 - \phi_0$ is independent of the choice of the starting point w_0. Further, we also note that the difference $\phi_1 - \phi_0$ is an integral multiple of 2π and the integer $(\phi_1 - \phi_0)/2\pi$, denoted by $n(\Gamma; 0)$, is the winding number of Γ around the origin in w-plane as z describes C once in the positive direction (see 4.5). If $n(\Gamma; 0) = -1$, then Γ winds around the origin once in the clockwise direction. If Γ does not enclose the origin, then it is obvious that $n(\Gamma; 0) = 0$.

For instance, consider the special case when

$$f(z) = z^n \ (n \in \mathbb{N})$$

where $C = \{z : z = e^{i\theta}, \ 0 \le \theta \le 2\pi\}$. The function f has a zero of order n at $z = 0$. Then, $\Gamma = \{w : w = e^{in\theta}, \ 0 \le \theta \le 2\pi\}$ which is the circle traversed n-times and so, we may decompose Γ as

$$\Gamma = \cup_{j=1}^{n} \Gamma_j, \quad \Gamma_j = \left\{ w : w = e^{in\theta}, \ \frac{2(j-1)\pi}{n} \le \theta \le \frac{2j\pi}{n} \right\}.$$

Note that each Γ_j is described in the positive direction and has the origin in its interior. Hence, we find that

$$n(\Gamma; 0) = \frac{1}{2\pi} \Delta_C \arg f(z) = n = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz.$$

8.35. Theorem. *If f has a zero of order m at $z = a$,*

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}; a\right] = m.$$

Proof. If f has a zero of order m at $z = a$ then $f(z) = (z-a)^m g(z)$, where g is analytic at $z = a$ and $g(a) \neq 0$. It follows that in a deleted neighborhood of a

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{(z-a)^{m-1}[mg(z) + (z-a)g'(z)]}{(z-a)^m g(z)} \\ &= \frac{m}{z-a} + \frac{g'(z)}{g(z)}, \ \ 0 < |z-a| < \delta, \text{ for some } \delta. \end{aligned}$$

As g'/g is analytic at $z = a$, the conclusion follows. ■

A proof analogous to that of the above theorem shows that if f has a pole of order n at $z = b$, then near $z = b$ we have

$$\frac{f'(z)}{f(z)} = -\frac{n}{z-b} + \text{an analytic function at } z = b.$$

Hence at each pole $z = b$ of f, f'/f has a simple pole at $z = b$ with residue equal to $-n$.

One of the important applications of Cauchy's residue theorem concerns the number of zeros and poles of meromorphic functions.

8.36. Theorem. (Argument Principle) *Let f be meromorphic in a domain $D \subseteq \mathbb{C}$ and have only finitely many zeros and poles in D. If C is a simple closed contour in D such that no zeros or poles of f lie on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P \tag{8.37}$$

where N and P denote, respectively, the number of zeros and poles of f inside C, each counted according to their order.

Proof. Define $F(z) = f'(z)/f(z)$. Then, the only possible singularities of F inside C are the zeros and poles of f. Therefore[13]

$$\frac{1}{2\pi i}\int_C F(z)\,dz = \sum \operatorname{Res}\,[F(z); C]. \tag{8.38}$$

If a_j is a zero of order n_j of f and if b_k is a pole of order p_k of f, then (see Theorem 8.35) it follows that

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}; a_j\right] = n_j \quad \text{and} \quad \operatorname{Res}\left[\frac{f'(z)}{f(z)}; b_k\right] = -p_k.$$

Thus (8.38) becomes

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_j n_j - \sum_k p_k = N - P. \qquad \blacksquare$$

8.39. Remark. If, in addition, ϕ is analytic on D, then under the hypotheses of Theorem 8.36 we easily get that

$$\frac{1}{2\pi i}\int_C \phi(z)\frac{f'(z)}{f(z)}\,dz = \sum_j n_j\phi(a_j) - \sum_k p_k\phi(b_k)$$

where a_j and b_k are the zeros of order n_j and the poles of order p_k for f, respectively. ●

Why is Theorem 8.36 known as an argument principle? Let us now restate Theorem 8.36 in terms of the properties of the logarithmic function $\log f(z)$. For this, under the hypotheses of Theorem 8.36, consider the transformation $w = \log f(z)$. Note that f is analytic on C and $f(z) \neq 0$ on C. Hence, $f(z) \neq 0$ in a neighborhood of C. For any analytic branch $\log f(z)$ of logarithm of $f(z)$, we have

$$\frac{d}{dz}(\log f(z)) = \frac{f'(z)}{f(z)}$$

[13] We use $\sum \operatorname{Res}\,[f(z); D]$ to denote the sum of the residues of f at the singularities a_j, where a_j belongs to the interior of D. Sometimes, we denote this simply by $\sum \operatorname{Res}\,[f(z); C]$ where C is a given closed contour.

and therefore,

$$\text{(8.40)} \qquad \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C d(\log f(z)) = \frac{1}{2\pi i}\Delta_C \log f(z).$$

We refer to this integral as the logarithmic integral of $f(z)$ along C. Here $\Delta_C \log f(z)$ denotes the increase in $\log f(z)$ when C is traversed once in the positive direction, and we say that the logarithmic integral measures the change of $\log f(z)$ along the contour C. Now, we express

$$\log f(z) = \ln|f(z)| + i \arg f(z)$$

where $\ln|f(z)|$ is single-valued and hence, $\Delta_C \ln|f(z)| = 0$, as $\ln|f(z)|$ returns to its original value when C is traversed. This observation implies that

$$\Delta_C \log f(z) = i\Delta_C \arg f(z).$$

Therefore, (8.40) yields

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_C \arg f(z)$$

where $\Delta_C \arg f(z)$ is referred to as the increase in the argument of $f(z)$ along C. Thus, the argument principle can be restated as follows.

8.41. Corollary. *Under the hypotheses of Theorem* 8.36*, we have*

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N - P.$$

8.42. Corollary. *If f is analytic inside and on a simple closed contour C and $f(z) \neq 0$ on C, then $(1/2\pi)\Delta_C \arg f(z) = N$.*

8.43. Example. Consider the integral $I = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f'(z)}{f(z)}\,dz$, where $C = \{z : |z-1-i| = 2\}$ and

$$\text{(i) } f(z) = \frac{z-2}{z(z-1)}, \qquad \text{(ii) } f(z) = \frac{z-2}{z(z-1)^2}, \qquad \text{(iii) } f(z) = \frac{z^2-9}{z^2+1}.$$

Then, we note that $0, 1, 2$ and i are inside C and $3, -3$ and $-i$ are outside C. Thus for (i), $I = 1 - 2 = -1$; for (ii), $I = 1 - 3 = -2$ and for the last case, $I = 0 - 1 = -1$. ●

8.44. Example. Note that $\cos z = 0 \iff z = (k + \frac{1}{2})\pi,\ k \in \mathbb{Z}$, the only zero of $\cos z$ inside the unit circle about $\pi/2$ is at $z = \pi/2$. Therefore, by the argument principle, we have

$$\int_{|z-\pi/2|=1} \tan z\,dz = -\int_{|z-\pi/2|=1} \frac{f'(z)}{f(z)}\,dz = -2\pi i, \quad f(z) = \cos z.$$

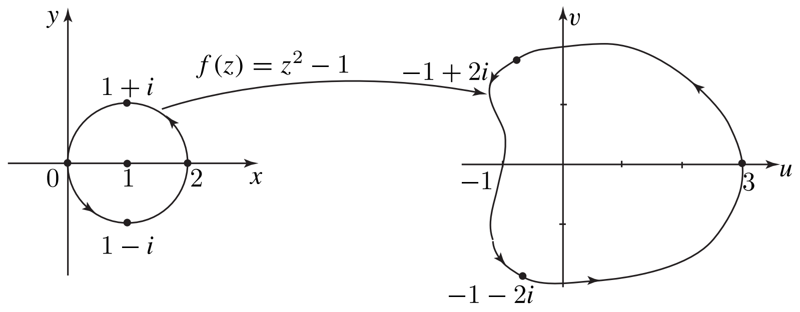


Figure 8.1: Mapping $w = z^2 - 1$.

From this integral, we also note that $\text{Res}\,[\tan z; \pi/2] = -1$. •

8.45. Examples. Let $f(z) = \tanh z$ and $C = \{z : |z| = 3\}$. Since $\cosh z = 0 \iff z = -i(k + \frac{1}{2})\pi$ $(k \in \mathbb{Z})$, we see that $\cosh z$ has 2 zeros at $z = \pm\pi i/2$ in the interior of C. Further, we note that

$$\frac{d}{dz}(\cosh z) = \sinh z, \quad \text{and} \quad \int_{|z|=3} \tanh z\, dz = 4\pi i,$$

by the argument principle. Similarly we easily obtain the following:

(i) $\displaystyle\int_{|z-1|=2} \tanh z\, dz = 4\pi i$

(ii) $\displaystyle\int_{|z|=3\pi} \frac{e^z}{e^z - 1}\, dz = 6\pi i$

(iii) $\displaystyle\int_{|z|=\pi} \tan \pi z\, dz = 12i$

(iv) $\displaystyle\int_{|z|=1} \frac{dz}{\sin z} = \int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 2\pi i$, with $f(z) = \tan(z/2)$. •

8.46. Example. Consider $f(z) = z^2 - 1$, $C = \{z : |z - 1| = 1\}$. Using the image points, it is easy to sketch the values which $w = f(z)$ assumes on C:

$$\begin{array}{cccc} z: & 2 & 1+i & 0 \\ & \downarrow & \downarrow & \downarrow \\ w: & 3 & -1+2i & -1. \end{array}$$

Since $f(\overline{z}) = \overline{f(z)}$ and since the contour C in the z-plane is symmetric about x-axis, the image curve Γ in the w-plane must be symmetric about u-axis. From Figure 8.1, we easily deduce that

$$\frac{1}{2\pi}\Delta_C \arg f(z) = 1$$

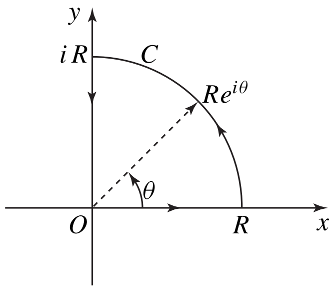


Figure 8.2: The curve $C = \{z : |z| = R, 0 \le \arg z \le \pi/2\}$.

as C is traversed once in the positive direction starting from a point and ending at the same point. Further, we also note that f has two simple zeros at $z = 1$ and $z = -1$. Only the zero at $z = 1$ lies inside C. Thus, $N = 1$.

Similarly, it is easy to see that if $f(z) = z^{-n}$ and C is a closed contour (or circle) enclosing the origin, then $\frac{1}{2\pi}\Delta_C \arg f(z) = -n$, which agrees with the fact that in the interior of C, the function has a pole of order n at $z = 0$. ●

8.47. Example. Consider $f(z) = z^4 + z^3 + 1$. If $z = x$ with $x > 0$, then

(8.48) $$f(z) = x^4 + x^3 + 1$$

and if $z = -x$ with $x > 0$, then

(8.49) $$f(z) = x^4 - x^3 + 1 = \begin{cases} x^3(x-1) + 1 & \text{if } x \ge 1, \\ x^4 + (1 + x + x^2)(1 - x) & \text{if } 0 < x \le 1. \end{cases}$$

Thus, (8.48) and (8.49) imply that $f(z)$ has no real roots. Further if $z = iy$ with y-real, then

$$f(z) = y^4 + 1 - iy^3$$

which shows that f has no purely imaginary roots. On the other hand if $C = \{z : z = Re^{i\theta}, 0 \le \theta \le \pi/2\}$, i.e. C is taken round the part of the first quadrant bounded by $|z| = R$ for sufficiently large R (see Figure 8.2), then

$$f(z) = R^4 e^{i4\theta} + R^3 e^{i3\theta} + 1 = R^4 e^{4\theta i}\left[1 + \frac{1}{Re^{i\theta}} + \frac{1}{R^4 e^{4\theta i}}\right].$$

If R is sufficiently large, then the square bracketed term is practically 1 and so

$$\Delta_C \arg f(z) \to 4\left(\frac{\pi}{2}\right) = 2\pi \text{ as } R \to \infty.$$

On the axis of y,

$$\arg f(iy) = \arctan\left(-\frac{y^3}{1 + y^4}\right).$$

Here y ranges from ∞ to 0 along the positive imaginary axis, the initial and final values of $\arg f(z)$ are zero. Thus, the total change when R is sufficiently large is given by $\Delta_C \arg f(z) = 2\pi - 0$ which means that there is a root lying in the open first quadrant. ●

8.5 Rouché's Theorem

The argument principle allows a comparison, under certain conditions, of the number of zeros of two analytic functions.

8.50. Theorem. (Rouché's Theorem) *Let f and g be meromorphic in a domain $D \subseteq \mathbb{C}$ and have only finitely many zeros and poles in D. Suppose that C is a simple closed contour in D such that no zeros or poles of f or g lie on C, and, in addition, assume that*

$$|g(z)| < |f(z)| \quad \text{on } C. \tag{8.51}$$

Then $\Delta_C f(z) = \Delta_C(f(z) + g(z))$; i.e. the difference between the number of zeros and number of poles is the same for f and $f + g$:

$$N_f - P_f = N_{f+g} - P_{f+g}.$$

Proof. By the hypotheses, both $f(z)$ and $f(z) + g(z)$ are nonzero on C. By the argument principle (see Corollary 8.41), we have

$$N_f - P_f = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz \quad\text{and}\quad N_{f+g} - P_{f+g} = \frac{1}{2\pi i}\int_C \frac{(f+g)'(z)}{(f+g)(z)}\,dz$$

so that a straightforward calculation gives that

$$\begin{aligned} &N_f - P_f - (N_{f+g} - P_{f+g}) \\ &\quad = \frac{1}{2\pi i}\int_C \left(\frac{f'(z)(f(z)+g(z)) - f(z)(f'(z)+g'(z))}{f(z)(f(z)+g(z))}\right) dz \\ &\quad = -\frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}\,dz, \quad F(z) = 1 + \frac{g(z)}{f(z)}. \end{aligned}$$

In view of (8.51), $|g(z)/f(z)| < 1$ on C so that, the meromorphic function $F(z)$ maps C into $|w-1| < 1$. Thus, as z describes C, the point $w = F(z)$ traverses a closed contour Γ lying completely inside the domain $|w-1| < 1$ (see Figure 8.3) so that Γ neither passes through the origin nor contains the origin. It follows that

$$\int_C \frac{F'(z)}{F(z)}\,dz = \int_\Gamma \frac{dw}{w} = 0$$

and so, $N_f - P_f - (N_{f+g} - P_{f+g}) = 0$. The result follows. ■

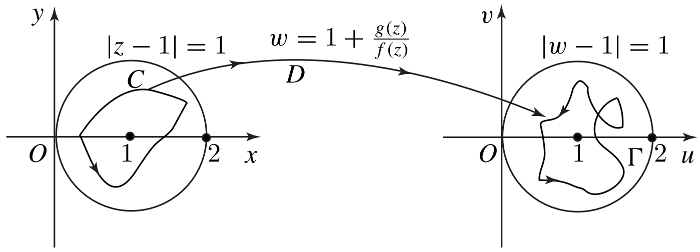


Figure 8.3: Illustration for the proof of Rouché's theorem.

Sometimes Rouché's theorem is given an equivalent formulation: Instead of assuming $|g(z)| < |f(z)|$ on C, it is assumed that

$$|g(z) - f(z)| < |f(z)| \quad \text{on } C.$$

Then the conclusion with respect to this assumption is that $\Delta_C g(z) = \Delta_C f(z)$, i.e. $N_f - P_f = N_g - P_g$.

8.52. Remark. If f and g have no poles in D, then the conclusion of Theorem 8.50 shows that f and $f + g$ have the same number of zeros inside the contour C. ●

Consider, $f(z) = z^6 - 5z^4 + 7$. Then we have

(i) $|f(z) - 7| < |z|^6 + 5|z|^4 < 7$ for $|z| = 1$, and so $f(z)$ has no zeros in $|z| < 1$

(ii) $|f(z) - (-5z^4)| < |z|^6 + 7 = 2^6 + 7 < 5|z|^4$ for $|z| = 2$, and so $f(z)$ has four zeros in $|z| < 2$

(iii) $|f(z) - z^6| < 5|z|^4 + 7 < |z^6|$ for $|z| = 3$, and so $f(z)$ has all the six zeros in the disk $|z| < 3$.

8.53. Remark. We next show that the fundamental theorem of algebra follows from Rouché's theorem. Consider $f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ and let $C = \{z : |z| = R\}$, $R > 1$. Then, for $z \in C$, we have

$$\begin{aligned} \left| \frac{f(z)}{z^n} - 1 \right| &\leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \cdots + \frac{|a_{n-1}|}{R} \\ &< [|a_0| + |a_1| + \cdots + |a_{n-1}|]\frac{1}{R} \text{ (since } R > 1). \end{aligned}$$

Thus, for sufficiently large R, i.e. for $R > \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}$, we see that

$$\left| \frac{f(z)}{z^n} - 1 \right| < 1; \quad \text{or} \quad |f(z) - z^n| < |z|^n \quad \text{on } |z| = R.$$

By Rouché's theorem, f has n zeros inside this circle. ●

8.54. Example. Take $f(z) = 2 + z^2$ and $g(z) = e^{iz}$. For $z = x \in \mathbb{R}$, we have

$$f(x) = 2 + x^2 > |e^{ix}| = 1 = |g(x)|, \quad \text{i.e. } |f(z)| > |g(z)|.$$

On the other hand if $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then

$$|f(Re^{i\theta})| = |2 + R^2 e^{i2\theta}| \geq R^2 - 2 \text{ and } |g(Re^{i\theta})| = |e^{iRe^{i\theta}}| = e^{-R\sin\theta}.$$

Therefore, on the upper semi-circle, we have $|f(z)| > |g(z)|$ if R satisfies the condition $R^2 - 2 > e^{-R\sin\theta}$. If we choose $R > \sqrt{3}$ then

$$R^2 - 2 > 1 \geq e^{-R\sin\theta} \text{ for } 0 \leq \theta \leq \pi.$$

By Rouché's theorem, the number of zeros of $f + g$ in

$$\Omega_R = \{z : |z| \leq R \text{ and } \operatorname{Re} z \geq 0\}$$

for $R > \sqrt{3}$ is equal to the number of roots of the equation $f(z) = 2 + z^2 = 0$ in Ω_R. Thus, in the entire upper half-plane equation $2 + z^2 - e^{iz} = 0$ has only one root. ●

8.55. Example. Let $f(z) = 1 + z^4$ and $g(z) = iz^3$. Then, on $|z| = R$, $|f(z)| \geq R^4 - 1$ and $|g(z)| = R^3$ so that the inequality $|f(z)| > |g(z)|$ holds on $|z| = R$ if $R^4 - 1 > R^3$. By Rouché's theorem, f and $f + g$ have same number of zeros in $|z| < R$ for whenever $R^4 - 1 > R^3$. For example, $z^4 + iz^3 + 1 = 0$ has all the four roots in $|z| < 3/2$. ●

8.56. Example. Consider $f(z) = 3 + \alpha z + (|\alpha|R - 3)z^n$, where $|\alpha|R - 3 > 0$ with $0 < R \leq 1$ and $n = 2, 3, \ldots$. Then, on $|z| = R$, we have

$$|3 + \alpha z| \geq |\alpha|R - 3 \geq (|\alpha|R - 3)R^n = |(|\alpha|R - 3)z^n|$$

and therefore, f has exactly one zero inside the circle $|z| = R$. For instance $f(z) = 3 + \alpha z + (|\alpha| - 3)z^n$ has one zero in $|z| < 1$ if $|\alpha| > 3$. ●

8.57. Example. Given $\alpha > 1$, we wish to prove that $e^{-z} + z - \alpha = 0$ has a unique solution in $\{z : \operatorname{Re} z > 0\}$. To do this, we take $f(z) = z - \alpha$ and $g(z) = e^{-z}$. Then, for $z = iy$, we have

$$|f(iy)| = |iy - \alpha| = \sqrt{y^2 + \alpha^2} > 1 = |e^{-iy}| = |g(iy)|$$

so that $|g(z)| < |f(z)|$ for z in the vertical line segment connecting iR and $-iR$. On the other hand if $|z| = R$, $\operatorname{Re} z \geq 0$, then we have

$$|f(z)| \geq |z| - \alpha = R - \alpha > 1 \geq |e^{-z}| = e^{-\operatorname{Re} z} = |g(z)| \tag{8.58}$$

(since $-\mathrm{Re}\, z \leq 0$), provided $R > 1+\alpha$. By Rouché's theorem, the equation $f(z) + g(z) = e^{-z} + z - \alpha = 0$ has only one root inside $\Omega = \{z : |z| = R,\ \mathrm{Re}\, z > 0\}$ for $R > \alpha + 1$. The root is real because the left hand side of the equation $e^{-z} + z - \alpha = 0$ for $z = x = 0$ gives $1 - \alpha$ which is negative and approaches $+\infty$ as $x \to +\infty$. This shows the root must be real. ●

8.59. Example. It is easy to prove that for each $R > 0$ there is an integer $N = N(R)$ such that

$$f_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

has no zeros in $|z| < R$ for $n \geq N$. Indeed, since $f_n(z) \to e^z$ on $\mathbb{C}$ as $n \to \infty$, for a given $\epsilon > 0$, there exists an N such that

$$|f_n(z) - e^z| < \epsilon \text{ for all } |z| = R,\ n \geq N.$$

That is, if we choose $\epsilon < \min_{|z|=R} |f_n(z)|$, then the last inequality gives

$$|e^z - f_n(z)| < |f_n(z)| \text{ for } |z| = R.$$

Our assertion therefore follows from Rouché's theorem (and the fact that $e^z \neq 0$ in $\mathbb{C}$). One can also use Hurwitz' theorem (see Theorem 12.4) to get an alternate treatment of this example.

On the other hand, we easily see that if $m, n \in \mathbb{N}$, then the polynomial

$$p(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + az^m$$

has m zeros in the unit disk whenever $e < |a|$. Indeed, as

$$\left| \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} = e < |a| = |az^m| \text{ for } |z|=1 ,$$

Rouché's theorem gives the desired result. ●

8.6 Exercises

8.60. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) If f and g are analytic in a deleted neighborhood of z_0, and if a and $b \in \mathbb{C}$, then $\mathrm{Res}\,[af(z) + bg(z); z_0] = a\,\mathrm{Res}\,[f(z); z_0] + b\,\mathrm{Res}\,[g(z); z_0]$.

(b) If $f(z)$ has an isolated singularity at $a \in \mathbb{C}$ with nonzero residue at $z = a$, then the residues of $f'(z)$ and $(z-a)f'(z)$ at $z = a$ are zero.

(c) If f has an isolated singularity at a, and c is a non-zero complex number, then $f(cz)$ has an isolated singularity at a/c, and

$$\mathrm{Res}\,[f(z); a/c] = (1/c)\mathrm{Res}\,[f(z); a].$$

(d) Let $f \in \mathcal{H}(\Delta(a;\delta) \setminus \{a\})$ for some $\delta > 0$ and f have a simple pole at a with residue a_{-1}. Then, for a circular arc $\gamma(\epsilon)$ of the form $a + \epsilon e^{i\theta}$ ($\theta \in [\theta_1, \theta_2]$, $0 \le \theta_1 < \theta_2 \le 2\pi$, $\epsilon < \delta$), one has

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\, dz = i a_{-1}(\theta_2 - \theta_1).$$

(e) If $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ and $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ have an isolated singularity at 0, then $\mathrm{Res}\,[f(z)g(z); 0] = \sum_{n\in\mathbb{Z}} a_n b_{-n-1}$.
Note: Is this result helpful to compute $\mathrm{Res}\,[e^{1/z^2}\sin(1/z); 0] = 0$?

(f) If $f \in \mathcal{H}(\mathbb{C}\backslash\{0\}$ and $\mathrm{Res}\,[f(z); 0] = a_{-1}$, then there exists $z \in \partial\Delta$ such that $|f(z) - 1/z| \ge |a_{-1} - 1|$.

(g) If f is analytic at ∞ and has a zero of order $n\,(\ge 2)$ at ∞, then $\mathrm{Res}\,[f(z); \infty] = 0$.

(h) If f is analytic at ∞ and has a simple zero at ∞, then $\mathrm{Res}\,[f(z); \infty] = -\lim_{z\to\infty} z f(z)$.

(i) If $D \subset \mathbb{C}$ is a domain, $M > 0$ and $f : D \to \Delta(M; M)$ is analytic, then for every closed contour γ in D, $\int_\gamma (f'(z)/f(z))\, dz = 0$.

(j) $\displaystyle\int_{|z|=1} \frac{1}{e^z - 1 - 2z}\, dz = -2\pi i.$

(k) If f is a rational function such that the degree of its denominator exceeds that of numerator by at least two, then the sum of residues at all the poles is zero.

(l) If p is a polynomial of degree at most n and $|a| < R$, then

$$\int_{|z|=R} \frac{p(z)}{z^{n+1}(z-a)}\, dz = 0.$$

(m) If $p(z)$ is a polynomial of degree $n\,(\ge 2)$, then $\int_{|z|=R} \frac{dz}{p(z)} = 0$ for large R, and the same is also true if $p(z)$ is a linear function, i.e. if $n = 1$.

(n) There does not exist an analytic function f defined on $\mathbb{C}\backslash\{0\}$ such that $f'(z) = 1/z$ (see also Example 4.89).

(o) The function $f(z) = [1 + z + z^2 + \cdots + z^{n-1}]^{-1}$, has simple poles at $z_k = e^{2k\pi i/n}$ ($k = 1, 2, \ldots, n-1$) and

$$\mathrm{Res}\,[f(z); z_k] = 2i\frac{e^{3k\pi i/n}}{n} \sin\left(\frac{k\pi}{n}\right).$$

(p) If k is a fixed integer and $C = \{z : |z| = (2|k|+1)\pi/2\}$, then

$$\frac{1}{2\pi i}\int_C \cosh z \cot z \, dz = \sum_{j=-k}^{k} \cosh j\pi.$$

(q) For $f(z) = e^{z+1/z}$, $\text{Res}\,[f(z); 0] = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} = -\text{Res}\,[f(z); \infty]$.

(r) For $a > 1$ and b-real,

$$\int_{|z|=1} \frac{e^{bz}}{a^2z^2+1}\, dz = \frac{2\pi i \sin(b/a)}{a}.$$

(s) The equation $e^z = 2 + 3z$ has at most one solution in the unit disk $|z| < 1$.

(t) If $|z| = 1$ and $a > 1$, then the equation $ze^{a-z} = 1$ has exactly one solution in $|z| < 1$.

(u) All the roots of the polynomial $p(z) = 1 + z + z^2 + z^3 + z^4$ have absolute value less than 2.

(v) If $|f(z)| > m$ on $|z| = 1$, f is analytic for $|z| \le 1$ and $|f(0)| < m$, then f has at least one zero in $|z| < 1$.

(w) Let f be analytic in a neighborhood of $\overline{\Delta} = \{z : |z| \le 1\}$. If $|f(z)| < 1$ for all $|z| = 1$, then f has exactly one fixed point in Δ.

(x) If $f_n(z) = 1 + \sum_{k=1}^{n} kz^{k-1}$ and $0 < R < 1$, then there exists an N such that f_n has no zeros in $|z| < R$ whenever $n > N$.

(y) All the roots of the equation $z^3 - 5z^2 + 10 = 0$ lie in the annulus $\{z : 1 < |z| < \sqrt{2}\}$.

(z) If f is a meromorphic function on the Riemann sphere, then $N_f = P_f$, where N_f and P_f are respectively the number of zeros and the number of poles of f, counted with multiplicity.

8.61. Find the residues at each of the isolated singularities of the following functions in $\mathbb{C}$ or $\mathbb{C}_\infty$:

(i) $\dfrac{z}{z^2+3z+3}$ (ii) $\left(\dfrac{z^2+z+1}{z+1}\right)^3$ (iii) $\dfrac{1}{(z^3+1)(z+1)^2}$.

8.62. Suppose f and g are analytic in a domain D and $f'(z) \neq 0$ in D. Let γ be a closed contour in D. Then for $a \notin \gamma$, show that

$$\frac{g(a)}{f'(a)} n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{f(z)-f(a)}\, dz.$$

Apply this result for $g(z) = 1$, e^z, $\cos z$.

8.63. Evaluate $\int_{|z|=1} f(z)\,dz$, where $f(z)$ is given by the following functions:

$$\frac{\sin^6 z}{(z-\pi/6)^3},\quad \frac{z}{z^4-6z^2+1},\quad \frac{z}{(z^2+4z+1)^2},\quad \frac{1-\cos z}{(e^z-1)\sin z},\quad \frac{(e^z-e^{-z})^2}{z^3}.$$

8.64. Evaluate the following integrals using the residue theorem:

(i) $\displaystyle\int_{|z|=1} \frac{dz}{z^2(e^z-e^{-z})}$ and $\displaystyle\int_{|z|=4} \frac{dz}{z^2(e^z-e^{-z})}$

(ii) $\displaystyle\int_{|z-1|=3} \frac{z}{(z^2-1)^3(1+z^2)}\,dz$

(iii) $\int_{|z|=\pi} \cot \pi z\,dz$

(iv) $\displaystyle\int_{|z|=3} \frac{e^{1/(z-1)}}{z-2}\,dz$

(v) $\displaystyle\int_C \frac{dz}{1+z^2}$, where C is any circle enclosing i and $-i$ inside.

8.65. Let C be a simple closed contour enclosing the points $0,1,2,\ldots,n$. If $f_m(z)=\prod_{k=1}^m (z-k)$ for $m=1,2,\ldots,n$, then compute the integrals

$$I_m = \int_C \frac{dz}{zf_m(z)} \quad\text{and}\quad J_m = \int_C \frac{f_m(z)}{z}\,dz \qquad \text{for } m=1,2,\ldots,n.$$

8.66. Suppose that $f\in\mathcal{H}(\overline{\Delta})$ and such that $f(\partial\Delta)\subset\mathbb{R}$. Show that f is a constant.

8.67. Define

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{7^n n! z^n}+\sum_{n=0}^{\infty}\frac{z^n}{5n^2+n!}.$$

Does the series converge for all $z\neq 0$? If so, find the value of $\int_{|z|=1} f(z)\,dz$.

8.68. Let $f(z)=\phi(z)/g(z)$, where ϕ and g are both analytic around z_0. If $\phi(z_0)\neq 0$, $g(z_0)=g'(z_0)=g''(z_0)=0$, and $g'''(z_0)\neq 0$, show that

$$\mathrm{Res}\,[f(z);z_0] = 3\left[\frac{\phi''(z)}{g'''(z)}-\frac{1}{2}\frac{\phi'(z)g^{(iv)}(z)}{(g'''(z))^2}\right]\Bigg|_{z=z_0}.$$

8.69. Define $f_\alpha(z)=\ln|z|+i\arg_\alpha z,\ z\in D_\alpha$, where $D_\alpha=\mathbb{C}\backslash\{Re^{i\alpha}: R>0\}$, $\alpha\in\mathbb{R}$ is fixed such that $\arg_\alpha z$ is the choice of $\arg z$ in $(\alpha-2\pi,\alpha)$. Assuming $\alpha\neq\pi/2, 3\pi/2$, find $\mathrm{Res}\,[(z^2+a^2)^{-1}f_\alpha(z);\pm ia]$.

8.70. Find the analog of (8.8) when f has a pole at ∞. More precisely, if f has a pole of order n at ∞, then show that

$$\operatorname{Res}[f(z);\infty] = \lim_{z\to\infty} \frac{(-1)^n}{(n+1)!} z^{n+2} f^{(n+1)}(z).$$

In the case when f is analytic at ∞, then this formula continues to hold if $n = 0$.

8.71. Let $p(z)$ and $q(z)$ be polynomials with no common zeros, and with degrees m and n, respectively. Set $f(z) = p(z)/q(z)$. Show that

(i) f has a removable singularity at ∞ if $n \geq m$
(ii) $\operatorname{Res}[f(z);\infty] = 0$ if $n \geq m + 2$.

8.72. Suppose that f is meromorphic on $\mathbb{C}_\infty$. Then prove or disprove the following: f has a pole only at ∞ iff $\operatorname{Res}[f(z),\infty] = 0$.

8.73. Suppose that f is analytic on $|z| \leq 1$, $|f(z)| < 1$ whenever $|z| = 1$, and that $|\alpha| < 1$. Find the number of solutions of $f(z) = ((z-\overline{\alpha})/(\alpha z - 1))^2$ in $|z| \leq 1$.

8.74. Let $|a| > e^R/R^n$ for a positive integer n. Prove that the equation $az^n - e^z = 0$ has n solutions (counting multiplicity) z satisfying $|z| < R$. In the case when $R = 1$, show that these solutions are simple roots with positive real part in $|z| < 1$.

8.75. If $f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$, and C is a simple closed contour enclosing all the zeros of f, then show that

$$\int_C \frac{zf'(z)}{f(z)}\, dz = -2\pi a_{n-1} i \quad \text{and} \quad \int_C \frac{z^2 f'(z)}{f(z)}\, dz = 2\pi i (a_{n-1}^2 - 2a_{n-1}).$$

8.76. Suppose that $f(z)$ is analytic in the disk $|z| < 2$ and

$$I_n = \frac{1}{2\pi i} \int_{|z|=1} z^n \frac{f'(z)}{f(z)}\, dz.$$

Find the value of I_n when $n = 0, 1, 2$.

8.77. Let f be analytic for $|z| < 2$. Show that

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{\overline{f(z)}}{z-a}\, dz = \begin{cases} \overline{f(0)} & \text{if } |a| < 1 \\ \overline{f(0)} - \overline{f(1/\overline{a})} & \text{if } |a| > 1. \end{cases}$$

Chapter 9

Evaluation of certain Integrals

This chapter describes some systematic methods to evaluate certain types of definite and improper integrals occurring in Real Analysis. The method of residue calculus, using the *Residue Theorem*, is a powerful tool for evaluating such integrals. Here we illustrate the methods together with a suitable function f and a suitable closed contour C; the choice, nevertheless, depends on the problem. In Section 9.1, we first discuss the evaluation of integrals of certain periodic functions over the interval $[\alpha, 2\pi+\alpha]$. In the remaining sections we apply the residue theorem to evaluate various types of real integrals whose integrands have no known explicit anti-derivatives. Let us start with a simple example:

$$I = \int_{|z|=r} f(z)\,dz, \quad f(z) = \frac{e^z}{z}.$$

An immediate consequence of the Cauchy residue theorem (or *the Cauchy integral formula*) gives $I = 2\pi i$ as f has a simple pole at 0 with $\mathrm{Res}\,[f(z);0] = 1$. If we substitute $z = re^{i\theta}$, then $dz = ire^{i\theta}d\theta = izd\theta$ so that

$$2\pi i = I = i\int_0^{2\pi} \exp(re^{i\theta})\,d\theta.$$

Equating real and imaginary parts, we have

$$\int_0^{2\pi} e^{r\cos\theta}\sin(r\sin\theta)\,d\theta = 0 \quad\text{and}\quad \int_0^{2\pi} e^{r\cos\theta}\cos(r\sin\theta)\,d\theta = 2\pi.$$

9.1 Integrals of Type $\int_\alpha^{2\pi+\alpha} R(\cos\theta, \sin\theta)\,d\theta$

This section provides a method of evaluating integrals of the form

$$I = \int_\alpha^{2\pi+\alpha} R(\cos\theta, \sin\theta)\,d\theta, \tag{9.1}$$

where $R(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ (with real coefficients) which is finite in the range of the integral.

Quite often the integrals of the above type can be evaluated by means of some substitutions such as $t = \tan\theta$, $t = \tan(\theta/2)$, etc., but sometimes the evaluation may prove to be difficult or even impossible with the real analytic methods at our disposal.

In equation (9.1), θ varies between α and $2\pi + \alpha$. Since θ varies over a range of 2π, we may consider θ as an argument of a point z on the unit circle $C = \{z : |z| = 1\}$. Therefore, we may write $z = e^{i\theta}$ so that

$$\cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2iz} \quad \text{and} \quad d\theta = \frac{dz}{iz}, \quad (0 \leq \theta \leq 2\pi).$$

Thus, the integral in (9.1) becomes

$$I = \int_C f(z)\,dz = \int_C R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

where f is a rational function of z that is finite on the path of integration C. By the residue theorem, we then have $I = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f(z); \alpha_k]$, where α_k denotes those poles of f which lie inside C and the integral along C is taken in the positive direction.

9.2. Remark. As pointed out above, by means of the substitution $t = \tan\theta/2$ or simply by $t = \tan\theta$, we can prove

(i) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{d\theta}{1 - 2\alpha\sin\theta + \alpha^2} = \frac{\pi}{1-\alpha^2} \quad (|\alpha| < 1)$

(ii) $\displaystyle\int_0^{\pi/2} \frac{1 + 2\sin^2\theta}{1 + 2\cos^2\theta}d\theta = \frac{\pi(4\sqrt{3}-3)}{6}.$ ●

9.3. Example. Let us evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta},$$

where a and b are real with $|b| < |a|$. First, we note that if $b = 0$ then $I = 2\pi/a$. If $b \neq 0$, then

$$I = \frac{1}{b}\int_0^{2\pi} \frac{d\theta}{(a/b) + \sin\theta} \quad (a, b \text{ real},\ 1 < |a/b|).$$

So, it suffices to compute the integral for $b = 1$. Now, putting $z = e^{i\theta}$ (and assuming $b = 1$ and $a \in \mathbb{R}$ with $|a| > 1$), we find that

$$I = 2\int_{|z|=1} f(z)\,dz, \quad f(z) = \frac{1}{z^2 + 2iaz - 1} =: \frac{1}{(z-\alpha)(z-\beta)}.$$

We see that the only singularities of f are the simple poles at

$$\alpha = -i(a+\sqrt{a^2-1}) \quad \text{and} \quad \beta = -i(a-\sqrt{a^2-1}) = -\frac{1}{\alpha}.$$

Observe that, since $1 < |a|$ and the product of the two roots is -1, one root lies inside the unit circle $|z| = 1$ while the other lies outside. In fact α lies in $|z| < 1$ whenever $a < -1$, and $\beta = -1/\alpha$ lies in $|z| < 1$ whenever $a > 1$. Now,

$$\text{Res}\,[f(z);\beta] = \frac{1}{\beta-\alpha} = \frac{1}{2i\sqrt{a^2-1}}, \quad \text{and} \quad \text{Res}\,[f(z);\alpha] = -\frac{1}{\beta-\alpha}.$$

Therefore, by the residue theorem, we have

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2\left[2\pi i \sum \text{Res}\,[f(z);C]\right] = \begin{cases} \dfrac{2\pi}{\sqrt{a^2-1}} & \text{if } a > 1 \\ -\dfrac{2\pi}{\sqrt{a^2-1}} & \text{if } a < -1. \end{cases} \quad \bullet$$

9.4. Example. In the following examples, we skip some steps. For $a > 1$,

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{4}{i}\int_{|z|=1} f(z)\,dz, \quad f(z) = \frac{z}{(z^2+2az+1)^2}.$$

Observe that f has two poles (each of order two) at

$$\alpha = -a+\sqrt{a^2-1} \quad \text{and} \quad \beta = -a-\sqrt{a^2-1}.$$

As $z^2+2az+1 = (z-\alpha)(z-\beta)$, we have $\alpha\beta = 1$ so that one pole lies inside the unit circle $|z| = 1$ while the other must lie outside. Clearly, α lies inside $|z| = 1$, and therefore, it suffices to compute

$$\begin{aligned} \text{Res}\,[f(z);\alpha] &= \lim_{z\to\alpha} \frac{d}{dz}\left((z-\alpha)^2 f(z)\right) \\ &= \lim_{z\to\alpha} \frac{d}{dz}\left(\frac{z}{(z-\beta)^2}\right) \\ &= -\frac{(\alpha+\beta)}{(\alpha-\beta)^3} \\ &= \frac{a}{4(a^2-1)^{3/2}}. \end{aligned}$$

Finally, by the Cauchy residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{4}{i}\left[2\pi i\left(\frac{a}{4(a^2-1)^{3/2}}\right)\right] = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

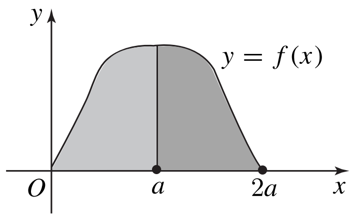


Figure 9.1: Illustration for an integral by area under a curve.

What happens to the integral if $a < -1$?

Similarly, for $a > 1$, we can write

$$\int_0^{2\pi} \frac{d\theta}{(a+\sin\theta)^2} = -\frac{4}{i}\int_{|z|=1} f(z)\,dz,$$

where

$$f(z) = \frac{z}{(z^2+2iaz-1)^2} = \frac{z}{(z-\alpha)^2(z-\beta)^2}$$

and α and β are as in Example 9.3. It follows that f has a double pole at $z = \alpha$ inside the unit circle, and

$$\mathrm{Res}\,[f(z);\alpha] = -\frac{(\alpha+\beta)}{(\alpha-\beta)^3} = -\frac{a}{4(a^2-1)^{3/2}}$$

so that

$$\int_0^{2\pi} \frac{d\theta}{(a+\sin\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \quad \text{for } a > 1.$$

What is the value of the integral when $a < -1$? ●

9.5. Remark. One may adopt the method of the above example to evaluate such integrals whose range of integration is not of length 2π. In this context we often use the following:

$$\text{“If } f(x) = f(2a-x), \text{ then } \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.\text{”}$$

One can be readily convinced of this fact by interpreting the integrals as areas under a curve (see Figure 9.1). Since the curve is symmetric about $x = a$, the two different shaded regions are equal in area. Alternately, it suffices to rewrite

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

and use the change of variable $x = 2a - t$ for the second integral on the right. An integral over $[0,\pi]$ can also be handled whenever $f(\theta)$ is even in θ and is 2π-periodic, since in this situation

$$\int_0^{\pi} f(\theta)\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} f(\theta)\, d\theta. \qquad \bullet$$

9.6. Remark. If a and b are constants, x a real parameter, $c \le x \le d$, and $R(\theta, x)$ a continuous function with a continuous partial derivative with respect to x for $a \le \theta \le b$, $c \le x \le d$, then, according to *Leibnitz's rule*, we have

$$\frac{d}{dx}\left(\int_a^b R(\theta, x)\, d\theta\right) = \int_a^b \frac{\partial R}{\partial x}\, d\theta.$$

Leibnitz's rule can be extended suitably to cases where the limits a and b are infinite or dependent on x. Using this rule, we can easily deduce from Example 9.3 that for $1 < |a|$,

$$\int_0^{2\pi} \frac{d\theta}{(a+\sin\theta)^2} = \begin{cases} 2\pi a/(a^2-1)^{3/2} & \text{for } a > 1 \\ -2\pi a/(a^2-1)^{3/2} & \text{for } a < -1. \end{cases} \qquad \bullet$$

9.7. Example. Set $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{1+\alpha^2 - 2\alpha\cos\theta}$, $1 \neq \alpha > 0$. As in the previous examples, to evaluate this integral, we may rewrite it as

$$I = \frac{i}{\alpha}\int_{|z|=1} f(z)\, dz, \quad f(z) = \frac{1}{(z-\alpha)(z-1/\alpha)}.$$

The only singularities of f are the simple poles at $z = \alpha$ and $z = 1/\alpha$. If $0 < \alpha < 1$, then $z = \alpha$ is inside $|z| < 1$ while the other is outside the unit circle. Therefore, for $0 < \alpha < 1$,

$$I = \frac{i}{\alpha}\{2\pi i \,\mathrm{Res}\,[f(z); \alpha]\} = -\frac{2\pi}{\alpha}\lim_{z\to\alpha}(z-\alpha)f(z) = \frac{2\pi}{1-\alpha^2}.$$

Similarly, we deduce that $I = 2\pi/(\alpha^2 - 1)$ for $\alpha > 1$. $\bullet$

9.8. Example. Let us show that

(i) $I = \displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \int_0^{2\pi} \sin^{2n}\theta\, d\theta = \frac{2\pi(2n)!}{2^{2n}(n!)^2}$

(ii) $\displaystyle\int_0^{2\pi} (a\cos\theta + b\sin\theta)^{2n}\, d\theta = \frac{2\pi(2n)!(a^2+b^2)^n}{2^{2n}(n!)^2}$ $(a, b$ are real).

As usual, let $z = e^{i\theta}$. Then the first integral, which we have already encountered in Chapter 4, becomes

$$I = \frac{1}{i2^{2n}}\int_{|z|=1} f(z)\, dz, \quad f(z) = \frac{(z^2+1)^{2n}}{z^{2n+1}} = \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}.$$

The only singularity of f is the pole at $z = 0$ of order $2n + 1$. Since

$$f(z) = \frac{1}{z}\sum_{k=0}^{2n} \binom{2n}{k} z^{2n-k}\left(\frac{1}{z}\right)^k = \sum_{k=0}^{2n}\binom{2n}{k} z^{2(n-k)-1}, \quad |z| > 0,$$

we see that the coefficient of z^{-1} is $a_{-1} = \binom{2n}{n}$. Consequently,

$$I = \frac{1}{i2^{2n}}\{2\pi i \text{ Re}\,[f(z); 0]\} = \frac{2\pi}{2^{2n}}\binom{2n}{n} = \frac{2\pi}{2^{2n}}\left\{\frac{(2n)!}{(n!)^2}\right\}$$

and the integration formula for the first part follows. Similarly, by considering functions of the form $z^k f(z)$ $(k \in \mathbb{Z})$, we can actually evaluate

$$\int_0^{2\pi} \cos^{2n}(\theta)\cos k\theta\, d\theta \quad \text{and} \quad \int_0^{2\pi}\cos^{2n}(\theta)\sin k\theta\, d\theta.$$

However, for the proof of (ii), we may rewrite (as there is nothing to prove if $(a, b) = (0, 0)$)

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right]$$

and observe that, given a pair $(a, b) \neq (0, 0)$, there exists a unique $\phi \in [0, 2\pi)$ such that

$$\cos\phi = a/\sqrt{a^2+b^2} \quad \text{and} \quad \sin\phi = b/\sqrt{a^2+b^2}.$$

Thus, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \phi)$ and

$$\begin{aligned}\frac{1}{(a^2+b^2)^n}\int_0^{2\pi}(a\cos\theta + b\sin\theta)^{2n}\, d\theta &= \int_0^{2\pi}\cos^{2n}(\theta-\phi)\, d\theta \\ &= \int_0^{2\pi}\cos^{2n}\theta\, d\theta\end{aligned}$$

so that (ii) follows from (i). ●

9.9. Remark. If $\cos n\theta$ or $\sin n\theta$ occurs in the integrand, we may use the formulas

$$\cos n\theta = \frac{z^n + z^{-n}}{2} \quad \text{and} \quad \sin n\theta = \frac{z^n - z^{-n}}{2i}$$

where $z = e^{i\theta}$ and $n \in \mathbb{Z}$. ●

Finally, for $a > 1$, we write

$$I = \int_0^{2\pi}\frac{\sin^2\theta}{a+\cos\theta}\, d\theta = \frac{i}{2}\int_{|z|=1} f(z)\, dz, \quad f(z) = \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)},$$

where

$$\alpha = -a + \sqrt{a^2-1} \text{ and } \beta = -a - \sqrt{a^2-1}$$

are the roots of the quadratic equation $z^2 + 2az + 1 = 0$. We note that $z = \alpha$ lies in $|z| < 1$ while $z = \beta$ lies outside the unit circle $|z| = 1$. Further, f has a pole of order two at $z = 0$. Also, it is easy to see that

$$\text{Res}\,[f(z);0] = -2a \text{ and } \text{Res}\,[f(z);\alpha] = 2\sqrt{a^2-1}.$$

Therefore, by the residue theorem,

$$I = \frac{i}{2}\left\{2\pi i\left(-2a + 2\sqrt{a^2-1}\right)\right\} = 2\pi(a - \sqrt{a^2-1}).$$

What happens if $a < -1$?

9.2 Integrals of Type $\int_{-\infty}^{\infty} f(x)\,dx$

In the previous section, we transformed certain real trigonometric integrals into contour integrals and then computed them with the help of residue theorem. This section proposes to evaluate certain types of improper and definite integrals, but by interpreting the given integral as $2\pi i$ times the sum of residues at the singularities of a properly chosen analytic function. We start with a simple example of evaluating

$$I = \int_{-\infty}^{\infty} f(x)\,dx, \tag{9.10}$$

where $f(x)$ is a continuous function on $\mathbb{R}$. As we know, this is an improper integral of f on $(-\infty, \infty)$ and has the meaning

$$I = \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{S\to\infty} \int_{0}^{S} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

provided these two limits exist. We can now write I as

$$I = \lim_{R,S\to\infty} \int_{-R}^{S} f(x)\,dx$$

with the understanding that R and S have to be allowed to run to ∞ independently of each other; i.e. the existence of $\lim_{R\to\infty} \int_{-R}^{R}$ does not imply the existence of $\int_{-\infty}^{\infty}$ as the function $f(x) = x$ demonstrates this fact. However, we next give a precise example illustrating this note.

Suppose f is continuous on interval (a, b) except at a point x_0 in (a, b), where f has a singularity (in which sense?), i.e. $f(x)$ is unbounded near x_0. Then,

$$\int_a^b f(x)\,dx \tag{9.11}$$

might not be defined and so we have to find a natural way to make a meaningful definition. For $\epsilon_1 > 0$ and $\epsilon_2 > 0$, consider

$$\int_a^{x_0-\epsilon_1} f(x)\,dx + \int_{x_0+\epsilon_2}^{b} f(x)\,dx$$

(note that both the integrals exist for each ϵ_1 and ϵ_2) and let $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. *If both limits exist, we say that the integral* (9.11) *is convergent.* It may happen that even though this limit does not exist, the limit when $\epsilon_1 = \epsilon_2 = \epsilon$ with $\epsilon \to 0$ may exist. For instance consider $f(x) = x^{-3}$, $x \in [-1, 1]\setminus\{0\}$. Then, with $x_0 = 0$, we have

$$\int_{-1}^{0-\epsilon_1} \frac{dx}{x^3} = \frac{1}{2}\left(1 - \frac{1}{\epsilon_1^2}\right) \quad \text{and} \quad \int_{0+\epsilon_2}^{1} \frac{dx}{x^3} = \frac{1}{2}\left(\frac{1}{\epsilon_2^2} - 1\right).$$

Clearly, the limit of each of these integrals does not exist as $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. This shows that the integrals $\int_{-1}^{0} x^{-3}\,dx$ and $\int_0^1 x^{-3}\,dx$ are not convergent. On the other hand, if we take $\epsilon_1 = \epsilon_2 = \epsilon$, we find that

$$\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3} = 0$$

for all ϵ such that $0 < \epsilon < 1$. Thus if we define

$$\int_a^b f(x)\,dx = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{\int_a^{x_0-\epsilon_1} f(x)\,dx + \int_{x_0+\epsilon_2}^{b} f(x)\,dx\right\} \tag{9.12}$$

and

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \left\{\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^{b} f(x)\,dx\right\} \tag{9.13}$$

we see that we may get different values for (9.11), depending on whether we use definition (9.12) or (9.13). Thus if $|f(x)| \to \infty$ as $x \to x_0$ and the limit in (9.13) exists, then $\int_a^b f(x)\,dx$ is called a convergent *improper integral*. The limit in (9.13), which is also of interest to us, is called the *Cauchy's principal value* of the integral (9.11). Note also that if the limit exists in (9.12), then it also exists in the sense defined in (9.13) and hence, both the limits are equal. When $f(x)$ is finite for all real values, then by the Cauchy principal value of $\int_{-\infty}^{\infty} f(x)\,dx$ we mean $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ (if it exists). For instance, since $\int_{-R}^{R} x\,dx = 0$ for every $R > 0$, the Cauchy principal value of $\int_{-\infty}^{\infty} x\,dx$ is zero.

In general if f is continuous on $\mathbb{R}$ except for a finite number of points $x_0, x_1, \dots, x_n$ $(x_0 < x_1 < \cdots < x_n)$ and if

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left\{\left(\int_{-R}^{x_0-\epsilon} + \int_{x_0+\epsilon}^{x_1-\epsilon} + \cdots + \int_{x_{n-1}+\epsilon}^{x_n-\epsilon} + \int_{x_n+\epsilon}^{R}\right) f(x)\,dx\right\}$$

exists and is finite, then we call this limit the Cauchy principal value of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ denoted by itself with the additional remark, if necessary, that the principal value is meant. Some books denote for brevity by

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx.$$

By (9.13), we see that for $a > 0$

$$\int_{-a}^{2a} \frac{dx}{x} = \lim_{\epsilon\to 0}\left\{\int_{-a}^{0-\epsilon} \frac{dx}{x} + \int_{0+\epsilon}^{2a} \frac{dx}{x}\right\} = \lim_{\epsilon\to 0}[\ln\epsilon - \ln a + \ln 2a - \ln\epsilon] = \ln 2.$$

On the other hand, using (9.12), we get

$$\int_{-a}^{2a} \frac{dx}{x} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{\ln 2 + \ln\left(\frac{\epsilon_1}{\epsilon_2}\right)\right\}. \tag{9.14}$$

Clearly, the limit in (9.14) on the right does not exist. This shows that

$$\mathrm{PV}\int_{-a}^{2a} \frac{dx}{x} = \ln 2$$

whereas the integral $\int_{-a}^{2a} \frac{dx}{x}$ does not exist as an improper integral. Similarly, we can easily see that

$$I = \mathrm{PV}\int_{-\infty}^{\infty} \frac{x^{n-1}}{x^n+1}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^{n-1}}{x^n+1}\,dx$$

so that $I = 0$ if n is even.

Now we start with $f(x) = 1/(1+x^2)$ and consider $\int_{-\infty}^{\infty} f(x)\,dx$, where the path of integration is the line $\mathrm{Im}\, z = y = 0$. If we want to use *the Cauchy residue formula*, we need to consider an integral along a closed contour. This observation suggests that we may have to start with an experiment

$$J = \int_C f(z)\,dz, \quad f(z) = \frac{1}{1+z^2},$$

where C is the semi-circular contour shown in Figure 9.2. Here we choose $R > 1$ so that $z = i$ lies inside C. Therefore, by the residue theorem, we easily get $J = 2\pi i \mathrm{Res}\,[f(z); i] = \pi$. Write $J = J_1 + J_2$, where

$$J_1 = \int_{-R}^{R} \frac{dx}{1+x^2} \quad\text{and}\quad J_2 = \int_{Re^{i0}}^{Re^{i\pi}} \frac{dz}{1+z^2} = \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{1+(Re^{i\theta})^2}.$$

Since

$$|J_2| \le \int_0^{\pi} \frac{R}{R^2-1}\,d\theta \to 0 \text{ as } R\to\infty,$$

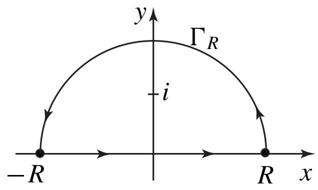


Figure 9.2: Contour $C = [-R, R] \cup \Gamma_R$.

we deduce that $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$. In particular, we have the well-known result

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Next, we use the same idea and show that

$$\int_0^\infty \frac{\cos ax}{m^2 + x^2}\, dx = \frac{\pi e^{-am}}{2m} \quad (a,\ m > 0). \tag{9.15}$$

Note that if we consider the most obvious complex function $\cos az/(m^2 + z^2)$, then we will not be able to achieve the desired result, because for $z = \pm iR$ (R large enough)

$$\left|\frac{\cos az}{m^2 + z^2}\right| \Bigg|_{z=iR} = \frac{e^{aR} + e^{-aR}}{2|m^2 - R^2|} \to \infty \quad \text{as } R \to \infty.$$

Nevertheless, as the term $e^{iaz} = e^{iax}e^{-ay}$ is bounded in the upper half-plane, it is natural to consider

$$f(z) = \frac{e^{iaz}}{m^2 + z^2} = \frac{(z + im)^{-1}e^{iaz}}{z - im}$$

and we use the same semi-circular contour as above. Then, we have

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}[f(z); im] = \frac{\pi e^{-am}}{m}.$$

Now, we write

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_0^\pi \frac{iRe^{i\theta}e^{iaRe^{i\theta}}}{m^2 + R^2e^{i2\theta}}\, d\theta = J_1 + J_2, \text{ say.} \tag{9.16}$$

Since $\sin x$ is odd and $\cos x$ is even, we have

$$J_1 = \int_{-R}^{R} f(x)\, dx = \int_{-R}^{R} \left(\frac{\cos ax}{m^2 + x^2} + i\frac{\sin ax}{m^2 + x^2}\right) dx = 2\int_0^R \frac{\cos ax}{m^2 + x^2}\, dx$$

and (as $e^{-aR\sin\theta} \leq e^0 = 1$ for $\theta \in [0, \pi]$, $a > 0$ and $R > 0$)

$$|J_2| \leq \int_0^{\pi} \frac{Re^{-aR\sin\theta}}{R^2 - m^2}\,d\theta \leq \frac{R}{R^2 - m^2}\int_0^{\pi} d\theta \ \to 0 \text{ as } \ R \to \infty.$$

The above observations prove (9.15), in view of (9.16). Further, using Leibnitz's rule, (9.15) readily gives

$$\int_0^{\infty} \frac{x\sin ax}{m^2 + x^2}\,dx = \frac{\pi e^{-am}}{2m}, \ a > 0.$$

In general, problems of this type may be solved using the same idea. Consider the integral $I = \int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is a rational function without real poles. Therefore, to evaluate such an integral, we let

$$J = \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz,$$

where $C = [-R, R] \cup \Gamma_R$ is the same contour as in Figure 9.2. Then

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz = 2\pi i \sum \mathrm{Res}\,[f(z); C].$$

As $R \to \infty$, the first integral on the left tends to I. We shall then have to show that the second integral tends to 0. If so, this would then imply that

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \mathrm{Res}\,[f(z); C].$$

Thus, we have to find a suitable condition under which $\int_{\Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$. Before we establish such a condition (see Theorem 9.23) let us first discuss a few more examples.

9.17. Example. We wish to prove that

$$\int_0^{\infty} \frac{x^{m-1}}{1 + x^n}\,dx = \frac{\pi}{n\sin(m\pi/n)}, \quad \text{for } m, n \in \mathbb{N} \text{ with } n > m > 0.$$

To do this, we let

$$f(z) = \frac{z^{m-1}}{1 + z^n}.$$

Then f has simple poles at $a_k = e^{i(1+2k)\pi/n}$ $(k = 0, 1, 2, \ldots, n-1)$ with

$$\mathrm{Res}\,[f(z); a_k] = \left.\frac{z^{m-1}}{nz^{n-1}}\right|_{z=a_k} = \left.\frac{z^m}{-n}\right|_{z=a_k} = -\frac{a_k^m}{n}.$$

Here we use a different contour to evaluate the integral.

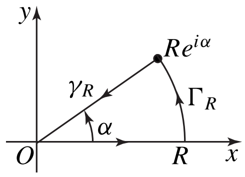


Figure 9.3: Contour $[0,R]\cup\Gamma_R\cup[Re^{i\alpha},0]$.

Note that these poles are simple and the only pole inside C (see Figure 9.3 with $\alpha = 2\pi/n$) is at a_0, where

$$\begin{aligned} C &= [0,R]\cup\{z=Re^{i\theta} : 0\le\theta\le 2\pi/n\}\cup\{z=re^{i2\pi/n} : 0\le r\le R\} \\ &= [0,R]\cup\Gamma_R\cup\gamma_R. \end{aligned}$$

Now (see Theorem 8.13), by the residue theorem

$$\int_C f(z)\,dz = \left(\int_{[0,R]}+\int_{\Gamma_R}+\int_{\gamma_R}\right) f(z)\,dz = -\frac{2\pi i a_0^m}{n}. \tag{9.18}$$

Since $|f(z)| \le \dfrac{|z|^{m-1}}{|z|^n-1} \le \dfrac{2|z|^{m-1}}{|z|^n}$ as $|z|\to\infty$, we have

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \frac{2}{R^{n+1-m}}\frac{2\pi R}{n} \to 0 \text{ as } R\to\infty. \tag{9.19}$$

Next,

$$\int_{\gamma_R} f(z)\,dz = -\int_0^R f(re^{i2\pi/n})\,d(re^{i2\pi/n}) = -e^{i2m\pi/n}\int_0^R \frac{r^{m-1}dr}{1+r^n}.$$

Using this equation and (9.19), (9.18) becomes as $R\to\infty$,

$$(1-a_0^{2m})\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = 2\pi i\left\{-\frac{a_0^m}{n}\right\}$$

and therefore,

$$\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n} = \left\{\frac{2\pi i}{n(a_0^m-a_0^{-m})}\right\} = \frac{\pi}{n\sin(m/n)\pi}. \qquad \bullet$$

9.20. Remark. If C is the rectangular contour with vertices at $\mp R$ and $\pm R+2\pi i$ (see Figure 9.4) and if

$$f(z) = \frac{e^{az}}{1+e^z} \quad (0<a<1),$$

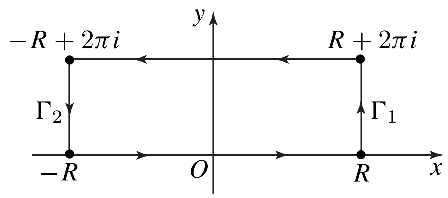


Figure 9.4: Rectangular contour with vertices at $\mp R,\ \pm R + 2\pi i$.

then we see that

$$\int_{-R}^{R} f(x)\,dx + \int_0^{2\pi} f(R+iy)\,d(R+iy) + \int_R^{-R} f(x+2\pi i)\,d(x+2\pi i)$$
$$+ \int_{2\pi}^0 f(-R+iy)\,d(-R+iy) = 2\pi i\{-e^{a\pi i}\},$$

as the only pole of f inside C is at $z = \pi i$. Since $f(z+2\pi i) = e^{2\pi ai} f(z)$, the above equation simplifies to

$$(1-e^{2\pi ai})\int_{-R}^{R} f(x)\,dx + i\int_0^{2\pi} [f(R+iy) - f(-R+iy)]\,dy = -2\pi i e^{\pi ai}.$$

Observe that

$$\left|\int_0^{2\pi} f(R+iy)\,dy\right| = \left|\int_0^{2\pi} \frac{e^{aR}e^{aiy}}{1+e^Re^{iy}}\,dy\right| \le \frac{2\pi e^{aR}}{e^R-1} = \frac{e^{(a-1)R}}{1-e^{-R}}.$$

Similarly, we find that

$$\left|\int_0^{2\pi} f(-R+iy)\,dy\right| \le \frac{2\pi e^{-aR}}{1-e^{-R}}.$$

As $R\to\infty$, we easily have (since $0<a<1$)

$$(1-e^{2\pi ai})\int_{-\infty}^{\infty} f(x)\,dx = -2\pi i e^{\pi ai}$$

from which we get

$$\text{(9.21)}\qquad \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi i}{e^{\pi ai}-e^{-\pi ai}} = \frac{\pi}{\sin \pi a}.$$

Using the new variable $e^x = t$, (9.21) reduces to

$$\text{(9.22)}\qquad \int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt = \frac{\pi}{\sin(\pi a)}.$$

Further, if we use the transformation $t = x^n$ with $a = m/n$ in (9.22), it becomes

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin(m\pi/n)} \quad \text{for} \quad n > m > 0.$$

However, we can obtain (9.22) directly by integrating a suitable function around a suitable contour. ●

We may use the same idea as in previous examples to solve more such problems (in general) by means of the residue theorem. They are mostly dealt with by application of the following

9.23. Theorem. *Suppose that f is analytic on $\mathbb{C}$ except for a finite number of poles such that none of its poles lies on the real axis. If there exist two positive numbers M and R_0 such that*

$$(9.24) \qquad |f(Re^{i\theta})| \le \frac{M}{R^\alpha} \quad \text{for } R > R_0,$$

for some $\alpha > 1$, then

$$(9.25) \qquad \int_{-\infty}^{\infty} f(x)\,dx = \begin{cases} 2\pi i \sum \mathrm{Res}\,[f(z);\mathbb{H}^+] \\ -2\pi i \sum \mathrm{Res}\,[f(z);\mathbb{H}^-], \end{cases}$$

where $\mathbb{H}^+, \mathbb{H}^-$ are respectively the upper and lower half-planes.

Proof. Let R be arbitrarily large enough so that all the poles of $f(z)$ are in $|z| < R$. Let $C = [-R,R] \cup \Gamma_R$, where $\Gamma_R = \{z : |z| = R,\ 0 \le \arg z \le \pi\}$, the semi-circular arc of radius R (see Figure 9.2). Then

$$(9.26) \qquad \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz.$$

But, by (9.24), the second integral on the right of (9.26) is such that

$$\left|\int_{\Gamma_R} f(z)\,dz\right| = \left|\int_0^\pi f(Re^{i\theta})iRe^{i\theta}\,d\theta\right| \le \frac{M}{R^\alpha}R\pi = \frac{M\pi}{R^{\alpha-1}},$$

which approaches zero as $R \to \infty$, since $\alpha > 1$. The conclusion now follows from (9.26) by letting $R \to \infty$. ■

9.27. Corollary. *The conclusion of the above theorem holds if f is a rational function, $f(z) = P(z)/Q(z)$, where*

(i) *f has no poles on the real axis, and*

(ii) *$P(z)$ and $Q(z)$ are polynomials of degree m and $m + n$, respectively, with $n \ge 2$.*

Proof. The condition (ii) implies that there exist two positive numbers M_1 and M_1' such that (see Exercise 6.88)

$$M_1'|z|^m \le |P(z)| \le M_1|z|^m \quad \text{for } |z| \ge R$$

and M_2 such that $|Q(z)| \ge M_2|z|^{m+n}$ for $|z| \ge R$ $(R \ge 1)$. Then

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{M_1|z|^m}{M_2|z|^{m+n}} = \frac{M_1}{M_2|z|^n} \le \frac{M_1}{M_2}\cdot\frac{1}{|z|^2} \le \frac{M}{|z|^2}, \text{ say},$$

for $|z| \ge R$, and Theorem 9.23 applies. ■

9.28. Example. Let us show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}\,dx = \frac{\pi(1-e^{-2})}{2}.$$

To do this, we consider (since $\sin^2 x = (1-\cos 2x)/2$)

$$f(z) = \frac{1-e^{i2z}}{1+z^2}.$$

This function has exactly two simple poles at $\pm i$ and

$$\text{Res}\,[f(z); i] = \lim_{z\to i} \frac{1-e^{i2z}}{z+i} = -\frac{i}{2}(1-e^{-2}).$$

Let $C = [-R, R] \cup \Gamma_R$, with $\Gamma_R = \{z : |z| = R,\ 0 \le \arg z \le \pi\}$. Then

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma_R} f(z)\,dz = 2\pi i\left\{-\frac{i}{2}(1-e^{-2})\right\} = \pi(1-e^{-2}).$$

Since $|f(z)| \le (1+e^{-2\text{Im}\,z})/(|z|^2-1) \le 2/(R^2-1)$ for $z \in \Gamma_R$, we have

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \frac{2}{R^2-1}\int_{\Gamma_R} |dz| = \frac{2\pi R}{R^2-1} \to 0 \quad \text{as } R \to \infty.$$

It follows that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{2\sin^2 x - i\sin 2x}{1+x^2}\,dx = \pi(1-e^{-2})$$

and equating the real parts yields the desired result. ●

9.29. Remark. Theorem 9.23 may not apply in some cases: for example to the well-known *Gauss Error Integral* $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. This is because e^{-z^2} does not have the required limiting behavior, as $z \to \infty$. In fact,

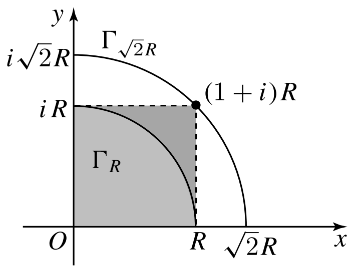


Figure 9.5: Illustration for Gauss Error Integral.

(i) $|e^{-z^2}| = 1$ on $\arg z = \pm\frac{\pi}{4}$ and $\arg z = \pm\frac{3\pi}{4}$.

(ii) $|e^{-z^2}| \to 0$ faster than reciprocal of any polynomial on the lines $\arg z = \pm\frac{\pi}{2}, 0, \pi$.

We include here a classical proof of the error integral. To do this, we let $I = \int_0^R e^{-x^2}\,dx$. Then

$$I^2 = \left(\int_0^R e^{-x^2}\,dx\right)\left(\int_0^R e^{-y^2}\,dy\right) = \int_0^R\int_0^R e^{-(x^2+y^2)}\,dx\,dy.$$

Here we are integrating along a square S in the first quadrant whose sides have length R. Let Γ_R and $\Gamma_{\sqrt{2}R}$ denote the quarter-circles in the first quadrant centered at the origin having radii R and $R\sqrt{2}$, respectively (see Figure 9.5). Evaluating along the circles in polar coordinates, we have

$$\int_0^{\pi/2}\int_0^R e^{-r^2} r\,dr\,d\theta < \int_0^R\int_0^R e^{-(x^2+y^2)}\,dx\,dy < \int_0^{\pi/2}\int_0^{R\sqrt{2}} e^{-r^2} r\,dr\,d\theta,$$

and so, for each $R > 0$, we have

$$\frac{\pi}{4}(1-e^{-R^2}) < I^2 = \left(\int_0^R e^{-x^2}\,dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2}).$$

Letting $R \to \infty$, we see that $I = \sqrt{\pi}/2$. •

9.30. Remark. One can evaluate the error integral using the residue theorem directly. To do this, we define

$$f(z) = \frac{e^{-z^2}}{1+e^{-2az}},\ a = (1+i)\sqrt{\frac{\pi}{2}}.$$

Since $a^2 = \pi i$, we note that $e^{-2a(a)} = 1$ and so a is a period of e^{-2az}. As $e^{-a^2} = e^{-\pi i} = -1$, we also note that

$$f(z) - f(z+a) = \frac{e^{-z^2}}{1+e^{-2az}}(1-e^{-a^2-2az}) = e^{-z^2}. \tag{9.31}$$

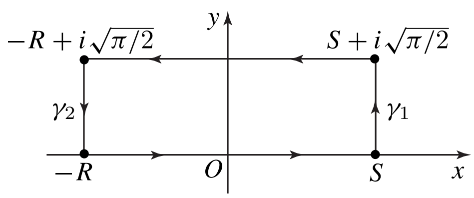


Figure 9.6: Rectangular contour $C=[-R,S]\cup\gamma_1\cup[S+i\sqrt{\pi/2},-R+i\sqrt{\pi/2}]\cup\gamma_2$.

Note that $f(z)$ has infinitely many simple poles in $\mathbb{C}$, namely at $-a/2+ka$, $k\in\mathbb{Z}$. Therefore, it is not advisable to choose a contour that includes many poles. Instead, we choose the rectangular contour as depicted in Figure 9.6. Since f has only the point $a/2$ inside the contour

$$C=[-R,S]\cup\gamma_1\cup[S+i\sqrt{\pi/2},-R+i\sqrt{\pi/2}]\cup\gamma_2,$$

we have

$$\text{(9.32)}\qquad \operatorname{Res}[f(z);a/2]=\left.\frac{e^{-z^2}}{-2ae^{-2az}}\right|_{z=a/2}=\frac{e^{-a^2/4}}{-2ae^{-a^2}}=-\frac{i}{2\sqrt{\pi}}.$$

Therefore, by the residue theorem, we have

$$\int_{-R}^{S} f(x)\,dx+\int_{\gamma_1} f(z)\,dz+\int_{S}^{-R} f(x+(\sqrt{\pi}/2+i\sqrt{\pi}/2)\,dx$$
$$+\int_{\gamma_2} f(z)\,dz=2\pi i\operatorname{Res}[f(z);a/2].$$

Because of (9.31) and (9.32) the above equation becomes

$$\int_{-R}^{S} e^{-x^2}\,dx+\int_{\gamma_1} f(z)\,dz+\int_{\gamma_2} f(z)\,dz=\sqrt{\pi}$$

and letting $R,S\to\infty$, we conclude that $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$. ●

9.3 Integrals of Type $\int_{-\infty}^{\infty} g(x)\cos mx\,dx$

This section discusses some other important improper integrals. Suppose that $f(z)=e^{iaz}g(z)$ for $a\in\mathbb{R}$ and for some analytic function g. Then the conditions on f in Theorem 9.23 can be weakened; that is in this case it is enough to assume that

$$|f(Re^{i\theta})|\le\frac{M}{R}\quad\text{for } R>R_0.$$

Observe that in most of the examples it is sufficient that $f(z) \to 0$ as $z \to \infty$ in $\arg z$, $0 \le \arg z \le \pi$. Therefore, Theorem 9.23 and Corollary 9.27 take the following form:

9.33. Theorem. *Suppose that g is analytic in $\mathbb{C}$ except possibly for a finite number of poles and none of them are real. If there exist two positive numbers M and R_0 such that*

$$|g(Re^{i\theta})| \le \frac{M}{R^\alpha} \quad \text{for } R \ge R_0\ , \tag{9.34}$$

for some $\alpha > 0$, then

$$\int_{-\infty}^{\infty} g(x)e^{iax}\,dx = \begin{cases} 2\pi i \sum \operatorname{Res}\,[g(z)e^{iaz}; \mathbb{H}^+] & \text{if } a > 0 \\ -2\pi i \sum \operatorname{Res}\,[g(z)e^{iaz}; \mathbb{H}^-] & \text{if } a < 0, \end{cases} \tag{9.35}$$

where $\mathbb{H}^+, \mathbb{H}$ are the upper and lower half-planes, respectively. Further, if $g(z) = P(z)/Q(z)$, where P, Q are polynomials such that

(i) *g has no poles on the real axis,*
(ii) *deg $Q \ge 1+$ deg $P \iff |g(z)| \le M/|z|$ for $|z| \ge R_0$,*
(iii) *g is real on the real axis,*

then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos ax\,dx = \textit{Real part of the R.H.S of } (9.35)$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin ax\,dx = \textit{Imaginary part of the R.H.S of } (9.35).$$

In establishing our theorem, we shall make use of *Jordan's Inequality* (see Figure 9.7).

9.36. Lemma. *For $\theta \in [0, \pi/2]$, we have $\sin\theta \ge (2/\pi)\theta$.*

Proof. Clearly, the inequality is true at the end points. Now,

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\right) = \frac{\psi(\theta)}{\theta^2} \quad \text{with } \psi(\theta) = \theta\cos\theta - \sin\theta.$$

We claim that $\phi(\theta) = \theta^{-1}\sin\theta$ is strictly decreasing on $(0, \pi/2)$. To do this, it suffices to observe that $\psi'(\theta) = -\theta\sin\theta < 0$ for $\theta \in (0, \pi/2)$ so that ψ is decreasing on $(0, \pi/2)$ and thus, $\psi(\theta) < \psi(0) = 0$ which shows that $\phi'(\theta) < 0$ and hence, $\phi(\theta) = \theta^{-1}\sin\theta$ is strictly decreasing on $(0, \pi/2)$. Finally, as

$$\lim_{\theta\to\pi/2} \phi(\theta) = \lim_{\theta\to\pi/2} \frac{\sin\theta}{\theta} = \frac{2}{\pi},$$

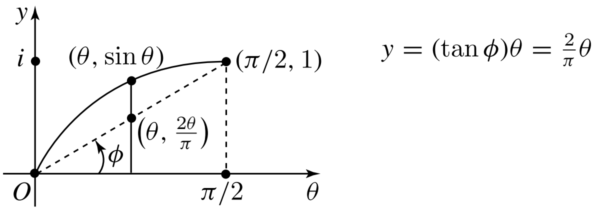


Figure 9.7: Geometric proof of Jordan's inequality.

we deduce that $\phi(\theta) > 2/\pi$ on $(0, \pi/2)$, as desired. ■

9.37. Remark. For a geometric proof of Lemma 9.36, we refer to Figure 9.7. Set $f(\theta) = \sin\theta$. Then $f''(\theta) = -\sin\theta < 0$ on $(0, \pi/2)$ so that $y = \sin\theta$ is concave down on $(0, \pi/2)$. Therefore, the graph of $y = \sin\theta$ lies above the straight lines connecting the end points $(0, 0)$ and $(\pi/2, 1)$. Note that the equation of the line passing through the point $(0, 0)$ and $(\pi/2, 1)$ is

$$y = (2/\pi)\theta.$$

Therefore, the inequality $\sin\theta \geq (2/\pi)\theta$ holds on $[0, \pi/2]$. ●

9.38. Remark. For $R > 0$, it follows that

$$I = \int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}.$$

Indeed, if $f(\theta) = e^{-R\sin\theta}$ then $f(\theta) = f(\pi - \theta)$ so that

$$\begin{aligned} I &= 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \quad \text{(see Remark 9.5)} \\ &\leq 2\int_0^{\pi/2} e^{-(2R/\pi)\theta}\,d\theta \quad (\text{since } \sin\theta \geq \tfrac{2}{\pi}\theta \text{ on } [0, \pi/2]) \\ &= 2\left(\frac{e^{-R} - 1}{-2R/\pi}\right) = \frac{\pi}{R}(1 - e^{-R}) < \frac{\pi}{R} \end{aligned}$$

and the desired inequality follows. ●

Proof of Theorem 9.33. First consider $a > 0$. Choose R large enough so that all the singularities of g in $\mathbb{H}^+$ lie inside the upper semi circle $C = [-R, R] \cup \Gamma_R$ (see Figure 9.2). By the residue theorem

$$\int_{-R}^{R} e^{iax}g(x)\,dx + \int_{\Gamma_R} e^{iaz}g(z)\,dz = 2\pi i \sum \text{Res}\,[g(z)e^{iaz}; \mathbb{H}^+]. \tag{9.39}$$

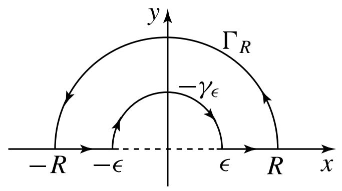


Figure 9.8: Contour $[-R,-\epsilon]\cup(-\gamma_\epsilon)\cup[\epsilon,R]\cup\Gamma_R$.

Now we estimate the absolute value of the second integral. Let $z=Re^{i\theta}\in\Gamma_R$. Then $|e^{iaz}|=e^{-aR\sin\theta}$. In view of Remark 9.38, taking $R\geq R_0$ so that $|g(z)|\leq M/R^\alpha$, we obtain

$$\left|\int_{\Gamma_R} e^{iaz}g(z)\,dz\right|\leq\frac{M}{R^{\alpha-1}}\int_0^\pi e^{-aR\sin\theta}\,d\theta<\frac{M}{R^{\alpha-1}}\left(\frac{\pi}{aR}\right)=\frac{\pi M}{aR^\alpha}.$$

Since a and $\alpha>0$, the R.H.S of the above inequality approaches zero as $R\to\infty$ and the required result follows from (9.39).

As for the case $a<0$, we simply consider the contour in the lower half-plane $\mathbb{H}^-$ and proceed similarly. The proof of the remaining part is similar to the proof of Corollary 9.27. ■

Using the idea described in Theorem 9.33, it is easy to show that

$$\int_0^\infty\frac{x\sin ax}{m^2+x^2}\,dx=\frac{\pi e^{-am}}{2}\quad(a,\ m>0).$$

9.4 Singularities on the Real Axis

We shall now discuss the case where f has simple poles on the real axis. Suppose that the only singularity of f on the real axis is a simple pole at the origin. Let C be the indented contour shown in Figure 9.8. Thus, C consists of the line segment $[-R,-\epsilon]$, $\epsilon<R$, the semi-circle $(-\gamma_\epsilon)$ from $-\epsilon$ to ϵ, the line segment $[\epsilon,R]$, and the semi-circle Γ_R from R to $-R$. Here, the small semi-circle $-\gamma_\epsilon$ is described to avoid the singularity of f at the origin. Assume further that C encloses all the singularities of f in the upper half-plane $\mathbb{H}^+$. By the residue theorem

$$\begin{aligned}\int_C f(z)\,dz &= \int_{-R}^{-\epsilon} f(x)\,dx-\int_{\gamma_\epsilon} f(z)\,dz+\int_\epsilon^R f(x)\,dx+\int_{\Gamma_R} f(z)\,dz\\ &= 2\pi i\sum\operatorname{Res}\,[f(z);\mathbb{H}^+].\end{aligned}\tag{9.40}$$

To evaluate integrals of this type we employ

9.41. Theorem. *Let* $\gamma_\epsilon=\{z:\ |z-z_0|=\epsilon$ *and* $\theta_1\leq\arg(z-z_0)\leq\theta_2\}$.

If f is continuous on $0 < |z - z_0| \le \epsilon$ and if $\lim_{z \to z_0} (z - z_0) f(z) = \ell$, then

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\, dz = i\ell(\theta_2 - \theta_1),$$

where γ_ϵ is positively oriented.

Proof. Write $(z - z_0) f(z) = \ell + \phi(z)$ and choose ϵ sufficiently small so that, for any arbitrary $\eta > 0$, $|\phi(z)| < \eta$ if $|z - z_0| = \epsilon$. Now,

$$\int_{\gamma_\epsilon} f(z)\, dz = \ell \int_{\gamma_\epsilon} \frac{dz}{z - z_0} + \int_{\gamma_\epsilon} \frac{\phi(z)}{z - z_0}\, dz = \ell \int_{\theta_1}^{\theta_2} i\, d\theta + \int_{\theta_1}^{\theta_2} i\phi(z_0 + \epsilon e^{i\theta})\, d\theta.$$

As $\left| \int_{\theta_1}^{\theta_2} i\phi(z_0 + \epsilon e^{i\theta})\, d\theta \right| \le \eta(\theta_2 - \theta_1)$, the last equation implies the desired result. ●

In particular, if $\theta_2 - \theta_1 = 2\pi$ then γ_ϵ becomes a positively oriented full circle and so,

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\, dz = 2\pi i \lim_{z \to z_0} (z - z_0) f(z).$$

9.42. Remark. If there exists a real number $\alpha > 1$ such that $|f(z)| \le K|z|^{-\alpha}$ as $|z| \to \infty$ in the upper half-plane then, as seen in the proof of Theorem 9.41, we obtain $\lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz = 0$. Therefore, (9.40) becomes

$$\lim_{\substack{R \to \infty \\ \epsilon \to 0}} \left\{ \int_{-R}^{-\epsilon} f(x)\, dx + \int_{\epsilon}^{R} f(x)\, dx \right\} + \lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz$$

$$- \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\, dz = 2\pi i \sum \operatorname{Res}[f(z); C].$$

From Theorem 9.41 (with $z_0 = 0$), this reduces to

$$\int_{-\infty}^{\infty} f(x)\, dx + 0 - i(\pi - 0) \operatorname{Res}[f(z); 0] = 2\pi i \sum \operatorname{Res}[f(z); \mathbb{H}^+]$$

from which we get the value of the integral when the only singularity of f on the real axis is a simple pole at the origin. ●

9.43. Example. Let us evaluate the integral

$$I = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

We have trouble if we proceed the way we did in the previous two types of problems as we are now faced with a problem at the origin. Again, for $z = iR$ (R large enough), we have

$$\left.\frac{\sin z}{z}\right|_{z=iR} = \frac{e^R - e^{-R}}{2R} \to \infty \quad \text{as } R \to \infty.$$

So, for evaluating the given integral, we have to consider a suitable function and a suitable contour which avoids the origin. First we rewrite

$$(9.44) \qquad I = \frac{1}{2i} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\, dx \right\}$$

and note that $|e^{iz}| = e^{-y} \leq 1$ on the upper half-plane. Thus to evaluate the given integral we consider

$$\int_C f(z)\, dz, \quad f(z) = \frac{e^{iz}}{z},$$

where C is the contour shown in Figure 9.8. Note that C is made up of the (large) upper semi circular contour $\Gamma_R = \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}$, the (small) semi-circular contour $-\gamma_\epsilon$, where $\gamma_\epsilon = \{z = \epsilon e^{i\theta} : 0 \leq \theta \leq \pi\}$, and the real axis intercepted between them, namely the segments $[-R, -\epsilon]$ and $[\epsilon, R]$. Note that f has a simple pole at the origin and there are no other singularities. Since $z = 0$ lies outside C, we have, by the Cauchy theorem, $\int_C f(z)\, dz = 0$; that is,

$$(9.45) \qquad \int_{-R}^{-\epsilon} f(x)\, dx - \int_{\gamma_\epsilon} f(z)\, dz + \int_{\epsilon}^{R} f(x)\, dx + \int_{\Gamma_R} f(z)\, dz = 0.$$

Since $\lim_{z\to 0} z f(z) = 1$, by Theorem 9.41, we note that

$$(9.46) \qquad \lim_{\epsilon\to 0} \int_{\gamma_\epsilon} f(z)\, dz = i(\pi - 0).$$

Alternatively, we can provide a direct proof. As $f(z)$ has a simple pole with $\text{Res}\,[f(z); 0] = 1$, we have $f(z) = 1/z + g(z)$ for z near 0, where $g(z)$ is analytic at $z = 0$. In particular,

$$\begin{aligned}
\int_{\gamma_\epsilon} f(z)\, dz &= \int_{\gamma_\epsilon} \frac{1}{z}\, dz + \int_{\gamma_\epsilon} g(z)\, dz \\
&= \int_0^{\pi} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta + \int_{\gamma_\epsilon} g(z)\, dz \\
&= i\pi + \int_{\gamma_\epsilon} g(z)\, dz.
\end{aligned}$$

The integral on the right tends to zero as $\epsilon \to 0$, because $g(z)$ is bounded near 0 and that

$$\left|\int_{\gamma_\epsilon} g(z)\, dz\right| \le \sup_{z\in\gamma_\epsilon} |g(z)|\pi\epsilon \to 0 \quad \text{as } \epsilon \to 0.$$

Thus, $\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)\, dz = i\pi$. We next claim that $\lim_{R\to\infty}\int_{\Gamma_R} f(z)\, dz = 0$. Note that $e^{-\sin\theta} \le e^0 = 1$ for $\theta \in [0,\pi]$. But then we cannot claim that the last integral approaches zero as $R \to \infty$. Instead, by Jordan's inequality (see Remark 9.38), we observe that

$$\left|\int_{\Gamma_R} f(z)\, dz\right| \le \int_0^\pi e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \to 0 \text{ as } R \to \infty. \tag{9.47}$$

So, if we allow $R \to \infty$ and $\epsilon \to 0$ in (9.45), by (9.46) and (9.47),

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{\int_{-R}^{-\epsilon} f(x)\, dx + \int_\epsilon^R f(x)\, dx\right\} = i\pi.$$

By (9.44), we then have $I = \pi/2$. ●

9.48. Example. By considering $f(z) = ze^{iz}(z^2-1)^{-1}$ $(a > 0)$, and $C = [-R, -1-\epsilon_1] \cup (-\gamma_{\epsilon_1}) \cup [-1+\epsilon_1, 1-\epsilon_2] \cup (-\gamma_{\epsilon_2}) \cup [1+\epsilon_2, R] \cup \Gamma_R$, it is easy to prove that

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2-1}\, dx = \pi\cos a.$$

Indeed, as usual, the Cauchy theorem gives

$$\left(\int_{-R}^{-1-\epsilon_1} - \int_{\gamma_{\epsilon_1}} + \int_{-1+\epsilon_1}^{1-\epsilon_2} - \int_{\gamma_{\epsilon_2}} + \int_{1+\epsilon_2}^{R} + \int_{\Gamma_R}\right) f(z)\, dz = 0.$$

As $\text{Res}\,[f(z); 1] = e^{ia}/2$ and $\text{Res}\,[f(z); -1] = e^{-ia}/2$, it can be seen that

$$\lim_{\epsilon_1\to 0}\int_{\gamma_{\epsilon_1}} f(z)\, dz = \frac{e^{ia}}{2} i\pi \quad \text{and} \quad \lim_{\epsilon_2\to 0}\int_{\gamma_{\epsilon_2}} f(z)\, dz = \frac{e^{-ia}}{2} i\pi.$$

Further,

$$\left|\int_{\Gamma_R} f(z)\, dz\right| \le \int_0^\pi \frac{R^2}{R^2-1} e^{-aR\sin\theta}\, d\theta \le \frac{R^2}{R^2-1}\left(\frac{\pi}{aR}\right) \to 0 \quad \text{as } R \to \infty.$$

The desired result follows by the limiting process.

Similarly, we can easily show that

$$\int_0^\infty \frac{\sin ax}{x(x^2+m^2)}\, dx = \frac{\pi(1-e^{-am})}{2m^2} \quad (a > 0, m > 0). \tag{9.49}$$

Then, by differentiating both sides of (9.49) with respect to m (keeping a as constant) and using Leibnitz's rule, we easily get

$$\text{(9.50)} \qquad \int_0^\infty \frac{\sin ax}{x(x^2+m^2)^2}\,dx = \frac{\pi}{4m^4}[2-(2+am)e^{-am}] \quad (a,m>0).$$

Similarly, it is easy to see from (9.49) that (by differentiating (9.49) with respect to a and keeping m as constant)

$$\text{(9.51)} \qquad \int_0^\infty \frac{\cos ax}{x^2+m^2}\,dx = \frac{\pi e^{-am}}{2m} \quad (a,m>0).$$

One can also give an independent proof for (9.50). In fact, (9.51) has already been proved in Section 9.2. ●

9.52. Remark. (i) For $0<R<\infty$, integration by parts yields

$$\begin{aligned}
\int_0^R \frac{\sin x}{x}\,dx &= \int_0^{R/2} \frac{\sin 2x}{x}\,dx \\
&= 2\int_0^{R/2} \frac{\sin x}{x}\,d(\sin x) \\
&= \left.\frac{2\sin^2 x}{x}\right|_0^{R/2} - 2\int_0^{R/2} \sin x d\left(\frac{\sin x}{x}\right) \\
&= \frac{4\sin^2(R/2)}{R} - 2\int_0^{R/2} \frac{\sin x}{x}\,d(\sin x) + 2\int_0^{R/2} \frac{\sin^2 x}{x^2}\,dx \\
&= \frac{4\sin^2(R/2)}{R} - \int_0^{R/2} \frac{\sin 2x}{x}\,dx + 2\int_0^{R/2} \frac{\sin^2 x}{x^2}\,dx.
\end{aligned}$$

Thus,

$$\int_0^R \frac{\sin x}{x}\,dx = \frac{2\sin^2(R/2)}{R} + \int_0^{R/2} \frac{\sin^2 x}{x^2}\,dx.$$

Since $\lim_{R\to\infty} R^{-1}\sin^2(R/2) = 0$, it follows that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

and hence, we have

$$\text{(9.53)} \qquad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

(ii) One can give an independent proof for (9.53) by considering the integral $\int_C f(z)\,dz$, where

$$f(z) = \frac{1-e^{2iz}}{z^2}, \quad \operatorname{Re} f(x) = \frac{1-\cos 2x}{x^2} = \frac{2\sin^2 x}{x^2},$$

and C is the contour shown in Figure 9.8. Since $\lim_{\epsilon\to 0} zf(z) = -2i$, we have

$$\lim_{z\to 0}\int_{\gamma_\epsilon} f(z)\,dz = i(-2i)(\pi - 0) = 2\pi,$$

where γ_ϵ is positively oriented. Further, $|1 - e^{2iz}| \leq 1 + e^{-2y} \leq 2$ for $y \geq 0$ and so,

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \frac{2\pi}{R}, \quad \text{i.e.} \quad \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = 0.$$

As in the above example, we easily deduce that

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\left[\int_{-R}^{-\epsilon} f(x)dx + \int_{\epsilon}^{R} f(x)\,dx\right] = 2\pi, \quad \text{i.e.} \quad \int_{-\infty}^{\infty}\frac{1 - e^{2ix}}{x^2}\,dx = 2\pi.$$

On equating real parts and taking account of the fact that $\sin^2 x/x^2$ is an even function we obtain (9.53).

(iii) Since

$$\sin^3 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{1}{4}\,\mathrm{Im}\,[(1 - e^{3ix}) - 3(1 - e^{ix})],$$

we have $\lim_{z\to 0} zf(z) = 3$, where

$$f(z) = \frac{(1 - e^{3iz}) - 3(1 - e^{iz})}{z^3}.$$

Therefore, proceeding as in the above example, it is easy to show that

$$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}. \qquad \bullet$$

9.54. Example. By the error integral, we have

$$\sqrt{\pi} = 2\int_0^\infty e^{-x^2}\,dx \tag{9.55}$$

Using (9.55), we can evaluate the integrals

$$\int_0^\infty \sin x^2\,dx \quad \text{and} \quad \int_0^\infty \cos x^2\,dx.$$

To do this, we consider the entire function $f(z) = e^{-z^2}$. The contour C shown in Figure 9.3 (with $\alpha = \pi/4$) shows that

$$\int_0^R f(x)\,dx + \int_{\Gamma_R} f(z)\,dz + \int_R^0 f(xe^{i\pi/4})\,d(xe^{i\pi/4}) = 0. \tag{9.56}$$

Note that $\int_R^0 f(xe^{i\pi/4})\, d(xe^{i\pi/4}) = -e^{i\pi/4}\int_0^R e^{-ix^2}\, dx$ and

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\, dz\right| &= \left|\int_0^{\pi/4} f(Re^{i\theta})iRe^{i\theta}\, d\theta\right| \\
&\le R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\, d\theta \\
&= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\cos\phi}\, d\phi \\
&= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\theta}\, d\theta \quad (\theta = \pi/2-\phi) \\
&< \frac{R}{2}\frac{\pi(1-e^{-R^2})}{2R^2} \to 0 \ \text{ as } R\to\infty.
\end{aligned}$$

Taking the limit in (9.56), as $R\to\infty$, we obtain

$$\int_0^\infty f(x)\, dx - e^{i\pi/4}\int_0^\infty e^{-ix^2}\, dx = 0.$$

So, by (9.55),

(9.57) $$\int_0^\infty e^{-ix^2}\, dx = e^{-i\pi/4}\left(\frac{\sqrt{\pi}}{2}\right).$$

Separating the real and the imaginary parts in (9.57), we conclude that

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$ ●

9.58. Remark. The above method helps us to prove a more general result by choosing $f(z) = e^{-z^2}$ and letting C be the contour shown in Figure 9.3. Then, proceeding as in Example 9.54, it is clear that

$$\int_0^\infty f(x)\, dx = e^{i\alpha}\int_0^\infty e^{-x^2(\cos 2\alpha + i\sin 2\alpha)}\, dx.$$

Using (9.55),

$$\int_0^\infty e^{-x^2\cos 2\alpha}e^{-ix^2\sin 2\alpha}\, dx = \frac{\sqrt{\pi}e^{-i\alpha}}{2}.$$

Equating the real and imaginary parts, we see that

$$\int_0^\infty e^{-x^2\cos 2\alpha}\cos(x^2\sin 2\alpha)\, dx = \frac{\sqrt{\pi}}{2}\cos\alpha$$

and

$$\int_0^\infty e^{-x^2\cos 2\alpha}\sin(x^2\sin 2\alpha)\, dx = \frac{\sqrt{\pi}}{2}\sin\alpha.$$

Note that $\alpha = \pi/4$ yields the Fresnel integrals. ●

9.59. Example. Finally, by integrating $e^{az}/\cosh bz$ $(-b < a < b)$ around the rectangular contour with vertices at $\pm R$ and $\pm R + i\pi/b$, we show that

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx}\, dx = \frac{\pi}{b} \frac{1}{\cos(a\pi/2b)}.$$

As $I = \frac{1}{b}\int_{-\infty}^{\infty} \frac{e^{(a/b)y}}{\cosh y}\, dy$, it suffices to prove the result for $b = 1$ and $a \in (-1, 1)$. So, we let

$$f(z) = \frac{e^{az}}{\cosh z}.$$

Then, inside the rectangular contour with vertices at $\pm R$ and $\pm R + i\pi$, f has only one pole, namely, at $z = i\pi/2$. Further,

$$\text{Res}\left[f(z); \frac{\pi}{2}i\right] = \left.\frac{e^{az}}{\sinh z}\right|_{z=\pi i/2} = -ie^{a\pi i/2}$$

and therefore, by the residue theorem, we see that

$$\int_{-R}^{R} [f(x) - f(x + i\pi)]\, dx + \int_{0}^{\pi} f(R + iy)i\, dy$$

$$+\int_{\pi}^{0} f(-R + iy)i\, dy = 2\pi e^{a\pi i/2}. \tag{9.60}$$

Consider the second integral in (9.60):

$$\left|\int_0^{\pi} f(R + iy)i\, dy\right| \le \int_0^{\pi} |f(R + iy)|\, dy = \int_0^{\pi} \frac{e^{aR}}{|\cosh(R + iy)|}\, dy.$$

Since

$$|\cosh(R + iy)| = \left|\frac{e^{(R+iy)} + e^{-(R+iy)}}{2}\right| \ge \frac{e^R - e^{-R}}{2} \ge \frac{e^R}{4}$$

and $-1 < a < 1$, the second integral in (9.60) approaches zero as $R \to \infty$.

Similarly, we can show that the third integral in (9.60) approaches zero as $R \to \infty$. Thus, integrals along γ_1 and γ_2, i.e. the second and third integrals in (9.60), approach zero with increasing R, so that (9.60) becomes

$$\int_{-\infty}^{\infty} [f(x) - f(x + i\pi)]\, dx = (1 + e^{ia\pi}) \int_{-\infty}^{\infty} f(x)\, dx = 2\pi e^{ia\pi/2},$$

that is,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, dx = 2\pi \left\{\frac{e^{ia\pi/2}}{1 + e^{ia\pi}}\right\} = \frac{\pi}{\cos(a\pi/2)}.$$ ●

9.5 Integrals Involving Branch Points

Integrals featuring x^λ and $\log x$ can in some cases be evaluated using contour integration. We illustrate this in detail by considering a special case of the real integral

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx \quad \text{for } 0<\lambda<1.$$

The process used to evaluate such integrals is often referred to as the integral around branch point. Clearly, the integral is improper for two reasons as it has a infinite discontinuity at the origin and has an infinite limit of integration. Moreover,

$$\frac{x^{\lambda-1}}{1+x}\sim x^{\lambda-1} \text{ for } x \text{ near } 0, \text{ and } \frac{x^{\lambda-1}}{1+x}\sim x^{\lambda-2} \text{ for } x \text{ near } \infty$$

so that the integral does converge for $0<\lambda<1$. Observe that when x is replaced by z, the integrand becomes

$$f(z)=\frac{z^{\lambda-1}}{1+z}$$

which is a multiple-valued function. The origin is a branch point of $f(z)$. We consider the branch of $f(z)$ on the slit plane $\mathbb{C}\setminus[0,\infty)$ so that the positive real axis has been chosen as the branch cut for $f(z)$ with

$$z^{\lambda-1}=e^{(\lambda-1)(\ln|z|+i\arg z)}, \quad 0<\arg z<2\pi.$$

This guarantees that $f(z)$ is single-valued and we can integrate along an appropriate contour. The function f has a simple pole at $z=-1$ with residue

$$\operatorname{Res}[f(z);-1]=\lim_{z\to-1}(1+z)f(z)=\lim_{z\to-1}z^{\lambda-1}=e^{(\lambda-1)\pi i}=-e^{\pi i\lambda}.$$

Further the residue theorem is applicable only to single-valued functions and the origin cannot be inside the simple closed contour C along which we integrate. Now, we let $R>1>\epsilon>0$ and set (see Figure 9.9)

$$C=[\epsilon+i\delta,R+i\delta]\cup\Gamma_R\cup[R-i\delta,\epsilon-i\delta]\cup(-\gamma_\epsilon)$$

so that the inside of C is a simply connected domain not containing the origin but containing the point $z=-1$. Thus, C consists of

(i) the horizontal line segment γ_+ from $\epsilon+i\delta$ to $R+i\delta$

(ii) the circular arc Γ_R of radius R centered at the origin traced counterclockwise from $R+i\delta$ to $R-i\delta$

(iii) the horizontal line segment γ_- from $R-i\delta$ to $\epsilon-i\delta$

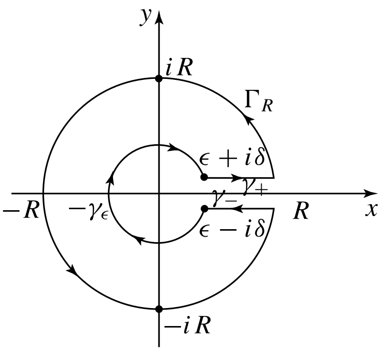


Figure 9.9: Contour for a multi-valued function.

(iv) the circular arc $(-\gamma_\epsilon)$ of radius ϵ centered at the origin traced clockwise from $\epsilon - i\delta$ to $\epsilon + i\delta$.

The residue theorem yields

(9.61) $$I = \int_C \frac{z^{\lambda-1}}{1+z}\, dz = -2\pi i e^{\pi i \lambda}.$$

The value of the integral is independent of δ, R and ϵ, but it depends only on the fact that $z = -1$ lies inside C. Therefore, it is natural to write (9.61) equivalently as

(9.62) $$I = \left(\int_{\gamma_+} + \int_{\Gamma_R} + \int_{\gamma_-} + \int_{\gamma_\epsilon} \right) \frac{z^{\lambda-1}}{1+z}\, dz = -2\pi i e^{\pi i \lambda}.$$

Our method of approach will then be to let $\delta \to 0$, $R \to \infty$, and $\epsilon \to 0$ to obtain the desired value of the given real integral. Note that, despite what is shown in Figure 9.9, it is possible to regard the slit $[0, \infty)$ as having an *upper side* of the positive real axis for which $\arg z = 0$ and *a lower side* of the positive real axis for which $\arg z = 2\pi$.

For the integrals over Γ_R and γ_ϵ, the standard ML-inequality gives

$$\left| \int_{\Gamma_R} \frac{z^{\lambda-1}}{1+z}\, dz \right| \leq \int_{|z|=R} \left| \frac{z^{\lambda-1}}{1+z} \right| |dz| \leq \frac{R^{\lambda-1}}{R-1} 2\pi R \sim R^{\lambda-1} \to 0$$

as $R \to \infty$ (since $0 < \lambda < 1$), and

$$\left| \int_{\gamma_\epsilon} \frac{z^{\lambda-1}}{1+z}\, dz \right| \leq \frac{\epsilon^{\lambda-1}}{1-\epsilon} 2\pi\epsilon \sim \epsilon^{\lambda} \to 0 \quad \text{as } \epsilon \to 0.$$

For the remaining two integrals, we proceed as follows. We have

$$\int_{\gamma_+} \frac{z^{\lambda-1}}{1+z}\, dz = \int_{\epsilon}^{R} \frac{(x+i\delta)^{\lambda-1}}{1+(x+i\delta)}\, dx.$$

Given $\eta > 0$, we can choose $\delta > 0$ small enough so that

$$\left| \int_{\gamma_+} \frac{z^{\lambda-1}}{1+z}\, dz - \int_{\epsilon}^{R} \frac{x^{\lambda-1}}{1+x}\, dx \right| < \eta$$

which gives

$$\lim_{\delta \to 0} \int_{\gamma_+} \frac{z^{\lambda-1}}{1+z}\, dz = \int_{\epsilon}^{R} \frac{x^{\lambda-1}}{1+x}\, dx.$$

As we integrate along γ_- (as $\delta \to 0$),

$$z^{\lambda-1} = e^{(\lambda-1)(\ln|x+i\delta|+i\arg(x+i\delta))} \to e^{(\lambda-1)(\ln x+i2\pi)} = x^{\lambda-1}e^{2\pi i\lambda}$$

so that

$$\lim_{\delta \to 0} \int_{\gamma_-} \frac{z^{\lambda-1}}{1+z}\, dz = e^{2\pi i\lambda} \int_{R}^{\epsilon} \frac{x^{\lambda-1}}{1+x}\, dx = -e^{2\pi i\lambda} \int_{\epsilon}^{R} \frac{x^{\lambda-1}}{1+x}\, dx.$$

Therefore,

$$\lim_{\delta \to 0} \left(\int_{\gamma_+} + \int_{\gamma_-} \right) \frac{z^{\lambda-1}}{1+z}\, dz = (1 - e^{2\pi i\lambda}) \int_{\epsilon}^{R} \frac{x^{\lambda-1}}{1+x}\, dx. \tag{9.63}$$

Allowing $R \to \infty$, $\epsilon \to 0$ and using (9.62), we obtain

$$(1 - e^{2\pi i\lambda}) \int_0^\infty \frac{x^{\lambda-1}}{1+x}\, dx = -2\pi i e^{\pi i\lambda}$$

which yields the required identity

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\, dx = \frac{2\pi i e^{\pi\lambda i}}{e^{2\pi\lambda i} - 1} = \frac{\pi}{\sin \pi\lambda}. \tag{9.64}$$

This identity can also be extended to complex values of the parameter λ. To do this, we consider

$$F(w) = \int_0^\infty \frac{x^{w-1}}{1+x}\, dx \quad \text{and} \quad G(w) = \frac{\pi}{\sin \pi w}.$$

Then, F and G are analytic on the strip $D = \{w : 0 < \operatorname{Re} w < 1\}$ and coincides on the interval $(0, 1)$. In view of this observation and the uniqueness theorem, (9.64) holds for all the complex values of the parameter λ with $0 < \operatorname{Re} \lambda < 1$

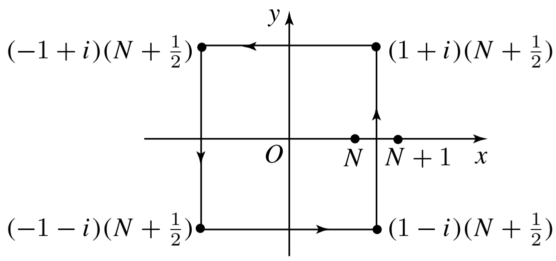


Figure 9.10: Square C_N.

9.6 Estimation of Sums

Convergence tests for series enable us to verify whether a given series converges to a finite limit, but they do not give the value of the sum. On the other hand, calculus of residues permits us to express certain integrals as a finite sum of the residues of the integrand. Therefore if an infinite sum, such as

$$\sum_{n=-\infty}^{\infty} f(n), \quad \text{and} \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n),$$

can be reorganized as a sum of the residues of f, then the calculus of residues may help us to evaluate it, provided f is a meromorphic function of a fairly simple kind.

Thus, for the first series, we must construct a function whose residues are given by $\{f(n) : n \in \mathbb{Z}\}$. For this, let f be a function which is analytic except for a finite number of poles $a_1, a_2, \ldots, a_m$ (each is not an integer). Suppose g is any function having simple poles at $z = n$ ($n \in \mathbb{Z}$) such that $\operatorname{Res}[g(z); n] = 1$ (for example, such functions are given by $\pi \cot \pi z$ and $2\pi i(e^{2\pi i z} - 1)^{-1}$). Then for each $n \in \mathbb{Z}$ ($n \neq a_k,\ k = 1, 2, \ldots, m$), we have

$$\operatorname{Res}[f(z)g(z); n] = f(n).$$

Thus if C_N is a closed contour enclosing points $z = 0, \pm 1, \pm 2, \ldots, \pm N$ and a_k ($k = 1, 2, \ldots, m$), we have (see Figure 9.10), by the Cauchy residue theorem,

$$\int_{C_N} f(z)g(z)\, dz = 2\pi i \sum \operatorname{Res}[f(z)g(z); C_N] = 2\pi i\{X + Y\}$$

where

$$X = \sum_{\substack{n=-N \\ n \neq a_k}}^{N} \operatorname{Res}[f(z)g(z); n] = \sum_{\substack{n=-N \\ n \neq a_k}}^{N} f(n)$$

and

$$Y = \sum \operatorname{Res}[f(z)g(z); \text{at the poles of } f \text{ in } C_N] = \sum_{k=1}^{m} \operatorname{Res}[f(z)g(z); a_k].$$

Let us now describe this idea by taking $g(z) = \pi \cot \pi z$, $\pi \csc \pi z$ and in either case show that

$$\lim_{N \to \infty} \int_{C_N} f(z)g(z)\,dz = 0.$$

9.65. Theorem. *Suppose f is meromorphic in $\mathbb{C}$, with a finite number of poles $a_1, a_2, \dots, a_m$. Suppose, moreover, that there exist two positive numbers M and R such that for $|z| > R$*

$$|z^r f(z)| \le M \quad \textit{for a fixed } r > 1. \tag{9.66}$$

Then

$$\lim_{N \to \infty} \sum_{\substack{n=-N \\ n \neq a_k}}^{N} f(n) = -\sum_{k=1}^{m} \mathrm{Res}\,[\pi \cot \pi z f(z); a_k]$$

and

$$\lim_{N \to \infty} \sum_{\substack{n=-N \\ n \neq a_k}}^{N} (-1)^n f(n) = -\sum_{k=1}^{m} \mathrm{Res}\,[\pi \csc \pi z f(z); a_k].$$

Proof. Equation (9.66) means that $|f(n)| \le Mn^{-r}$ for $|n| > R$ and so the series $\sum f(n)$ and $\sum (-1)^n f(n)$ are convergent since the n-th term is dominated by Mn^{-r} for large n and $r > 1$. By hypothesis, for $n \neq a_k$, $k = 1, 2, \dots, m$,

$$\mathrm{Res}\,[\pi \cot \pi z f(z); n] = f(n) \quad \text{and} \quad \mathrm{Res}\,[\pi \csc \pi z f(z); n] = (-1)^n f(n).$$

Let C_N be the square with vertices at $(N + 1/2)(\pm 1 \pm i)$ enclosing all the poles of f (see Figure 9.10), where N is a positive integer (take for instance $N > |a_k|$ for all $k = 1, 2, \dots, m$). Therefore, by the Cauchy residue theorem,

$$\frac{1}{2\pi i} \int_{C_N} \pi \cot \pi z f(z)\,dz = \sum_{\substack{n=-N \\ n \neq a_k}}^{N} f(n) + \sum_{k=1}^{m} \mathrm{Res}\,[\pi \cot \pi z f(z); a_k]$$

and

$$\frac{1}{2\pi i} \int_{C_N} \pi \csc \pi z f(z)\,dz = \sum_{\substack{n=-N \\ n \neq -a_k}}^{N} (-1)^n f(n) + \sum_{k=1}^{m} \mathrm{Res}\,[\pi \csc \pi z f(z); a_k].$$

The result will follow if we can show that

$$\lim_{N\to\infty} \int_{C_N} \pi \cot \pi z f(z)\,dz = 0 = \lim_{N\to\infty} \int_{C_N} \pi \csc \pi z f(z)\,dz.$$

For this we require that there exist two constants K_1 and K_2 such that $|\cot \pi z| < K_1$ and $|\csc \pi z| < K_2$ for all N and for all z on C_N. In fact, we will now prove the following inequalities:

(i) $|\cot \pi z| < 2$ for all z on C_N

(ii) $|\csc \pi z| < 1$ for all z on C_N.

First we set $\alpha = N + 1/2$. If z is on the horizontal sides of C_N, then we can write $z = x \pm i\alpha$, where $|x| \leq \alpha$. For $z = x \pm i\alpha$ on these horizontal lines, we have

$$|\cot \pi z| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\frac{e^{i\pi x}e^{\mp\alpha\pi} + e^{-i\pi x}e^{\pm\alpha\pi}}{e^{i\pi x}e^{\mp\alpha\pi} - e^{-i\pi x}e^{\pm\alpha\pi}}\right|.$$

By the triangle inequality,

$$\begin{aligned}
|e^{i\pi x}e^{\mp\alpha\pi} + e^{-i\pi x}e^{\pm\alpha\pi}| &\leq e^{\mp\alpha\pi} + e^{\pm\alpha\pi} = e^{\alpha\pi} + e^{-\alpha\pi}, \\
|e^{i\pi x}e^{\mp\alpha\pi} - e^{-i\pi x}e^{\pm\alpha\pi}| &\geq |e^{\mp\alpha\pi} - e^{\pm\alpha\pi}| = e^{\alpha\pi} - e^{-\alpha\pi}
\end{aligned}$$

so that

$$|\cot \pi z| \leq \frac{e^{\alpha\pi} + e^{-\alpha\pi}}{e^{\alpha\pi} - e^{-\alpha\pi}} = \coth(\alpha\pi) \leq \coth(3\pi/2) < 2$$

(since the later expression is maximized at $\alpha = \pi/2$, i.e. at $N = 0$).

Similarly, if z lies on the vertical sides then $z = \pm\alpha + iy$ with $|y| \leq \alpha$, and so for such z, we have

$$\begin{aligned}
|\cot \pi z| &= |\cot \pi(\pm\alpha + iy)| \\
&= \frac{|\cos(\pm\pi\alpha)\cosh \pi y - i\sin(\mp\alpha\pi)\sinh \pi y|}{|\sin(\pm\pi\alpha)\cosh \pi y + i\cos(\pm\pi\alpha)\sinh \pi y|} \\
&= |\tanh \pi y| = \frac{|\sinh \pi y|}{\sqrt{1 + \sinh^2 \pi y}} < 1.
\end{aligned}$$

Thus, $|\cot \pi z| < 2$ on C_N and this proves (i). To prove (ii), we first note that $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$. When $z = x \pm i\alpha$ lies on the horizontal sides of C_N,

$$|\csc^2 \pi z| = \frac{1}{\sin^2 \pi x + \sinh^2(\alpha\pi)} \leq \frac{1}{-1 + \sinh^2 \pi/2} < 1$$

and when $z = \pm\alpha + iy$ lies on the vertical sides of C_N, we have

$$|\csc^2 \pi z| = \frac{1}{\sinh^2 \pi y + \sin^2(N + 1/2)\pi} \leq \frac{1}{1 + \sinh^2 \pi y} \leq 1.$$

This proves (ii). By the usual ML-estimate now it follows that

$$\begin{aligned}
\left|\int_{C_N} \pi \cot \pi z f(z)\, dz\right| &\leq \sup_{z \in C_N} |\pi \cot \pi z f(z)| \times \text{ length of } C_N \\
&\leq 2\pi \frac{M}{N^r} \times \{4(2N + 1)\}
\end{aligned}$$

(since the length of C_N is $4(2N+1)$), which tends to zero as $N \to \infty$. Similarly, we have $\lim_{N\to\infty} \int_{C_N} \pi \csc \pi z f(z)\, dz = 0$. ■

9.67. Example. We easily get the following:

(a) $\operatorname{Res}\left[\frac{\cot \pi z}{z^2}; 0\right] = -\frac{\pi}{3}$ (see also Example 8.16(g)).

(b) $\operatorname{Res}\left[\frac{\csc \pi z}{z^2}; 0\right] = \frac{\pi}{6}$.

(c) For the function $f(z) = \cot \pi z/(z+a)^2$ (a-real and not an integer), we directly obtain that

$$\operatorname{Res}[f(z); -a] = \lim_{z \to -a} \left\{ \frac{d}{dz}(\cot \pi z) \right\} = -\pi \csc^2 \pi a$$

and for $k \in \mathbb{Z}$ we obtain that

$$\operatorname{Res}[f(z); k] = \lim_{z\to k} (z-k) f(z) = \frac{\cos k\pi}{(k+a)^2} \lim_{z\to k} \frac{z-k}{\sin \pi z} = \frac{1}{\pi(k+a)^2}.$$

(d) When $f(z) = \csc \pi z/(z^2+a^2)$, $a > 0$, a direct calculation yields

$$\operatorname{Res}[f(z); ia] = -\frac{1}{2a \sinh \pi a} = \operatorname{Res}[f(z); -ia]$$

and for $k \in \mathbb{Z}$,

$$\operatorname{Res}[f(z); k] = \frac{1}{(k^2+a^2)} \left\{ \frac{1}{\pi \cos \pi k} \right\} = \frac{(-1)^k}{\pi(k^2+a^2)}.$$

Thus, for $a > 0$, we have

$$\operatorname{Res}\left[\frac{\cot \pi z}{z^2+a^2}; ia\right] = -\frac{\coth \pi a}{2a} = \operatorname{Res}\left[\frac{\cot \pi z}{z^2+a^2}; -ia\right].$$

(e) Similarly if a is a non-zero real and not an integer, then, for the function $f(z) = 1/[(z+a)^2 \sin \pi z]$, we easily get

$$\operatorname{Res}[f(z); -a] = -\pi \csc \pi a \cot \pi a$$

and for $k \in \mathbb{Z}$, $\operatorname{Res}[f(z); k] = (-1)^k/[\pi(k+a)^2]$. ●

9.68. Example. The function $f(z) = z^{-2m}$ ($m \in \mathbb{N}$) satisfies the condition (9.66). Since f is even and has a pole of order $2m$ at 0, Theorem 9.65 gives

$$2\sum_{n=1}^{\infty} \frac{1}{n^{2m}} + \operatorname{Res}\left[\frac{\pi \cot \pi z}{z^{2m}}; 0\right] = 0$$

and

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2m}} + \operatorname{Res}\left[\frac{\pi\csc\pi z}{z^{2m}};0\right] = 0.$$

For $m = 1$, we find that (see Example 9.67)

$$\operatorname{Res}\left[\frac{\pi\cot\pi z}{z^2};0\right] = -\frac{\pi^2}{3},\quad \operatorname{Res}\left[\frac{\pi\csc\pi z}{z^2};0\right] = \frac{\pi^2}{6},$$

and hence, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{9.69}$$

More generally, since $\phi(z) = \pi z\cot\pi z$ has a removable singularity at 0 and $\lim_{z\to 0}\phi(0) \neq 0$, we note that

$$\begin{aligned}\operatorname{Res}\left[\frac{\pi\cot\pi z}{z^{2m}};0\right] &= \operatorname{Res}\left[\phi(z)z^{-2m-1};0\right]\\ &= \frac{1}{2\pi i}\int_{|z|=r}\frac{\phi(z)}{z^{2m+1}}\,dz,\quad r>0 \text{ is small},\\ &= \frac{\phi^{(2m)}(0)}{(2m)!}.\end{aligned}$$

Proceeding as above and taking $m = 2$ we easily derive

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

If we let $f(z) = 1/(z^2 + a^2)$, then Theorem 9.65, for $a \notin i\mathbb{Z}$, yields that

$$\operatorname{Res}\left[\frac{\pi\cot\pi z}{z^2+a^2};ia\right] + \operatorname{Res}\left[\frac{\pi\cot\pi z}{z^2+a^2};-ia\right] + \sum_{n=-\infty}^{\infty}\operatorname{Res}\left[\frac{\pi\cot\pi z}{z^2+a^2};n\right] = 0.$$

Thus, by Example 9.67(d), we conclude that

$$-\frac{\pi\coth\pi a}{a} + \sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = 0 \tag{9.70}$$

which gives

$$2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth\pi a - \frac{1}{a^2},\quad\text{i.e.}\quad \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi a\coth\pi a - 1}{2a^2}.$$

Note that if a is real and positive then, by using uniform convergence and making $a \to 0$, we get the first equation in (9.69). Differentiation of the last equality with respect to a immediately yields (since the convergence is uniform on any compact set disjoint from the set $\{ia : a \in \mathbb{Z}\}$, term by term differentiation is permitted),

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{1}{4a^4}\left[\frac{\pi^2 a^2}{\sinh^2 \pi a} + \pi a \coth \pi a - 2\right].$$

Further, from (9.70) we easily obtain that, for $b \notin \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2-b^2} = \frac{\pi}{ib}\coth(\pi(ib)) = -\frac{\pi \cot b\pi}{b}.$$

Thus by grouping the terms with $\pm n = m$, $m = 1, 2, \ldots$, we have

$$-\frac{1}{b^2} + \frac{1}{2b}\sum_{m=1}^{\infty}\left[\frac{1}{m-b} - \frac{1}{m+b}\right] = -\frac{1}{b^2} + 2\sum_{m=1}^{\infty}\frac{1}{m^2-b^2} = -\frac{\pi \cot \pi b}{b},$$

that is

$$\pi \cot \pi b = \frac{1}{b} + \sum_{m=1}^{\infty} \frac{2b}{b^2-m^2}. \tag{9.71}$$

Again, differentiation of the above result with respect to b, yields

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m-b)^2} = \pi^2 \csc^2 \pi b \quad (b \notin \mathbb{Z}). \tag{9.72}$$

Similarly, by applying Theorem 9.65, we see that for $a \neq 0, \pm i, \pm 2i, \ldots$

$$\text{Res}\left[\frac{\pi \csc \pi z}{z^2+a^2}; ia\right] + \text{Res}\left[\frac{\pi \csc \pi z}{z^2+a^2}; -ia\right] + \sum_{n=-\infty}^{\infty} \text{Res}\left[\frac{\pi \csc \pi z}{z^2+a^2}; n\right] = 0.$$

Thus, by Example 9.67(d), this implies

$$-\frac{\pi \csc \pi a}{a} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = 0; \quad \text{i.e.} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a}\csc \pi a.$$

We can also prove (9.72) by considering the function $f(z) = (z-b)^{-2}$ (b not an integer). Indeed, from Theorem 9.65, it follows that (see also Example 9.67(c))

$$\text{Res}\left[\frac{\pi \cot \pi z}{(z-b)^2}; b\right] + \sum_{n=-\infty}^{\infty} \text{Res}\left[\frac{\pi \cot \pi z}{(z-b)^2}; n\right] = 0;$$

that is

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-b)^2} = \pi^2 \csc^2 \pi b, \quad b \notin \mathbb{Z}.$$

Taking $b = -1/2$ in the last identity, we find that

$$\sum_{n=-\infty}^{-1} \frac{1}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$$

so that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{9.73}$$

From (9.73) we can easily deduce the first equation in (9.69), because

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$ •

9.7 Exercises

9.74. Prove the following integrals

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad (a, b \in \mathbb{R},\ |b| < |a|).$
2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a + b\cos^2\theta)^2} = \frac{(2a+b)\pi}{[a(a+b)]^{3/2}}, \quad a, b > 0.$
3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{1 + \alpha^2 - 2\alpha\cos 2\theta}\, d\theta = \frac{(1 + \alpha^2 - \alpha)\pi}{1 - \alpha}, \quad -1 < \alpha \neq 0 < 1.$
4. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1 + \alpha^2 - 2\alpha\cos\theta}\, d\theta = -\frac{(\alpha^4 - \alpha^2 + 1)\pi}{\alpha^2(\alpha^2 - 1)}, \quad 0 < |\alpha| < 1.$
5. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}, \quad a^2 > b^2 + c^2.$
6. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta + c\sin\theta)^2} = \frac{2\pi a}{\sqrt[3]{a^2 - b^2 - c^2}}, \quad a^2 > b^2 + c^2.$

9.75. If a and b are real with $|a| > |b|$ and n is a positive integer, prove that

$$I = \int_0^{2\pi} \frac{\cos n\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi b(a - \sqrt{a^2 - b^2})^n}{\sqrt{a^2 - b^2}}.$$

9.76. Show the following:

1. $\displaystyle I = \int_{|z|=3} \frac{3z+1}{z(z+2)(z-i)^2}\, dz = 0$
2. $\displaystyle\int_{|z|=1} \frac{\cos(e^{-z})}{z^2}\, dz = 2\pi i \sin(1).$

9.77. Prove the following real integrals:

1. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = \frac{5\pi}{12}$
2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{a^2 + 2b^2x^2 + c^2x^4} = \frac{\pi/(2\sqrt{2})}{a\sqrt{b^2 - ac}} \quad (a, b, c > 0 \text{ and } b^2 - ac > 0)$
3. $\displaystyle\int_{0}^{\infty} \frac{\cos x}{(1 + x^2)^{n+1}}\, dx = \frac{\pi e^{-1}}{n!2^{2n+1}} \sum_{k=0}^{n} \frac{(2n-k)!2^k}{k!(n-k)!}, \quad n \in \mathbb{N}.$
4. $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2 + m^2)(x^2 + n^2)} = \frac{\pi}{2mn(m+n)}, \ m, n > 0.$
5. $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2 + m^2)(x^2 + n^2)^2} = \frac{\pi(m + 2n)}{4mn^3(m+n)^2}, \ m, n > 0, \ m \neq n.$
6. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{(x^2 + m^2)^2}\, dx = \frac{\pi}{4m^3}(1 + am)e^{-am}, \quad a, m > 0.$
7. $\displaystyle\int_{0}^{\infty} \frac{x \sin ax}{(x^2 + m^2)^2}\, dx = \frac{\pi a e^{-am}}{4m}, \quad a, m > 0.$
8. $\displaystyle\int_{0}^{\infty} \frac{dx}{(1 + x^2)^n} = \frac{(2(n-1))!}{2^{2(n-1)}((n-1)!)^2} \cdot \frac{\pi}{2}, \quad n \in \mathbb{N}.$
9. $\displaystyle\int_{0}^{\infty} \frac{\cos ax\, dx}{(x^2 + m^2)(x^2 + n^2)} = \frac{\pi(me^{-an} - ne^{-am})}{2(m^2 - n^2)mn}, \quad m > n > 0, \ a > 0$
   and hence deduce that
   $$\int_{0}^{\infty} \frac{x \sin ax}{(x^2 + m^2)(x^2 + n^2)}\, dx = \frac{\pi(e^{-an} - e^{-am})}{2(m^2 - n^2)}.$$

9.78. Prove the following real integrals:

1. $\displaystyle\int_{0}^{\infty} e^{-(1+i\alpha)^2 t^2}\, dt = \left(\frac{1}{1 + i\alpha}\right) \frac{\sqrt{\pi}}{2} \text{ for } -1 \leq \alpha \leq 1.$
2. $\displaystyle\int_{0}^{\infty} \frac{x^\lambda}{(1 + x)^2}\, dx = \frac{\pi\lambda}{\sin \pi\lambda} \text{ for } -1 < \lambda \neq 0 < 1.$
3. $\displaystyle\int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\, dx = \frac{\pi \ln a}{2a} \text{ for } a > 0.$

Chapter 10

Analytic Continuation

Analytic continuation is an important idea because it provides a method for making the domain of definition of an analytic function as large as possible. Usually analytic functions are defined by means of some mathematical expressions such as polynomials, infinite series, integrals etc. The domain of definition of such an analytic function is often restricted by the manner of defining the function. For instance, the power series representation of such analytic functions does not provide any direct information as to whether we could have a function analytic in a domain larger than disk of convergence which coincides with the given function. In Section 10.1, we discuss a general method (such as the power series method). In Section 10.2, we present a technique for carrying out the continuation process and prove the Monodromy theorem. In Section 10.3, we develop the Poisson integral formula for harmonic functions on the open unit disk (and hence, for arbitrary disks). We use the Poisson integral to solve the Dirichlet problem for the unit disk and, as a consequence, we characterize harmonic functions by the mean value property just as Morera's theorem characterizes analytic functions. Later in Section 10.4, we use this characterization to establish the Symmetry principle (due to Schwarz) for harmonic functions which enables one to find an analytic continuation explicitly under a special situation.

10.1 Direct Analytic Continuation

What do we mean by an *"analytic continuation"*? It is simply a process of extending the domain of analyticity to larger domains. For example, if Ω is a domain and $f_j \in \mathcal{H}(\Omega)$ for $j = 1, 2$, such that $f_1(z) = f_2(z)$ for all points z in an open subset $D \subseteq \Omega$, then, by the uniqueness theorem (see Theorem 4.106), one has $f_1 \equiv f_2$ on Ω. So, a natural question is the following: *Is it always possible to have an extension?* Clearly, not. For example,

$$f(z) = \frac{1}{z} \quad \text{for} \quad z \in D = \mathbb{C} \setminus \{0\}$$

does not have an extension to $\mathbb{C}$. Similarly, if $D = \mathbb{C} \setminus \{x : x \leq 0\}$ is the cut plane and

$$\begin{aligned} f_1(z) &= \operatorname{Log} z, \quad z \in D \\ f_2(z) &= z^{1/2} = e^{(1/2)\operatorname{Log} z} = |z|^{1/2} e^{i(1/2)\operatorname{Arg} z} \quad (-\pi < \operatorname{Arg} z < \pi), \\ f_3(z) &= -z^{1/2} = e^{(1/2)(\operatorname{Log} z + 2\pi i)} = -e^{(1/2)\operatorname{Log} z} \quad (-\pi < \operatorname{Arg} z < \pi), \end{aligned}$$

then no extension from D to $\mathbb{C}$ is possible in each case. However, if the extension is possible then there are ways to carry out the process of continuation so that the given analytic function becomes analytic on a larger domain. To make this point more precise, let us start by examining the analytic continuation of the function

$$f(z) = \sum_{n \geq 0} z^n. \tag{10.1}$$

The series on the right defined by (10.1), as is well known, is convergent for $|z| < 1$ and diverges for $|z| \geq 1$. On the other hand, we know that the series given by the formula (10.1) represents an analytic function for $|z| < 1$ and the sum of the series (10.1) for $|z| < 1$ is $1/(1-z)$. However, the function F defined by the formula

$$F(z) = \frac{1}{1-z}$$

is analytic for $z \in \mathbb{C}_\infty \setminus \{1\} = D$ (since $F(1/z) = (1 - z^{-1})^{-1} = z/(z-1)$ is analytic at 0, $F(z)$ is analytic at $z = \infty$). Now

$$f(z) = F(z) \quad \text{for} \quad z \in \Delta \cap D$$

and we call F an analytic continuation of f from Δ into the domain D, i.e. function f, given at first for $|z| < 1$, has been extended to the extended complex plane but for the point $z = 1$ at which the function has a simple pole. Thus, it appears that F which is analytic globally is represented by a power series only locally.

Next we consider another function g defined by

$$g(z) = \int_0^\infty \exp[(z-1)t]\, dt. \tag{10.2}$$

If $\operatorname{Re} z < 1$, then the integral converges and

$$\int_0^\infty \exp[(z-1)t]\, dt = \left.\frac{e^{(z-1)t}}{z-1}\right|_0^\infty = -\frac{1}{z-1} = \frac{1}{1-z}.$$

Thus the integral defined by (10.2) is convergent in the half-plane $H = \{z : \operatorname{Re} z < 1\}$ and represents the same function $1/(1-z)$ for $z \in H$. Consequently, we have $F(z) = g(z)$ for $z \in H \cap D$ and hence we call F the continuation of g into the half-plane $\operatorname{Re} z \geq 1$ with the exception of point at $z = 1$. Similarly if $D_1 = \mathbb{C} \setminus \{z = x : x \geq 1\}$, then $-\operatorname{Log}(1 - z)$ is the analytic continuation of the power series $\sum_{n \geq 1} \frac{z^n}{n}$ from Δ into D_1.

10.3. Definition. Suppose that f and F are two functions such that

(i) f is analytic on some domain $D \subset \mathbb{C}$,

(ii) F is analytic in a domain D_1 such that $D_1 \cap D \neq \emptyset$ and $D_1 \supset D$, such that $f(z) = F(z)$ for $z \in D \cap D_1$.

Then we call F *an analytic continuation* or *a holomorphic extension* of f from D into D_1. In other words, f is said to be analytically continuable into D_1.

For a given analytic function f on D, if there exists an analytic continuation F of f into D_1, by the uniqueness theorem (see Theorem 3.75), then it is uniquely determined. Thus, we raise

10.4. Problem. When does a power series represent a function which is analytic beyond the disk of convergence of the original series?

One way to provide an affirmative answer is by "power series method". Let us start our discussion on this method and see how one can use the power series to go beyond the boundary of the disk of convergence. A fundamental fact about a function $f \in \mathcal{H}(\Omega)$ is that for each $a \in \Omega$, there exists a sequence $\{a_n\}_{n \geq 0}$ and a number $r_a \in (0, \infty]$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for all} \quad z \in \Delta(a; r_a).$$

To extend f, we choose a point b other than a in the disk of convergence $\Delta(a; r_a)$. Then $|b - a| < r_a$ and

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n (z-a)^n &= \sum_{n=0}^{\infty} a_n [z - b + b - a]^n \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} (b-a)^{n-k} (z-b)^k \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (b-a)^{n-k} \right) (z-b)^k \\
&= \sum_{k=0}^{\infty} A_k (z-b)^k.
\end{aligned}$$

The interchange of the summation is justified, since

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |b-a|^{n-k}|z-b|^k = \sum_{n=0}^{\infty} |a_n|(|z-b|+|b-a|)^n < \infty$$

whenever $|z-b|+|b-a| < r_a$. Therefore, the series about b converges at least for $|z-b| < r_a - |b-a|$. However, it may happen that the disk of convergence $\Delta(b;r_b)$ for this new series extends outside $\Delta(a;r_a)$, i.e. it may be possible that $r_b > r_a - |b-a|$. In this case, the function can be analytically continued to the union of these two disks. This process may be continued. For example, if $f(z) = \frac{1}{1-z}$ then (for $z \in \Delta$ with $a=0,\ r_a = 1$) we have

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n, \quad z \in \Delta.$$

Take $b = i$. In order to get the expression for $z \in \Delta(i;r_b)$, we write

$$\begin{aligned} \frac{1}{1-z} &= \frac{1}{1-i-(z-i)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}}(z-i)^n, \quad |z-i| < |1-i| = \sqrt{2}, \\ &= \sum_{n=0}^{\infty} A_n(z-i)^n, \quad A_n = \frac{(1+i)^{n+1}}{2^{n+1}},\ |z-i| < r_b = \sqrt{2}. \end{aligned}$$

Thus, $\sum_{n=0}^{\infty} A_n(z-i)^n$ is an analytic continuation of $\sum_{n=0}^{\infty} z^n$ in Δ to the disk $\Delta(i;\sqrt{2})$. Similarly, one can see that $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(z+1)^n$ is an analytic continuation of $\sum_{n=0}^{\infty} z^n$ from Δ to the disk $\Delta(-1;2)$.

10.5. Remark. The convergence or divergence of a power series $f(z) = \sum_{n\geq 0} a_n z^n$ at a point does not determine whether it can or cannot be extended beyond that point. For example, consider

$$f_1(z) = \sum_{n\geq 0} z^n \quad \text{and} \quad f_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}.$$

Recall that the first series diverges for $|z| = 1$ and its sum $f_1(z) = 1/(1-z)$ defined by this series is analytic in $\mathbb{C}\setminus\{1\}$. On the other hand, the second series converges at all points on $|z| = 1$. However, in both cases, the series cannot be continued analytically to a domain D with $1 \in D$. Note that

$$f_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt \quad \text{and} \quad f_2''(z) \to \infty \text{ as } z \to 1^- \text{ along reals.} \quad \bullet$$

10.6. Example. As another example of analytic continuation, we consider the power series

$$f_0(z) = \sum_{n\geq 0} (-1)^n z^{2n}. \tag{10.7}$$

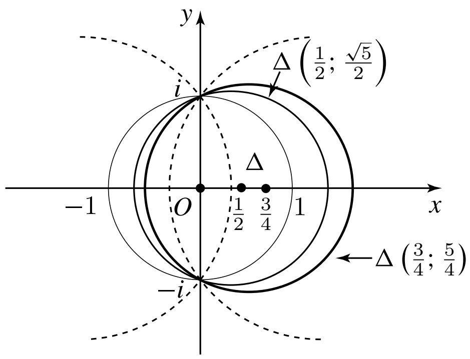


Figure 10.1: Direct analytic continuation.

Then (10.7) is absolutely convergent for $z \in \Delta$ and diverges outside the unit disk. Further, if we let

$$F_0(z) = \frac{1}{1+z^2} = \frac{i}{2}\left[\frac{1}{i+z} + \frac{1}{i-z}\right], \tag{10.8}$$

then it is easy to see that F_0 is analytic in the extended plane except at $z = \pm i$ and $f_0(z) = F_0(z)$ for all $z \in \Delta$. We also note that $F_0(x) = 1/(1+x^2)$ is well defined for all real values of x, which is expandable as a real power series about any point on the real axis. Yet the power series (e.g. about 0) given by

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - \cdots$$

has the unit interval as the interval of convergence and so, this illustrates an interesting fact about Taylor's series on the real axis.

Suppose we want to continue f_0 on Δ to a disk about z_0. Then, we write

$$F_0(z) = \frac{i}{2}\left[\frac{1}{i+z_0}\left(\frac{1}{1+\frac{z-z_0}{i+z_0}}\right) + \frac{1}{i-z_0}\left(\frac{1}{1-\frac{z-z_0}{i-z_0}}\right)\right].$$

Thus if

$$F_1(z) = \frac{i}{2}\sum_{n\geq 0}\left(\frac{(-1)^n}{(i+z_0)^{n+1}} + \frac{1}{(i-z_0)^{n+1}}\right)(z-z_0)^n, \tag{10.9}$$

where $R = \min\{|i+z_0|, |i-z_0|\}$ is the radius of convergence for (10.9), then $F_1(z)$ is analytic for $|z-z_0| < R$. For instance if $z_0 = 1/2$, then $R = \sqrt{5}/2$ and hence, in this choice, we see that (10.9) converges for some values of z for which (10.7) diverges (see Figure 10.1). Therefore, (10.7) and (10.9) both represent parts of (10.8) which encompasses all possible extensions.

However, there is no extension which is analytic at $z = \pm i$ which agrees with F_0 in a deleted neighborhood of $\pm i$, because

$$\left|\frac{1}{1+z^2}\right| \to \infty \quad \text{as} \quad z \to \pm i. \qquad \bullet$$

An analytic function f on a domain D will be called a *function element*, written as (f, D).

If (f_1, D_1) and (f_2, D_2) are two function elements such that $D_1 \cap D_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. Then $(f, D_1 \cup D_2)$ is also a function element, where

$$f(z) = \begin{cases} f_1(z) & \text{for } z \in D_1 \\ f_2(z) & \text{for } z \in D_2. \end{cases}$$

With the observation just made, (f_1, D_1) and (f_2, D_2) are called a *direct analytic continuation* of each other, thereby defining an analytic function in $D_1 \cup D_2$. For instance, if $f_1(z) = a/(a-z)$ for $z \in \mathbb{C}\backslash\{a\}$, and

$$f_2(z) = \sum_{n\geq 0} \left(\frac{z}{a}\right)^n \quad \text{for} \quad z \in D_2 = \{z : |z| < |a|\},$$

then $(f_1, \mathbb{C}\backslash\{a\})$ is a direct analytic continuation of (f_2, D_2).

10.10. Example. Define

$$f_0(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\cos \zeta}{\zeta - z}\, d\zeta$$

for $z \in \Delta$. Then, by the Cauchy integral formula, we have $f_0(z) = \cos z$ for each $z \in \Delta$. But, since $\cos z$ is analytic in $\mathbb{C}$, $(\cos z, \mathbb{C})$ is a direct analytic continuation of (f_0, Δ). ●

In general, the following theorem holds:

10.11. Theorem. *Let C be a simple closed contour with interior D and g be an entire function. If, for $n \in \mathbb{N}$,*

$$f_0(z) = \frac{n!}{2\pi i} \int_C \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in D,$$

then $(g^{(n)}, \mathbb{C})$ is a direct analytic continuation of (f_0, D).

Proof. By the Cauchy integral formula, $f_0(z) = g^{(n)}(z)$ for all $z \in D$. Since g is entire, so is $g^{(n)}$; the result follows. ∎

10.12. Definition. Let f be analytic on a domain D. If f cannot be continued analytically across the boundary ∂D, then ∂D is called *natural* *boundary* of f. A point $z_0 \in \partial D$ is said to be a regular point of $f(z)$ if f can be continued analytically to a region D_1 with $z_0 \in D_1$. Otherwise, $f(z)$ is said to have a *singular point* at z_0.

For instance, consider the power series

$$f(z) = \sum_{k \geq 0} z^{2^k}. \tag{10.13}$$

A direct consequence of the Root test is that the radius of convergence of (10.13) is 1 and so, f defined by (10.13) is analytic for $|z| < 1$. If $|z| \geq 1$, then $\lim_{n\to\infty} |z^{2^n}| \neq 0$ and therefore, the series diverges for $|z| \geq 1$.

Let $\zeta = e^{2\pi i m/2^n}$, $m = 0, 1, \ldots, 2^n - 1$ ($n \in \mathbb{N}$), be the 2^n-th root of unity. If $z = re^{2\pi i m/2^n} \in \Delta$, then

$$f(z) = \sum_{k=0}^{n-1} z^{2^k} + \sum_{k=n}^{\infty} z^{2^k}$$

and so for $r \to 1^-$, we have

$$f(\zeta r)| \geq \sum_{k=n}^{\infty} r^{2^k} - \left| \sum_{k=0}^{n-1} z^{2^k} \right| \geq \sum_{k=n}^{\infty} r^{2^k} - n,$$

and hence, for every 2^n-th root of unity ζ, we have $\lim_{r\to 1^-} |f(\zeta r)| = \infty$. Therefore if D is a domain containing points of Δ and of its complement, then D contains the points $\zeta = e^{2\pi i m/2^n}$ and so any function F in D which coincides with f in $D \cap \Delta$ cannot be continued analytically through $\zeta^{2n} = 1$ for each $n \in \mathbb{N}$. In other words any root of the equation

$$z^2 = 1, \;\; z^4 = 1, \ldots, z^{2n} = 1 \;\; (n \in \mathbb{N})$$

is a singular point of f and hence any arc, however small it may be, of $\partial\Delta$ contains an infinite number of singularities. Thus, f on Δ cannot be continued analytically across the boundary $\partial\Delta$ of Δ. This observation shows that the unit circle $|z| = 1$ is a natural boundary for the power series defined by (10.13).

Similarly if

$$f(z) = \sum_{k \geq 0} z^{k!} \tag{10.14}$$

then, $f \in \mathcal{H}(\Delta)$. Upon taking $\zeta = e^{2\pi i m/n}$, $m = 0, 1, 2, \ldots, n-1$; $z = r\zeta$, (where m/n is the irreducible fraction), and choosing r close to 1 from below along a radius of the unit circle it can be seen that $\lim_{r\to 1^-} |f(\zeta r)| = \infty$. Hence, f is singular at every n-th root of unity for any $n \in \mathbb{N}$.

Since every point on $|z| = 1$ is a singular point, f cannot be continued analytically through the n-th root of unity for any natural number n. In

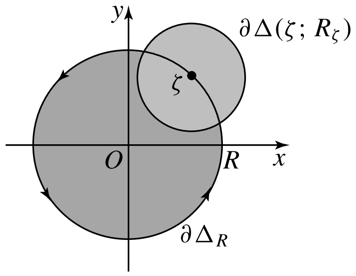


Figure 10.2: Illustration for singularity on circle $|z| = R$.

other words, there can be no continuation anywhere across $|z| = 1$ and hence, $|z| = 1$ is a natural boundary for the power series defined by (10.14).

10.15. Theorem. *If $f(z) = \sum_{n\geq 0} a_n z^n$ has a radius of convergence $R > 0$, then f must have at least one singularity on $|z| = R$.*

Proof. Suppose, on the contrary that f has no singularity on $|z| = R$. Then f must be analytic at all points of $|z| = R$. This, together with Theorem 3.71, implies that f is analytic for $|z| \leq R$. It follows, from the definition of analyticity at a point, that for each $\zeta \in \partial\Delta_R$ there exists some $R_\zeta > 0$ and a function f_ζ which is analytic in $\Delta(\zeta; R_\zeta)$ (see Figure 10.2) and

$$f = f_\zeta \quad \text{on} \quad \Delta(\zeta; R_\zeta) \cap \Delta_R.$$

In this way, if ζ_k and $\zeta_l \in \partial\Delta_R$ $(k \neq l)$ with $G = \Delta(\zeta_k; R_{\zeta_k}) \cap \Delta(\zeta_l; R_{\zeta_l}) \neq \emptyset$ then we have two functions f_{ζ_k} and f_{ζ_l} which are respectively analytic in $\Delta(\zeta_k; R_{\zeta_k})$ and $\Delta(\zeta_l; R_{\zeta_l})$ such that

$$f = f_{\zeta_k} = f_{\zeta_l} \quad \text{on} \quad G \cap \Delta_R.$$

Since G is connected and $G \cap \Delta_R$ is an open subset of G, by the uniqueness theorem, $f_{\zeta_k} = f_{\zeta_l}$ on G. Since $|\zeta| = R$ is compact, by the Heine-Borel theorem, we may select a finite number of $\Delta(\zeta_1; R_{\zeta_1}), \Delta(\zeta_2; R_{\zeta_2}), \ldots, \Delta(\zeta_n; R_{\zeta_n})$ from the collection $\{\Delta(\zeta; R_\zeta) :\, \zeta \in \partial\Delta_R\}$ such that it covers the circle $\partial\Delta_R$. Let

$$\Omega = \cup_{k=1}^n \Delta(\zeta_k; R_{\zeta_k}) \quad \text{and} \quad \delta = \text{dist}\,(\partial\Delta_R, \Omega).$$

Then, as $R_{\zeta_k} > 0$ for each k, we have $\delta > 0$. Moreover,

$$\{z :\, R - \delta < |z| < R + \delta\} \subset \Omega \quad \text{and} \quad \Delta_{R+\delta} \subset D = \Delta_R \cup \Omega.$$

Then g defined by

$$g(z) = \begin{cases} f(z) & \text{for } |z| < R \\ f_{\zeta_k}(z) & \text{for } |z - \zeta_k| < R_{\zeta_k},\ k = 1, 2, \ldots, n, \end{cases}$$

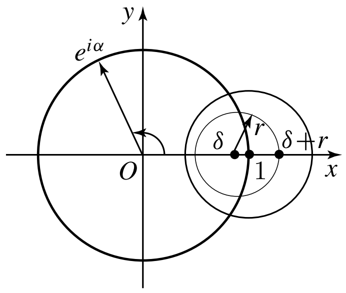


Figure 10.3: Existence of a singularity on the circle of convergence.

is well defined, single-valued and analytic on D and has same power series representation as f for $|z| < R$. Thus there exists an analytic function, say ϕ, in $\Delta_{R+\delta}$, which coincides with f on Δ_R. But then by Taylor's Theorem we have the power series expansion

$$\phi(z) = \sum_{n\geq 0} b_n z^n \quad \text{for } z \in \Delta_{R+\delta}.$$

Since $f = g$ on Δ_R, by the uniqueness theorem (see Theorem 3.75), we have $a_n = b_n$ for each n. This shows that the radius of convergence of f is $R + \delta$, which is a contradiction. ■

10.16. Theorem. *If $a_n \geq 0$ and $f(z) = \sum_{n\geq 0} a_n z^n$ has the radius of convergence 1, then (f, Δ) has no direct analytic continuation to a function element (F, D) with $1 \in D$.*

Proof. For each $z = re^{i\theta} \in \Delta$ $(0 < r < 1,\ \theta \in [0, 2\pi))$, we have

$$f^{(k)}(z) = \sum_{n\geq k} n(n-1)\cdots(n-(k-1))a_n z^{n-k} \tag{10.17}$$

so that (since $a_n \geq 0$)

$$|f^{(k)}(re^{i\theta})| \leq \sum_{n\geq k} n(n-1)\cdots(n-(k-1))a_n r^{n-k} = f^{(k)}(r). \tag{10.18}$$

We have to show that 1 is a singular point of f. Suppose, on the contrary, that 1 is a regular point of f. Then, f can be analytically continued in a neighborhood of $z = 1$ and so there is a δ with $0 < \delta < 1$ (see Figure 10.3) for which the Taylor's series expansion of f about δ, namely the series

$$\sum_{k\geq 0} \frac{f^{(k)}(\delta)}{k!}(z-\delta)^k, \tag{10.19}$$

would be convergent for $|z-\delta|<r$ with $\delta+r>1$. Now, by (10.18), we find that

$$\frac{|f^{(k)}(\delta e^{i\theta})|}{k!} \le \frac{f^{(k)}(\delta)}{k!}.$$

From this, the Root test and the comparison test with (10.19), it follows that the radius of convergence of the Taylor series about $\delta e^{i\theta}$ is at least r. This observation implies that the Taylor series

$$\sum_{k\ge 0} \frac{f^{(k)}(\delta e^{i\theta})}{k!}(z-\delta e^{i\theta})^k$$

would be convergent in the disk $|z-\delta e^{i\theta}|<r$ for each θ, with $\delta+r>1$. In other words, the Taylors series

$$\sum_{k\ge 0} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

about each z_0 with $|z_0|=\delta$ would have radius of convergence $\ge r>1-\delta$. Since this contradicts Theorem 10.15, 1 must be a singular point of f. This completes the proof. ■

Notice that the last series is actually a rearrangement of $\sum_{n\ge 0} a_n z^n$. Indeed, by (10.17),

$$\begin{aligned}\sum_{k\ge 0}\left(\sum_{n\ge k}\binom{n}{k}a_n z_0^{n-k}\right)(z-z_0)^k &= \sum_{n\ge 0}\sum_{k=0}^{n}\binom{n}{k}a_n z_0^{n-k}(z-z_0)^k\\ &= \sum_{n\ge 0} a_n(z-z_0+z_0)^n\\ &= \sum_{n\ge 0} a_n z^n.\end{aligned}$$

10.20. Corollary. *If $a_n\ge 0$ and $f(z)=\sum_{n\ge 0} a_n z^n$ has the radius of convergence $R>0$, then $z=R$ is a singularity of $f(z)$.*

Finally, we state the following result whose proof may be found in standard advanced texts (e.g. [24]).

10.21. Theorem. *Let $f(z)=\sum_{k\ge 0} a_k z^{n_k}$ and $\liminf_{k\to\infty}\frac{n_{k+1}}{n_k}>1$. Then the circle of convergence of the power series is the natural boundary for f.*

By this theorem, it is easy to see that the unit circle $|z|=1$ is the natural boundary for

$$f_1(z)=\sum_{k=0}^{\infty}\frac{z^{3^k}}{3^k} \quad\text{and}\quad f_2(z)=\sum_{k=0}^{\infty}\frac{z^{2^k}}{2^{k^2}}.$$

10.2 Monodromy Theorem

We start with a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ that represents a function f with $\Delta(z_0;R)$ as its disk of convergence. A function element of the type $(f,\Delta(z_0;R))$ is called *"an analytic germ of f at z_0"*, or briefly a *"germ at z_0"*. Obviously, an arbitrary function element (f,D) determines a germ at each point of D.

If $\gamma:[0,1]\to\mathbb{C}$ is a curve with $z_0=\gamma(0)$ as its initial point and $f(z)$ is a germ at z_0, then we say that $f(z)$ is *continued analytically along* γ if for every $t\in[0,1]$ there is an (analytic) germ at $\gamma(t)$, i.e. there is a convergent power series

$$f_t(z)=\sum_{n=0}^{\infty} a_n(t)(z-\gamma(t))^n \tag{10.22}$$

for $z\in D_t=\{z:|z-\gamma(t)|<R(t)\}$, such that

(i) $f_0(z)$ is the power series representing the function $f(z)$ at z_0

(ii) for each $t\in[0,1]$, D_t is the disk of convergence with center at $\gamma(t)$

(iii) whenever s and t in $[0,1]$ are near to each other, then $f_s(z)=f_t(z)$ for all $z\in D_s\cap D_t$, where D_s and D_t are the disks of convergence of f_s and f_t, respectively (Note that $D_s\cap D_t\neq\emptyset$); i.e. when s is near t, (f_s,D_s) and (f_t,D_t) are direct analytic continuations of each other.

In this way, we obtain a one parameter family of germs $\{f_t\}$ and refer to $f_1(z)$ as the analytic continuation of $f_0(z)=f(z)$ along γ, where we regard $f_t(z)$ either as a series or as an analytic function defined near $\gamma(t)$.

Clearly, for s near t, the condition (i) implies that

$$a_n(s)=\frac{f_t^{(n)}(\gamma(s))}{n!}$$

and so the Taylor coefficients in (10.22) depends continuously on the parameter t. Further, in the domain D_t of the germ f_t, the radius of convergence depends continuously on the center of the expansion of the power series. More precisely, we have

10.23. Lemma. *Let $\gamma:[0,1]\to\mathbb{C}$ be a curve and f be a germ at $z_0=\gamma(0)$. Assume that $f(z)$ can be continued analytically along γ with a convergent power series given by* (10.22). *Then either $R(t)=\infty$ for all $t\in[0,1]$, or*

$$|R(s)-R(t)|\leq|\gamma(s)-\gamma(t)|=:|z_s-z_t|$$

whenever s and t are such that $|s-t|<\delta$ for some $\delta>0$, i.e. the radius of convergence of f_t is a continuous function of t.

Proof. Fix t so that

$$f_t(z)=\sum_{n=0}^{\infty} a_n(t)(z-z_t)^n,\quad |z-z_t|<R(t),$$

where $R(t)$ is the radius of convergence for the series about $z_t := \gamma(t)$ to which f_t extends analytically. Thus, f_t does not extend analytically to a larger disk containing D_t. If we consider the power series

$$\sum_{n=0}^{\infty} a_n(s)(z - z_s)^n$$

and choose $z_s \in D_t$ so that $|z_s - z_t| < R(t)$, then the radius of convergence $R(s)$ of this new series is at least $R(t) - |z_s - z_t|$. Consequently,

$$R(s) \geq R(t) - |z_s - z_t|, \quad \text{i.e.} \quad R(t) - R(s) \leq |z_s - z_t|.$$

Interchanging the roles of z_s and z_t, we see that

$$R(s) - R(t) \leq |z_s - z_t|.$$

From the last two inequalities, we obtain that either $R(t) = \infty$ for all $t \in [0,1]$, or

$$|R(s) - R(t)| \leq |z_s - z_t|$$

holds for all s and t nearby. In particular, the continuity of γ implies that $R(t)$ is a continuous function of t. ■

Suppose that f is analytic at z_0, and $\gamma : [0,1] \to \mathbb{C}$ is a curve with $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$, along which f has an analytic continuation f_t. Then the message of Lemma 10.23 is that the radius of convergence $R(t)$ of the power series about $\gamma(t)$ that represents f_t given by (10.22), is a continuous function of t on $[0,1]$. In fact, $R(t)$ is uniformly continuous on $[0,1]$. As $R(t) > 0$ for each $t \in [0,1]$, we conclude that

$$R = \min\{R(t) : t \in [0,1]\} > 0$$

and so, $R(t) \geq R$ for all $t \in [0,1]$.

10.24. Example. Consider the principal branch of $\log z$, where its Taylor series expansion about $z = 1$ is given by

$$f(z) = \operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \quad z \in D = \{z : |z-1| < 1\}.$$

Note that $\operatorname{Log} z$ is analytic in $\mathbb{C}\backslash\{z : \operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$. Let $\gamma : [0,1] \to \mathbb{C}$ be the curve given by $\gamma(t) = e^{2\pi i t}$, starting from $z_0 = \gamma(0) = 1$. Then $f(z)$ actually has an analytic continuation along γ. In fact, for each $t \in [0,1]$, there is a collection of function elements $\{f_t, D_t\}$ such that

(i) $(f_0, D_0) = (f, D)$

(ii) for each $t \in [0,1]$, D_t is the disk of convergence with center $\gamma(t)$

(iii) for each $t \in [0,1]$, and s near t, we have $D_s \cap D_t \neq \emptyset$ and $f_s(z) = f_t(z)$ for all $z \in D_s \cap D_t$; i.e. (f_s, D_s) is a direct analytic continuation of (f_t, D_t).

More precisely, we have

$$f_t(z) = 2\pi it + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^{-2\pi int}}{n}(z - e^{2\pi it})^n$$

valid for $z \in D_t = \{z : |z - e^{2\pi it}| < 1\}$. In particular, for $t = 0$, we have $f_0(z) = f(z)$ and for $t = 1$

$$f_1(z) = 2\pi i + \operatorname{Log} z \quad \text{for } z \in D = D_1,$$

which is actually another branch of $\log z$. Thus, we can see that $\operatorname{Log} z$ has an analytic continuation along any curve in the punctured plane $\mathbb{C}\setminus\{0\}$. Note that $(\operatorname{Log} z, D_0)$ and $(\operatorname{Log} z + 2\pi i, D_0)$ are the initial and the final function elements, respectively. ●

10.25. Theorem. *Let (f, D_0), $D_0 = \Delta(z_0; R(0))$, be a germ at z_0 and γ be a curve with initial point $\gamma(0) = z_0$. Then any two analytic continuations of f along γ coincide in the following sense: if (f_t, D_t) and (g_t, U_t), $t \in [0,1]$, are two analytic continuations of (f, D_0) with (f_1, D_1) and (g_1, U_1) as the terminal elements of (f_t, D_t) and (g_t, U_t), respectively, then $f_1 = g_1$ on $D_1 \cap U_1$. Here $D_t = \Delta(\gamma(t); R(t))$ and $U_t = \Delta(\gamma(t); r(t))$ are the disks of definition for the germs (f_t, D_t) and (g_t, U_t), respectively.*

Proof. Set $E = \{t_0 \in [0,1] : f_t = g_t \text{ in } D_t \cap U_t \text{ for all } t \in [0, t_0]\}$. By the definition, $f_0 = g_0$ in $D_0 \cap U_0$ and so $0 \in E$. Thus, $E \neq \emptyset$. To complete the proof, we need to show that E is both open (in $[0,1]$) and closed. Then the connectedness of $[0,1]$ will imply $E = [0,1]$.

First we show that E is closed. Take an arbitrary $t' \in \overline{E}$ and an increasing sequence $\{t_n\}$ of elements of E such that $t_n \to t'$. Choose n so large that $|t_n - t'| < \min\{R(t'), r(t')\}$. Then

$$\gamma(t_n) \in D_{t_n} \cap U_{t_n} \cap D_{t'} \cap U_{t'}.$$

Also since $t_n \in E$, $f_{t_n} = g_{t_n}$ on $D_{t_n} \cap U_{t_n}$. Therefore,

$$f_{t'} = f_{t_n} = g_{t_n} = g_{t'} \quad \text{on } D_{t_n} \cap U_{t_n} \cap D_{t'} \cap U_{t'}$$

which gives $f_{t'} = g_{t'}$ on $D_{t'} \cap U_{t'}$ showing that $t' \in E$. Hence E is closed.

To see that E is open in $[0,1]$, we consider the complement $E^c = [0,1]\setminus E$ and show that E^c is closed. As before it suffices to show that if $\{t_n\}$ is a convergent sequence in E^c with $t_n \to t'$, then $t' \in E^c$. To see this we note that $t_n \notin E$ and so by definition of E, there exists $s_n \in [0,1]$ such that $t_n \geq s_n$ for all n and

$$f_{s_n} \not\equiv g_{s_n} \quad \text{on } D_{s_n} \cap U_{s_n}. \tag{10.26}$$

Passing to the subsequence if necessary, we may assume that s_n converges to some s'. Clearly $t' \geq s'$. Now we easily see that $t' \in E^c$. To confirm this it suffices to show that

$$f_{s'} \not\equiv g_{s'} \quad \text{on } D_{s'} \cap U_{s'}.$$

But if $f_{s'} = g_{s'}$ on $D_{s'} \cap U_{s'}$, then the previous reasoning shows that $f_{s_n} = g_{s_n}$ on $D_{s_n} \cap U_{s_n}$ for n large enough. This contradicts (10.26) and so E^c is open. We have that $E = [0, 1]$. ∎

Theorem 10.25 asserts that the analytic continuation of a given function element along a given curve is unique if it exists.

10.27. Theorem. **(Monodromy Theorem)** *Let z_0 and z_1 be two points in a domain D. Let f be an analytic germ at z_0 such that f can be continued analytically along every curve in D that begins at z_0 and ends at z_1. Assume that*

(i) *γ_0 and γ_1 are two curves with initial point z_0 and terminal point z_1.*
(ii) *γ_0 and γ_1 are homotopic in D.*

Let f_{γ_0} and f_{γ_1} be the analytic continuations of f along γ_0 and γ_1, respectively. Then f_{γ_0} and f_{γ_1} agree in a neighborhood of z_1, i.e. the analytic continuations of f along γ_0 and γ_1 produce the same terminal germ at z_1.

Proof. Let the homotopy connecting γ_0 and γ_1 be $F : [0,1]\times[0,1] \to \mathbb{C}$, and fixes the end points of γ_0 and γ_1 (i.e. $F(0, u) = z_1$ and $F(1, u) = z_2$ for all $u \in [0, 1]$). We denote by γ_u, the intermediate path associated with F: $\gamma_u(t) = F(t, u)$, where

$$F(t, 0) = \gamma_0(t) \quad \text{and} \quad F(t, 1) = \gamma_1(t) \quad \text{for all } t \in [0, 1].$$

Note that

$$z_0 = \gamma_0(0) = \gamma_1(0) \quad \text{and} \quad z_1 = \gamma_0(1) = \gamma_1(1).$$

Thus, for each fixed $u \in [0, 1]$, let $f_{t,u}$ denote an analytic continuation of $f(z)$ along the curve $\gamma_u = F(., u)$. Then the conclusion of the theorem is that $f_{1,0} = f_{1,1}$ (see Figure 10.4): Set $E = \{u \in [0, 1] : f_{1,\lambda} = f_{1,0} \text{ for all } \lambda \in [0, u]\}$. Since $0 \in E$, $E \neq \emptyset$. If we can show that E is both open (in $[0, 1]$) and closed, then the connectedness of E would imply that $E = [0, 1]$. In particular, $1 \in E$ so that $f_{1,1} = f_{1,0}$ which completes the proof.

To prove that E is open, take an arbitrary $u_0 \in E$. Then, by Lemma 10.23, there exists an $R > 0$ such that for each $t \in [0, 1]$, the radius of convergence of the power series expansion of $f_{u_0,t}$ about $\gamma_{u_0}(t)$ is at least R. By the uniform continuity of F on the compact set $[0, 1] \times [0, 1]$, there is a $\delta > 0$ such that for every $t \in [0, 1]$,

$$|\gamma_u(t) - \gamma_{u_0}(t)| = |F(t, u) - F(t, u_0)| < R = \epsilon$$

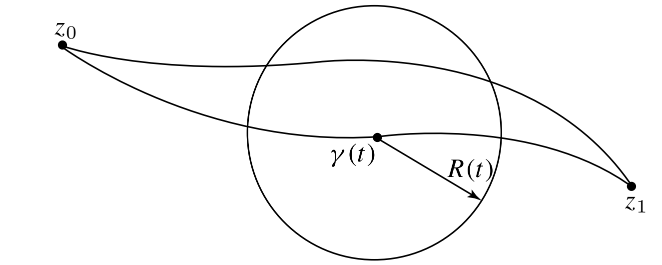


Figure 10.4: Illustration for Monodromy theorem.

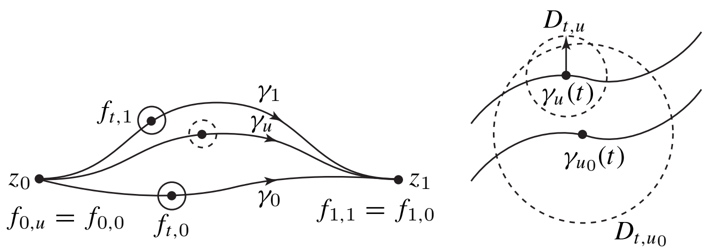


Figure 10.5: Illustration for Monodromy theorem.

whenever $|u - u_0| < \delta$ and $u \in [0,1]$. We fix $u_1 \in (u_0 - \delta, u_0 + \delta) \cap [0,1]$ and set

$$T = \{t \in [0,1] : f_{t,u_1} = f_{t,u_0} \text{ on } D_{t,u_1} \cap D_{t,u_0}\}$$

where D_{t,u_1} and D_{t,u_0} are the disks of definition of f_{t,u_1} and f_{t,u_0}, respectively. Since $0 \in T$, $T \neq \emptyset$. By Theorem 10.25, $T = [0,1]$ and so (see Figure 10.5)

$$f_{1,u_1} = f_{1,u_0} = f_{1,0}.$$

Thus, every $u_1 \in (u_0 - \delta, u_0 + \delta) \cap [0,1]$ belongs to E. It follows that E is open in $[0,1]$. To show that E is closed, let $\{u_n\}$ be a sequence in E such that $u_n \to u'$. A similar argument shows that there exists a $\delta > 0$ such that if $u \in (u' - \delta, u' + \delta) \cap [0,1]$ then

$$f_{1,u} = f_{1,u'} \text{ on } D_{1,u} \cap D_{1,u'}.$$

For large n, $u_n \in (u' - \delta, u' + \delta) \cap [0,1]$, so it follows that

$$f_{1,u_n} = f_{1,u_1} = f_{1,0}.$$

So $u' \in E$ and hence E is closed. We have that $E = [0,1]$. ∎

10.3 Poisson Integral Formula

For $f \in \mathcal{H}(\Delta)$, the Cauchy integral formula gives

$$f(0) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\theta})\,d\theta \quad (0<\rho<1).$$

If, in addition, f is continuous on $\overline{\Delta}$ then $f(z)$ is uniformly continuous on $\overline{\Delta}$ so that, allowing $\rho \to 1^-$,

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta. \tag{10.28}$$

That is, the value of f at the center of the unit circle $|z|=1$ is the mean value of f on $|z|=1$. Our procedure for computing $f(a)$ for an arbitrary $a \in \Delta$ will be as follows: consider

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z}.$$

We know that $\phi_a \in \mathcal{H}(\overline{\Delta})$ and maps the unit circle $|z|=1$ *onto* itself. Also, $\phi_a(a)=0$, $\phi_a(\Delta)=\Delta$, $\phi_a^{-1}=\phi_a$,

$$\phi_a'(z) = -\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right), \quad \text{and} \quad \frac{\phi_a'(z)}{\phi_a(z)} = \frac{1-|a|^2}{(1-\overline{a}z)(z-a)}.$$

Clearly, $F = f\circ\phi_a^{-1} \in \mathcal{H}(\Delta)$, $F(0)=f(a)$ and F is continuous on $\overline{\Delta}$ (as f is continuous on $\overline{\Delta}$ and $\phi_a \in \mathcal{H}(\overline{\Delta})$). Therefore, by (10.28), we get

$$f(a) = F(0) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{iT})\,dT = \frac{1}{2\pi}\int_0^{2\pi} f\left(\phi_a^{-1}(e^{iT})\right)\,dT. \tag{10.29}$$

If we let $\psi(T) = f\left(\phi_a^{-1}(e^{iT})\right)$, then $\psi(T)$ is 2π-periodic and so, we may change the variable of integration by setting

$$\phi_a^{-1}(e^{iT}) = e^{i\theta}, \quad \text{i.e.} \quad e^{iT} = \phi_a(e^{i\theta}),$$

so that $ie^{iT}\,dT = i\phi_a'(e^{i\theta})e^{i\theta}\,d\theta$. Thus,

$$dT = \frac{\phi_a'(e^{i\theta})e^{i\theta}}{\phi_a(e^{i\theta})}\,d\theta = \frac{(1-|a|^2)e^{i\theta}}{(1-\overline{a}e^{i\theta})(e^{i\theta}-a)}\,d\theta = \frac{1-|a|^2}{|e^{i\theta}-a|^2}\,d\theta.$$

Substituting this into (10.29), we obtain

10.30. Theorem. *Suppose that $f \in \mathcal{H}(\Delta)$ and continuous on $\overline{\Delta}$. Then we have*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|a|^2}{|e^{i\theta}-a|^2}\,d\theta \quad (a\in\Delta).$$

The factor appearing after $f(e^{i\theta})$ under the integral sign has a special notation: with $a = re^{it}$,

$$\mathrm{Re}\left(\frac{e^{i\theta}+a}{e^{i\theta}-a}\right) = \frac{1-|a|^2}{|e^{i\theta}-a|^2} =: P_r(\theta - t)$$

where $P_r(\theta)$ is known as the *Poisson kernel*:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

The extension of Theorem 10.30 for $|z| < R$ follows.

10.31. Theorem. *Suppose that* $f \in \mathcal{H}(\Delta_R)$ *and continuous on* $\overline{\Delta}_R$. *Then we have*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\frac{R^2-|a|^2}{|Re^{i\theta}-a|^2}\,d\theta \quad (a \in \Delta_R).$$

Proof. Define $g(z) = f(Rz)$ with $a = bR$ so that $|a| < R \Longleftrightarrow |b| < 1$ and $f(a) = f(bR) = g(b)$. Further, $g \in \mathcal{H}(\Delta)$ and continuous on $\overline{\Delta}$. By Theorem 10.30,

$$g(b) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\frac{1-|b|^2}{|e^{i\theta}-b|^2}\,d\theta \quad (b \in \Delta)$$

which is equivalent to the desired formula. ■

10.32. Example. If we let $f(z) \equiv 1$ in Theorem 10.31, it follows that

$$1 = \frac{1}{2\pi}\int_{|\zeta|=R} \frac{R^2-|a|^2}{|Re^{i\theta}-a|^2}\,d\theta \quad (\zeta = Re^{i\theta},\ d\zeta = i\zeta d\theta,\ |a| < R);$$

or equivalently (since $|d\zeta| = |i\zeta d\theta| = Rd\theta$)

$$\int_{|\zeta|=R} \frac{|d\zeta|}{|\zeta-a|^2} = \frac{2\pi R}{R^2-|a|^2} \quad \text{if } |a| < R.$$

How does one handle the problem if $|a| > R$? (see Example 8.31). ●

Theorem 10.31 is called the Poisson integral formula for analytic functions in disks, rather than just the unit disk. The integral in Theorem 10.31 is called *Poisson integral* for analytic functions in Δ_R and can be equivalently written as

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})P_r(R, \theta - t)\,d\theta \quad (z = re^{it} \in \Delta_R), \tag{10.33}$$

where $P_r(R,\theta)$ is the *Poisson kernel* given by

$$P_r(R,\theta) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}.$$

Set $f(z) = f(re^{it}) = u(z) + iv(z) := u(r,t) + iv(r,t)$. Since $P_r(R,\theta - t)$ is real-valued, we may equate the real parts of both sides of (10.33) and obtain the Poisson integral formula for harmonic functions in the circular domain $|z| < R$ and continuous on $|z| \le R$. The Poisson integral formula then reads (with $z = re^{it} \in \Delta_R$)

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})P_r(R,\theta - t)\,d\theta$$

and a similar expression holds for $v(z)$. Equivalently, we may write

$$u(r,t) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)P_r(R,\theta - t)\,d\theta \quad (0 \le r < R).$$

This formula implies that if u is harmonic in Δ_R and continuous on $\overline{\Delta}_R$, then its value $u(z)$ $(= u(r,t),\ r < R)$ at an interior point $z = re^{it}$ is completely determined by its boundary values $u(Re^{i\theta})$ $(= u(R,\theta))$ on the circle $|\zeta| = R$. Several extensions of this formula are immediate. For example if $u(z)$ is harmonic in $\Delta(z_0;R)$ and continuous on $\overline{\Delta}(z_0;R)$, then for each $z \in \Delta(z_0;R)$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\frac{R^2 - |z - z_0|^2}{|Re^{i\theta} - (z - z_0)|^2}\,d\theta.$$

It means that the value of a harmonic function at a point z can be expressed as $1/2\pi$ times the integration of the Poisson kernel with the values of the harmonic function on the boundary of the disk.

10.34. Theorem. *Suppose that $f = u + iv \in \mathcal{H}(\Delta_R)$ and is continuous on $\overline{\Delta}_R$. Then for each $z \in \Delta_R$ we have*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\,d\theta + iv(0).$$

Proof. Set $\zeta = Re^{i\theta}$, $z = re^{it}$ $(r < R)$ so that $d\zeta = i\zeta\,d\theta$, and let

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{\zeta + z}{\zeta - z}\,d\theta = \frac{1}{2\pi i}\int_{|\zeta|=R} u(\zeta)\frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta}.$$

It is easy to see that $g(z)$ is analytic with

$$g'(z) = \lim_{h\to 0}\frac{g(z+h) - g(z)}{h} = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{2\zeta}{(\zeta - z)^2}\,d\theta.$$

Indeed, as $(\zeta+z)/(\zeta-z) = 1+2\sum_{n=1}^{\infty}(z/\zeta)^n$ for $|z|<|\zeta|$, we can perform the term by term integration by the uniform convergence of the series (with $z \in \Delta_R$ fixed) and obtain

$$g(z) = b_0 + 2\sum_{n=1}^{\infty} b_n z^n, \quad b_n = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{u(\zeta)}{\zeta^{n+1}}\, d\zeta.$$

Now, with $M = \sup_{|\zeta|=R} |u(\zeta)|$,

$$|b_n| \le \frac{1}{2\pi}\int_{|\zeta|=R} \frac{|u(\zeta)|}{|\zeta|^{n+1}}\, |d\zeta| \le \frac{M}{R^n}$$

and so,

$$\limsup_{n\to\infty} |2b_n|^{1/n} \le \frac{1}{R}\lim_{n\to\infty}(2M)^{1/n} = \frac{1}{R}$$

showing that g is analytic in Δ_R. Now, since

$$\mathrm{Re}\, g(z) = \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{\zeta+z}{\zeta-z}\, d\theta\right\} = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\mathrm{Re}\,\frac{\zeta+z}{\zeta-z}\, d\theta,$$

a comparison (see Equation (10.33)) shows that f and g have the same real parts

$$\mathrm{Re}\, f(z) = \mathrm{Re}\, g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}\, d\theta.$$

Hence, an application of the Cauchy-Riemann equations gives

$$f(z) = g(z) + i\lambda, \quad \text{or} \quad f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\, d\theta + i\lambda.$$

Setting $z=0$, we have $\lambda = v(0)$ (since, by the Mean value theorem, the integral on the right equals $u(0)$), and the result follows. ∎

Equating the imaginary part on both sides of the last integral shows that

$$v(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{2rR\sin(t-\theta)}{R^2-2rR\cos(t-\theta)+r^2}\, d\theta + v(0)$$

or equivalently

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{2\mathrm{Im}\,(z\overline{\zeta})}{|\zeta-z|^2}\, d\zeta + v(0) \quad (\zeta = Re^{i\theta},\ |z|<R)$$

which is of course another representation for harmonic functions. The quantity $Q(\zeta,z)$ defined by

$$Q(\zeta,z) = \frac{2\mathrm{Im}\,(z\overline{\zeta})}{|\zeta-z|^2}$$

is known as the *conjugate Poisson kernel.*

10.35. Example. . We wish to provide an alternate proof of Exercise 6.89. For $|z| = R/2 < R$, where R is large enough, Theorem 10.34 gives that

$$f(z) = \frac{1}{2\pi}\int_{|\zeta|=R} \mathrm{Re}\, f(\zeta)\left(\frac{\zeta+z}{\zeta-z}\right)\frac{d\zeta}{i\zeta} + i\mathrm{Im}\, f(0)$$

and therefore,

$$|f(z)| \leq \frac{1}{2\pi}(\beta R^{\alpha})\left(\frac{R+R/2}{R-R/2}\right)2\pi + |\mathrm{Im}\, f(0)| = 3\beta R^{\alpha} + |\mathrm{Im}\, f(0)|$$

so that there exist constants A and B such that $|f(z)| \leq A|z|^{\alpha} + B$ for all $|z|$ sufficiently large. The desired conclusion follows from the generalized Liouville's theorem (see Theorem 6.60). •

We have shown that a harmonic function on Δ which is continuous on $\overline{\Delta}$ has the property that its values at any point in Δ can be achieved by its values on the boundary $\partial\Delta$. We wish to work the other way around. First we remark that, the Cauchy integral formula not only reproduces analytic functions but also creates them. Indeed, let us suppose that $f(\zeta)$ is continuous on $|\zeta| = 1$. Then there exists an $M > 0$ such that $|f(\zeta)| \leq M$ on $|\zeta| = 1$. Now define $g(z)$ on Δ by

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\zeta f(\zeta)}{\zeta - z}\,d\theta \quad \text{and let} \quad b_n = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

Then

$$g(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=1}\left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{\zeta^{n+1}}\right)z^n\,d\zeta = \sum_{n=0}^{\infty} b_n z^n.$$

Note that $|b_n| \leq M$ and $\limsup_{n\to\infty}|b_n|^{1/n} \leq \lim_{n\to\infty} M^{1/n} = 1$. So, g has a power series representation with radius of convergence not smaller than 1 and therefore, g is analytic for $z \in \Delta$. Also, the fact that g is analytic in Δ is a consequence of Morera's theorem, since $1/(\zeta - z)$ $(|z| < 1)$ is an analytic function of z for each ζ with $|\zeta| = 1$ and g is continuous on $|\zeta| = 1$. This observation shows that the Cauchy integral formula creates analytic functions. But, in general, there exists no direct connection between f and g. Is it then possible to recover f as the boundary limit of g? The function $f(\zeta) = \overline{\zeta} = \frac{1}{\zeta}$ for $|\zeta| = 1$ says 'no' as $g(z) = 0$. On the other hand, for harmonic functions the situation differs. The Poisson integral formula both reproduces and creates harmonic functions. Suppose that $f(\theta) := f(e^{i\theta})$ is a real-valued continuous function of θ, $\theta \in (0, 2\pi]$ and define a new function

$u(z)$ according to the Poisson integral formula for a harmonic function on Δ, namely,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, d\theta \quad \text{if } |z| < 1.$$

What property will the function u have? In contrast to the analytic case, there is a simple connection between f and the created function u, which we shall soon see is harmonic in Δ. This connection is precisely given by the next theorem which is often referred to as the "solution to the Dirichlet problem for disks." What is a Dirichlet problem? Given a bounded domain D and a continuous (more generally piecewise continuous) function $f : \partial D \to \mathbb{R}$, the simplest version of the Dirichlet problem is to find a function $u : \overline{D} \to \mathbb{R}$ such that u is harmonic on D, continuous on $\overline{D}$, and equal to f on ∂D. An important consequence of the maximum and minimum principles is that the solution u to the Dirichlet problem is unique. Indeed, if U is another such function, then $u - U$ is harmonic on D, continuous on $\overline{D}$, and vanishes on the boundary ∂D, but it takes its maximum and its minimum on the boundary, so it is identically zero, i.e. $u \equiv U$ on D. Our special emphasize here on the Dirichlet problem is when D is the unit disk Δ.

10.36. Theorem. (Solution to a Dirichlet Problem) *Suppose that $f : \partial\Delta \to \mathbb{R}$ is continuous. For $0 \le r < 1$, let $z = re^{it}$ and*

$$u(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) P_r(\theta - t)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \mathrm{Re} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) d\theta.$$

Then u is the unique function that is continuous on $\overline{\Delta}$ such that

(a) *$u(z)$ is harmonic for $|z| < 1$ and $u(z) = f(z)$ for $|z| = 1$*
(b) *$u(re^{it}) \to f(e^{it})$ uniformly as $r \to 1^-$.*

Proof. Set $\zeta = e^{i\theta}$, $z = re^{it}$ $(r < 1)$ so that $d\zeta = i\zeta\, d\theta$, and let

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta.$$

Then (for example as in the proof of Theorem 10.34), g is analytic in Δ and so $u(z) = \mathrm{Re}\, g(z)$, being the real part of an analytic function in Δ, is harmonic in Δ. For a proof of (b), we shall use only the following basic properties of the Poisson kernel $P_r(\theta)$:

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

(i) $P_r(\theta) > 0$ for $\theta \in \mathbb{R}$ and $0 \le r < 1$. Moreover, $P_r(\theta) = P_r(-\theta)$ and $P_r(\theta)$ is a 2π-periodic function.

(ii) The Poisson integral in Theorem 10.30 for the constant function 1 shows that

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\,d\theta = 1 \quad \text{for } 0 \le r < 1.$$

(iii) If $0 < \delta < \pi$, then $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\delta \le |\theta| \le \pi$. This property follows from the following observation. For each fixed δ, $0 < \delta < \pi$, we see that

$$P_r'(\theta) = -\frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2} \begin{cases} \le 0 & \text{for } \delta \le \theta \le \pi \\ \ge 0 & \text{for } -\pi \le \theta \le -\delta \end{cases}$$

and so $P_r(\theta)$ is increasing for $-\pi \le \theta \le -\delta$ and decreasing for $\delta \le \theta \le \pi$. This implies that

$$0 < P_r(\theta) \le P_r(\delta) = \frac{1-r^2}{1-2r\cos\delta + r^2} \quad \text{whenever } \delta \le |\theta| \le \pi$$

and the assertion follows.

Next, according to (ii), we can write

$$\begin{aligned} u(re^{it}) - f(e^{it}) &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)[f(e^{i\theta}) - f(e^{it})]\,d\theta \\ &= \frac{1}{2\pi}\int_{-t}^{2\pi - t} P_r(\phi)[f(e^{i(\phi+t)}) - f(e^{it})]\,d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi)[f(e^{i(\phi+t)}) - f(e^{it})]\,d\phi, \end{aligned}$$

because the integrand is a periodic function of period 2π. It remains to show that

$$\lim_{\substack{z\in\Delta \\ z\to e^{it}}} u(z) = \lim_{r\to 1^-} u(re^{it}) = f(e^{it}).$$

Since f is continuous on $[-\pi, \pi]$, it is bounded. So, we may let $M = \max_\theta |f(e^{i\theta})|$. By the uniform continuity of f, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(e^{i(\phi+t)}) - f(e^{it})| < \epsilon \text{ for all } t, \text{whenever } |\phi| < \delta.$$

Now, choose $\delta > 0$ sufficiently small to satisfy this condition and split the interval of integration as

$$\{\phi : |\phi| \le \pi\} = I_1 \cup I_2 := \{\phi : 0 \le |\phi| \le \delta\} \cup \{\phi : \delta \le |\phi| \le \pi\}$$

so that

$$\begin{aligned} u(re^{it}) - f(e^{it}) &= \frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\phi)[f(e^{i(\phi+t)}) - f(e^{it})]\,d\phi \\ &\qquad + \frac{1}{2\pi}\int_{\delta\le|\phi|\le\pi} P_r(\phi)[f(e^{i(\phi+t)}) - f(e^{it})]\,d\phi \\ &=: J_1 + J_2. \end{aligned}$$

On the set I_1,

$$|J_1| \leq \frac{1}{2\pi} \int_{I_1} \epsilon P_r(\phi)\, d\phi \leq \frac{\epsilon}{2\pi} \int_0^{2\pi} P_r(\phi)\, d\phi = \epsilon.$$

On the set I_2, we have $\delta \leq |\phi| \leq \pi$ and $P_r(\phi) \leq P_r(\delta)$ so that

$$|J_2| \leq \frac{2M}{2\pi} P_r(\delta) 2\pi = 2M P_r(\delta).$$

We get the estimate

$$|u(re^{it}) - f(e^{it})| \leq 2M P_r(\delta) + \epsilon$$

and, as r is sufficiently close to 1,

$$\limsup_{r \to 1^-} |u(re^{it}) - f(e^{it})| \leq \epsilon.$$

Hence, $\lim_{r \to 1^-} |u(re^{it}) - f(e^{it})| = 0$ and the uniqueness is a consequence of the maximum modulus principle for harmonic functions. ∎

10.37. Corollary. *Suppose that $a \in \mathbb{C}$, $R > 0$ and $h : \partial\Delta(a; R) \to \mathbb{R}$ is continuous. Then there is a unique continuous function $w(z)$ on $\overline{\Delta}(a; R)$ such that $w(z) = h(z)$ on $\partial\Delta(a; R)$ and $w(z)$ is harmonic on $\Delta(a; R)$.*

Proof. Consider $f(e^{i\theta}) = h(a + Re^{i\theta})$. By the previous theorem, the function w with the desired properties is given by $w(z) = u((z - a)/R)$, where u is defined as in Theorem 10.36. ∎

The proof of a general case for piecewise continuous functions f uses similar ideas but is slightly technical which we avoid in this text. So we state the result without proof.

10.38. Theorem. *Theorem 10.36 continues to hold under a weaker condition that $f : \partial\Delta \to \mathbb{R}$ is a piecewise continuous (i.e. continuous except for finitely many points) bounded function.*

As an application of Poisson's integral formula, we prove

10.39. Theorem. **(Harnack's Inequality for the unit disk)** *Let $u = u(z)$ be harmonic on Δ and continuous on $\overline{\Delta}$. If $u(e^{i\theta}) \geq 0$ for all θ, then for $z = re^{i\theta} \in \Delta$ we have*

$$u(0)\frac{1-r}{1+r} \leq u(re^{i\theta}) \leq u(0)\frac{1+r}{1-r} \quad (r < 1).$$

The estimate is sharp.

Proof. Since $u(e^{i\theta}) \geq 0$, by the Poisson integral formula for u, it follows that $u(z) \geq 0$ for all $z \in \Delta$. The proof depends on the estimate

$$\frac{1-r}{1+r} \leq \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} = P_r(\theta-t) \leq \frac{1+r}{1-r} \quad \text{for } 0 \leq r < 1.$$

Multiplying this by $\frac{1}{2\pi}u(e^{i\theta})\,(\geq 0)$ and integrating, we obtain

$$\frac{1-r}{1+r}\left(\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta\right) \leq u(re^{i\theta}) \leq \frac{1+r}{1-r}\left(\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta\right).$$

The desired inequality is a consequence of the mean value property, namely, $\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta = u(0)$. The above inequalities become an equality for the function $u(z) = \mathrm{Re}\,((1+z)/(1-z))$ at $z = \pm r$. ∎

A suitable scaling and translation of Harnack's inequality immediately implies the following:

10.40. Corollary. *Let $u = u(z)$ be harmonic on $\Delta(a;R)$ and continuous on $\overline{\Delta}(a;R)$. If $u(a+Re^{i\theta}) \geq 0$ for all θ, then for $z \in \Delta(a;R)$ we have*

$$u(a)\frac{R-r}{R+r} \leq u(z) \leq u(a)\frac{R+r}{R-r} \quad (0 \leq r < R).$$

The estimates in this corollary is referred to as Harnack's inequality for arbitrary disks. In particular, if u is harmonic and non-negative on $\Delta(a;R)$ then $u(z) \in [u(a)/3, 3u(a)]$ on $\Delta(a;R/2)$.

We know that if u is a harmonic function in a domain D and $\overline{\Delta}(a;r)$ is contained in D, then u satisfies the Mean value property

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta. \tag{10.41}$$

Conversely, it is easy to show that a continuous function with the Mean value property is necessarily harmonic.

10.42. Theorem. *Let $u : D \to \mathbb{R}$ be continuous on a domain D such that for each point $a \in D$,* (10.41) *holds whenever $\overline{\Delta}(a;r) \subset D$. Then $u(z)$ is harmonic on D.*

Proof. The function u is obviously continuous on $\overline{\Delta}(a;r)$. Also, if $\overline{\Delta}(a;r) \subset D$, by Corollary 10.37, there exists a function $U(z)$ harmonic on $\Delta(a;r)$, continuous on $\overline{\Delta}(a;r)$, and equal to $u(z)$ on the circle $\partial\Delta(a;r)$. Since $U(z) - u(z)$ is continuous on $\overline{\Delta}(a;r)$ and satisfies the Mean value property, by Theorem 6.16, $U(z) - u(z)$ attains both its maximum and minimum on the boundary $\partial\Delta(a;r)$. As $U(z) - u(z) \equiv 0$ on $|z-a| =$

r, it follows that $U(z) \equiv u(z)$ on $\Delta(a;r)$. Hence, $u(z)$ is harmonic in a neighborhood of a. Since a is arbitrary, the result follows. ∎

10.43. Corollary. *A continuous function in a domain D is harmonic in D iff it satisfies the Mean value property at each point of D.*

As an application, we prove the following analog to Weierstrass' theorem for sequences (see Theorem 4.84) of harmonic functions.

10.44. Theorem. *If $\{u_n(z)\}$ is a sequence of real-valued harmonic functions that converges uniformly on all compact subsets of a domain D to a function $u : D \to \mathbb{R}$, then $u(z)$ is harmonic in D.*

Proof. Since the limit of a uniformly convergent sequence of continuous functions is continuous (see Theorem 2.57), $u(z)$ is continuous on D. Also, if $\overline{\Delta}(a;r) \subset D$, then

$$u_n(a) = \frac{1}{2\pi}\int_0^{2\pi} u_n(a+re^{i\theta})\,d\theta$$

for each n. By the uniform convergence (see the proof of Theorem 4.84),

$$u(a) = \lim_{n\to\infty} u_n(a) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(a+re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta.$$

So, by Theorem 10.42, u is harmonic on $\Delta(a;r)$ (and hence on D). ∎

The Harnack inequality allows us to draw some striking conclusion, for example, the following 'convergence theorem' for sequences of harmonic functions.

10.45. Theorem. (Harnack's Principle/Theorem) *Let $\{u_n(z)\}_{n\geq 1}$ be an increasing (i.e. $u_{n+1} \geq u_n$ on D and for each $n \geq 1$) sequence of harmonic functions defined on a domain D. Then either $u_n(z) \to +\infty$ for each $z \in D$ (and uniformly on compact subsets of D) or $u_n(z)$ converges to a harmonic function u on D, uniformly on compact subsets.*

Proof. We may assume that $u_1(z) \geq 0$ (for otherwise, replace $u_n(z)$ by the nonnegative sequence $\{u_n(z) - u_1(z)\}$). By the monotonicity property, for each z in D either the pointwise limit $\lim_{n\to\infty} u_n(z)$ exists or $\lim_{n\to\infty} u_n(z) = +\infty$. The function $u(z)$ can be defined by this limit as a map $u : D \to \mathbb{R}$. Define

$$A = \{z \in D : u_n(z) \to \infty\} \quad \text{and} \quad B = \{z \in D : \{u_n(z)\} \ \text{converges}\}.$$

Given $a \in D$, choose a disk $\overline{\Delta}(a;R) \subset D$. Then, for all $z \in \Delta(a;R/2)$, Harnack's inequality gives

$$(1/3)u_n(a) \leq u_n(z) \leq 3u_n(a). \tag{10.46}$$

If a is such that $u_n(a) \to \infty$, then the left hand inequality of (10.46) shows that $u_n(z) \to \infty$ for $|z-a| \le R/2$, and that the convergence is uniform in this disk.

If a is such that $\{u_n(a)\}$ converges to a finite limit, then the right hand inequality shows that $\{u_n(z)\}$ converges for $|z-a| \le R/2$. Hence, A and B are both open sets with $A \cup B = D$. Since D is connected, either $A = \emptyset$ or $B = \emptyset$. If $A = \emptyset$, then $B = D$, and so $\{u_n(z)\}$ converges for all z in D. Next we must show that $\{u_n(z)\}$ converges uniformly on compact subsets of D. By using the right hand part of Harnack's inequality (10.46), we find that

$$u_{n+p}(z) - u_n(z) \le 3[u_{n+p}(a) - u_n(a)] \tag{10.47}$$

for $|z-a| \le R/2$ and $p = 1, 2, \dots$. By the Cauchy criterion, this inequality in turn implies that the convergence at a point a implies the uniform convergence in a neighborhood of a. Since every compact subset of D can be covered by finitely many such neighborhoods, $\{u_n(z)\}$ is uniformly Cauchy and so it converges uniformly on every compact subset of D to $u(z)$. Finally, it follows from Theorem 10.44 that the limit function is harmonic throughout D. ∎

The function $u_n(z) = x + n$, or $y + n^2$, each of which is harmonic in every domain, satisfies the conditions of the theorem.

10.4 Analytic Continuation via Reflection

A domain Ω is said to be *symmetric* with respect to the origin if for every $z \in \Omega$, the point $-z \in \Omega$. For example, disks centered at the origin. Now, we consider the following two sets of functions

$$\mathcal{F} = \{1+z^2, \cos z, \sin z, \exp z\}, \quad \text{and} \quad \mathcal{G} = \{i+z^2, \cos(iz), i\sin z, i\exp z\}.$$

For each $f \in \mathcal{F}$, we have $\overline{f(\overline{z})} = f(z)$ and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. For example, if we allow the three steps for $\cos z$ we find that

$$z \mapsto \overline{z} \mapsto \frac{e^{i\overline{z}} + e^{-i\overline{z}}}{2} \mapsto \frac{e^{-iz} + e^{iz}}{2} = \cos z.$$

On the other hand, for each $g \in \mathcal{G}$, we have

$$g(\overline{z}) \neq \overline{g(z)}, \quad \text{and } g(x) \notin \mathbb{R} \text{ if } x \in \mathbb{R}.$$

Note that the domain of analyticity for each $f \in \mathcal{F}$ and each $g \in \mathcal{G}$ is the whole complex plane. Further, the reflection of z with respect to the real axis of z-plane does not correspond to the reflection of $g(z)$ with respect to the real axis of the w-plane. We are interested in the following problem:

10.48. Problem. Under what conditions, does an analytic function f on some domain Ω possess the following property: $f(\overline{z}) = \overline{f(z)}$, i.e. the reflection of z with respect to the real axis of z-plane equals the reflection of $w = f(z)$ with respect to the real axis of the w-plane.

Our first result actually describes how it is possible to use two reflections to construct a new analytic function from an old one.

10.49. Theorem. *Let Ω be a domain which is symmetric with respect to the real axis. Then, $f(z) \in \mathcal{H}(\Omega)$ iff $\overline{f(\overline{z})} \in \mathcal{H}(\Omega)$.*

Proof. Define $f(z) = u(x,y) + iv(x,y)$ and $F(z) = \overline{f(\overline{z})}$. Then,

$$F(z) = u(x,-y) - iv(x,-y) =: U(x,y) + iV(x,y).$$

Note that u and v belong to $C^2(\Omega)$ iff U and V belong to $C^2(\Omega)$. Further,

$$\begin{aligned} U_x(x,y) &= u_x(x,-y), & U_y(x,y) &= -u_y(x,-y), \\ V_y(x,y) &= v_y(x,-y), & V_x(x,y) &= -v_x(x,-y), \end{aligned}$$

which, in particular, give that u and v satisfy the C-R equations on Ω iff U and V satisfy the same property on Ω. Now, the desired conclusion is a consequence of Theorem 3.26 and the fact that the real and imaginary parts of analytic functions in Ω are C^∞-functions on Ω. ■

Theorem 10.49 leads to a general symmetric principle for analytic functions.

10.50. Theorem. *Let Ω be a domain such that it contains a segment (a,b) of the real axis, and is symmetric with respect to the real axis. Suppose that $f(z) \in \mathcal{H}(\Omega)$. Then, $f(z)$ is real on (a,b) iff $f(z) = \overline{f(\overline{z})}$ for all $z \in \Omega$.*

Proof. Suppose that $f(z)$ is real on (a,b). Consider the difference

$$h(z) = f(z) - \overline{f(\overline{z})}.$$

Then, by Theorem 10.49, $h \in \mathcal{H}(\Omega)$. Moreover, $h(z) = 0$ for all $z \in (a,b)$. By the uniqueness theorem, $h(z) \equiv 0$ on Ω.

Conversely, if $f(z) = u(x,y) + iv(x,y)$ and $f(z) = \overline{f(\overline{z})}$ on Ω then it follows that

$$u(x,y) + iv(x,y) = u(x,-y) - iv(x,-y).$$

In particular, when $z = x + iy \in \Omega$ is real, this equation gives

$$u(x,0) + iv(x,0) = u(x,0) - iv(x,0)$$

so that $v(x,0) = 0$. Therefore, $f(z)$ is real when z is real. ■

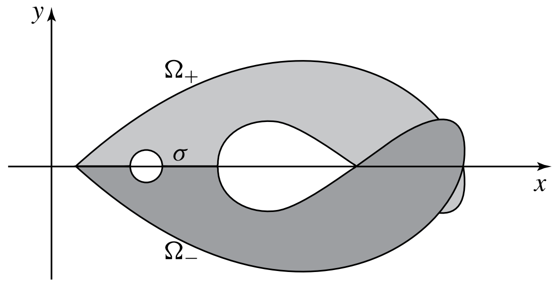


Figure 10.6: Reflection Principle

10.51. Lemma. (Reflection Principle for Harmonic Functions) *Let Ω be a domain which is symmetric with respect to the real axis and define Ω_+, Ω_-, and σ as the intersection of Ω with the upper half-plane $\{z : \operatorname{Im} z > 0\}$, the lower half-plane $\{z : \operatorname{Im} z < 0\}$, and the real axis, respectively (see Figure 10.6). Let $v(z)$ be a real-valued continuous function on $\Omega_+ \cup \sigma$, harmonic on Ω_+, and zero on σ. Then v admits a unique harmonic extension V on all of Ω and the extension satisfies the symmetry relation*

$$V(\overline{z}) = -V(z) \quad \text{for } z \in \Omega. \tag{10.52}$$

Proof. First we note that if such an extension exists it is certainly unique, so it suffices to show that the stated extension defines a harmonic function in Ω. Secondly, we note that Ω is a disjoint union of Ω_+, Ω_- and σ. We extend $v(z)$ to Ω by setting

$$V(z) = \begin{cases} v(z) & \text{for } z \in \Omega_+ \\ 0 & \text{for } z \in \sigma \\ -v(\overline{z}) & \text{for } z \in \Omega_-. \end{cases}$$

Then, V is continuous on Ω, harmonic on $\Omega_+ \cup \Omega_-$, and (10.52) holds. We claim that V has the mean value property. Fix $a \in \Omega$. If $a \in \Omega_+$ or Ω_-, then $V(z)$ has the mean value property for small disks centered at a, since $V(z)$ is harmonic near a. For $a \in \sigma$, we have

$$\begin{aligned} \int_{-\pi}^{\pi} V(a + re^{i\theta})\, d\theta &= -\int_{-\pi}^{0} v(a + re^{-i\theta})\, d\theta + \int_{0}^{\pi} v(a + re^{i\theta})\, d\theta \\ &= -\int_{0}^{\pi} v(a + re^{i\theta})\, d\theta + \int_{0}^{\pi} v(a + re^{i\theta})\, d\theta = 0 \end{aligned}$$

Thus, V satisfies the mean value property on Ω and hence, it is harmonic on Ω. ■

Lemma 10.51 not only provides a method of extending a harmonic function from a given open set to a larger open set but also gives a proof of a stronger Schwarz's reflection/symmetry principle for analytic functions.

10.53. Theorem. (Schwarz's Reflection Principle for Analytic Functions) *Let* Ω, Ω_+, Ω_-, *and* σ *be as above. Suppose that* $f \in \mathcal{H}(\Omega_+)$, *continuous on* σ *and* $f(z)$ *is real on* σ. *Then* f *extends to be analytic on* Ω *and the extension satisfies the symmetry relation* $f(z) = \overline{f(\overline{z})}$, $z \in D$.

Proof. Set $f(z) = u(z) + iv(z)$, $z \in \Omega_+$. By the previous result applied to $v(z) = \operatorname{Im} f(z)$, $v(z)$ extends to be harmonic on Ω with $v(\overline{z}) = -v(z)$, $z \in \Omega$. Again, we fix $a \in \sigma$ and let $\Delta(a;\delta)$ be a disk that is contained in Ω. Since $v(z)$ is harmonic in this disk, which is simply connected, it has a harmonic conjugate u_1 in $\Delta(a;\delta)$ such that $f_1 = u_1 + iv$ is analytic in $\Delta(a;\delta)$. Thus,

$$\operatorname{Im}(f(z) - f_1(z)) = 0 \quad \text{on } \Omega_+ \cap \Delta(a;\delta).$$

We conclude that $f(z) = f_1(z) + c$, c a real constant, on $\Omega_+ \cap \Delta(a;\delta)$. But f_1 is analytic in $\Omega_+ \cap \Delta(a;\delta)$ and therefore, f is also analytic in $\Delta(a;\delta)$. Consequently, the original function $f(z)$ extends to be analytic in $\Delta(a;\delta)$. By Theorem 10.49, $\overline{f(\overline{z})}$ is also analytic in $\Delta(a;\delta)$. Moreover, $f(z) = \overline{f(\overline{z})}$ on the interval $(a-\delta, a+\delta)$, and hence on $\Delta(a;\delta)$ by the uniqueness theorem. Extend $f(z)$ to Ω_- by setting $f(z) = \overline{f(\overline{z})}$ for $z \in \Omega_-$. Then the extended function f is analytic on Ω_- and coincides with the analytic continuation of $f(z)$ across σ from Ω_+. Hence, f is analytic on Ω and satisfies $f(z) = \overline{f(\overline{z})}$ on Ω. ∎

We observe that the line of reflection could be an arbitrary line, not just the real axis. Thus if Ω is a domain symmetric with respect to a line and f is defined and analytic on one side of the line, and real on the line, then f can be extended to be analytic on the whole domain. The proof can be obtained by translating and rotating the domain to the standard case. Similarly, we can also translate and rotate the image, so it is not necessary that $f(z)$ be real on the symmetry line, it is sufficient that it maps the line into some other line.

10.5 Exercises

10.54. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) If f is analytic for $|z| < R$ and satisfies the relation $f(2z) = 2f(z)f'(z)$ for $|z| < R/2$, then f can be continued analytically into the whole plane.

(b) For $0 \neq \alpha \in \mathbb{R}$, the functions defined by the series

$$\sum_{n=0}^{\infty} (\alpha z)^n \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \frac{(1-\alpha)^n z^n}{(1-z)^{n+1}}$$

are analytic continuations of each other.

(c) The series

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{and} \quad i\pi + \sum_{n=1}^{\infty} (-1)^n \frac{(z-2)^n}{n}$$

have no common region of convergence, but they represent the same function $-\operatorname{Log}(1-z)$ in their respective disks of convergence.

(d) The function f defined by the series $\sum_{n=1}^{\infty}(-1)^{n-1}z^n/n$ on Δ, and the function F defined by the series

$$\ln 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2\cdot 2^2} - \frac{(1-z)^3}{3\cdot 2^3} - \cdots$$

on $\Delta(1;2)$ are analytic continuations of each other.

(e) The imaginary axis is a natural boundary for $\sum_{n\geq 0}\exp(-n!z)$.

(f) The unit circle $|z|=1$ is the natural boundary for the series $\sum_{k=0}^{\infty} z^{3^k}$.

(g) If $f(z)=\sum_{n\geq 0} a_n z^n$ is analytic for $|z|\leq R$, then R cannot be the radius of convergence of this series defined by the sum $f(z)$.

(h) The function element $(1/(z(1+z)), \mathbb{C}\backslash\{0,1\})$ is the analytic continuation of

$$f(z) = \sum_{n=0}^{\infty}(1-z)^n - \sum_{n=0}^{\infty}\frac{(1-z)^n}{2^{n+1}}, \quad \text{for } |1-z|<1\ .$$

10.55. Show that the function $f(z) = \sum_{n\geq 1}\frac{z^{2^n}}{n^2}$ is continuous for $|z|\leq 1$, but every point on $|z|=1$ is a singular point.

10.56. Let f be analytic in the cut plane $D=\mathbb{C}\setminus(-\infty,0]$ such that $f(x)=x^x$ for all $x>0$. Show that $f(z)=\overline{f(\overline{z})}$ for all $z\in D$.

10.57. Suppose that $f(z)$ is analytic for $|z|<1$ and $|f^{(n)}(0)|\leq 3^n$ for all $n\in\mathbb{N}$. Show that f can be extended to an entire function g such that $f(z)=g(z)$ for $|z|<1$.

10.58. Find a function $u(x,y)$ that is harmonic in the region of the right half-plane between the curves $xy=1$ and $xy=2$ and takes the value 3 when $xy=1$, and the value 7 when $xy=2$.

10.59. Use Schwarz's reflection principle to show that every function which is bounded and analytic in the upper half-plane $\operatorname{Im}(z)\geq 0$, and real on the real axis is constant.

10.60. Use Schwarz's reflection principle to show that

(i) $\overline{\sin z} = \sin\overline{z}$ for $z\in\mathbb{C}$

(ii) $\overline{z^4-2z+2\cos z} = \overline{z}^4 - 2\overline{z} + 2\cos\overline{z}$ for $z\in\mathbb{C}$.

Chapter 11

Representations for Meromorphic and Entire Functions

The central importance of this chapter is to study representations of meromorphic functions by an infinite decomposition into partial fractions as well as representations of analytic functions by infinite products. These will be done in many different ways and we shall also see several applications.

In Section 11.1, we discuss the Mittag-Leffler expansion which gives a formula for representing a meromorphic function f as a series involving the principal parts of f at each of the poles of f. We begin Section 11.2 by presenting a brief introduction and basic properties of infinite products of sequences of complex numbers and show that their convergence properties are similar to those of infinite series. In particular, we prove several important tests for the convergence of infinite products In Section 11.3, we consider the convergence of infinite products of analytic functions which will be useful in expressing a non-constant entire function f as a product of the form $f(z) = P(z)H(z)$, $z \in \mathbb{C}$, where

- P and H are entire functions
- P and f have exactly the same zeros with prescribed multiplicities
- $H(z)$ has no zeros in $\mathbb{C}$.

In Section 11.4, we present an interesting and useful result called the Weierstrass product formula which gives a way of factoring certain entire functions into an infinite product. Clearly, some entire functions (such as polynomials $p(z)$ and the transcendental entire function $p(z)e^z$) have only a finite product representation. A comparison of products and series expansions provides us a number of interesting identities. In the Weierstrass product expansion the principal part of f has no role to play whereas the Mittag-Leffler expansion emphasizes the principal parts at the poles of f, but gives no information about its zeros. In Section 11.5, we study the gamma function as a meromorphic function in $\mathbb{C}$ having simple poles at $z = 0, -1, -2, \ldots$. Section 11.6 discusses the zeta function and its various properties. In Section 11.7 we prove Jensen's formula. In Section 11.8, we continue our discussion on entire and meromorphic functions, and introduce the concept of order and the genus of entire functions. Also, we study certain basic properties of entire functions of finite order and of finite genus in order to prove the much waited Hadamard factorization theorem which provides an interesting relationship between the order and the genus of entire functions.

11.1 Infinite Sums and Meromorphic Functions

In this section we are interested in the construction of meromorphic functions by their poles. Concerning poles there are two possibilities:

- meromorphic functions with a finite number of poles
- meromorphic functions with infinite number of poles.

A meromorphic function which has a pole of order m at a is $(z-a)^{-m}$ and a slightly different meromorphic function may be obtained by taking linear combinations of finitely many such simpler ones. For example,

$$\frac{1}{(z-1)^2} + \frac{4}{(z-3)^3} - \frac{2}{(z-2)^8}.$$

Let us now discuss the construction of meromorphic functions in the finite case. Suppose that f is a meromorphic function which has a finite number of poles at a_j $(1 \le j \le n)$ of order m_j $(1 \le j \le n)$. Then, by Laurent's expansion of f around each a_j, there is an associated principal part

$$P_j\left(\frac{1}{z-a_j}\right) = \sum_{k=1}^{m_j} \frac{A_{-k}^{(j)}}{(z-a_j)^k}$$

which may be thought of as a polynomial in the variable $1/(z-a_j)$. Define

$$g(z) = \sum_{j=1}^{n} P_j\left(\frac{1}{z-a_j}\right).$$

Clearly, g is analytic in $\mathbb{C}\setminus\{a_1, a_2, \ldots, a_n\}$. Since, for $s \ne r$, $P_s(1/(z-a_s))$ is analytic at a_r, the principal part of g about a_r is $P_r(1/(z-a_r))$. Thus, f and g are meromorphic functions in $\mathbb{C}$ with poles at a_j and have the same principal parts at a_j, $1 \le j \le n$. In particular, if we let $\phi(z) = f(z) - g(z)$ then ϕ is analytic in $\mathbb{C} \setminus \{a_1, a_2, \ldots, a_n\}$ and has removable singularities at $a_1, a_2, \ldots, a_n$. Consequently, ϕ can be extended to an entire function. Thus, $f(z) = g(z) + \phi(z)$ where ϕ is entire. Finally, because $g(z) \to 0$ as

$z \to \infty$, g has a removable singularity at ∞ which shows that ϕ and f have the same principal part at ∞. The above discussion gives

11.1. Theorem. *Let f be a meromorphic function with only poles at a_j $(1 \le j \le n)$. If $P_j(1/(z-a_j))$ denotes the principal part of $f(z)$ at a_j $(1 \le j \le n)$, then there exists an entire function $\phi(z)$ such that*

$$f(z) = \sum_{j=1}^{n} P_j\left(\frac{1}{z-a_j}\right) + \phi(z).$$

In addition, ϕ and f have the same principal part at ∞.

11.2. Corollary. *Every proper rational function of the form*

$$R(z) = \frac{p(z)}{q(z)} := \frac{p(z)}{c(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_n)^{m_n}}, \tag{11.3}$$

where p and q are polynomials with $\deg p(z) < \deg q(z)$, can be expanded as a sum of polynomials in $1/(z-a_k)$. Here a_k's denote the poles of $R(z)$ with order $m_k \ge 1$ $(k = 1, 2, \ldots, n)$.

Proof. By Theorem 11.1, $R(z) - \sum_{j=1}^{n} P_j(1/(z-a_j)) = \phi(z)$ is an entire function. Furthermore, ϕ is bounded since $R(z)$ and each of its principal part about each a_j approaches 0 as $z \to \infty$. As a consequence of Liouville's theorem, $\phi(z)$ is equal to a constant, in fact, $\phi(z) \equiv 0$ so that $R(z) = \sum_{j=1}^{n} P_j(1/(z-a_j))$ as asserted. ∎

Suppose that $R(z)$ is a proper rational function of the form (11.3). Then the partial fraction expansion of $R(z)$ may be written uniquely in the form

$$R(z) = \sum_{j=1}^{n}\left(\sum_{k=1}^{m_j} \frac{A_{-k}^{(j)}}{(z-a_j)^k}\right). \tag{11.4}$$

In general, a meromorphic function f in $\mathbb{C}$ may have an infinite number of poles at a_k, $k \in \mathbb{N}$. In that case the series (whose terms are the principal parts of these poles of f), $\sum_k P_k(\frac{1}{z-a_k})$, need not converge. It is then necessary to modify the terms to produce a convergent series. For example, suppose we wish to find a meromorphic function in $\mathbb{C}$ having a simple pole at k $(k \in \mathbb{N})$ with residue 1 at each k. From the prescription, our function corresponding to the principal part at $z = k$ is

$$P_k\left(\frac{1}{z-k}\right) = \frac{1}{z-k} = -\frac{1}{k(1-z/k)}$$

but the sum $\sum_{k=1}^{\infty} \frac{1}{z-k}$ does not converge in $\mathbb{C}\backslash\mathbb{N}$. We need to modify the series suitably so that it becomes convergent. The constant term in the series expansion of $1/(z-k)$ about 0 is $-1/k$. So we can try with

$$\sum_{k=1}^{\infty}\left\{P_k\left(\frac{1}{z-k}\right)-\left(-\frac{1}{k}\right)\right\}=\sum_{k=1}^{\infty}\left(\frac{1}{z-k}+\frac{1}{k}\right)=\sum_{k=1}^{\infty}\frac{z}{k(z-k)}.$$

This series does converge uniformly on every compact subset of $\mathbb{C}$ except at k, $k\in\mathbb{N}$ (by a comparison with the convergent series $\sum_{k\geq 1}k^{-2}$). Indeed,

$$\left|\frac{1}{z-k}\right|\leq\frac{1}{k-|z|}<\frac{2}{k}\ \text{ whenever } |z|<k/2$$

so that, for $|z|\leq R$ and $R<k/2$, we have

$$\sum_{k=1}^{\infty}\left|\frac{1}{k(z-k)}\right|\leq 2\sum_{k=1}^{\infty}\frac{1}{k^2}$$

and hence, by the Weierstrass M-test, the series $\sum_{k=1}^{\infty}\frac{z}{k(z-k)}$ represents an analytic function on $\mathbb{C}\backslash\{k:\ k\in\mathbb{N}\}$.

Given a meromorphic function f in $\mathbb{C}$ with an infinite number of poles at a_k, we may always assume that the poles of f are indexed so that $\{|a_k|\}$ is non-decreasing. Let us now state and prove a general theorem of Mittag-Leffler although its simple form (namely, Theorem 11.7) suffices for our purposes.

11.5. Theorem. (Mittag-Leffler) *Let $\{a_n\}_{n\geq 1}$ be a sequence of distinct non-zero complex numbers such that $|a_n|<|a_{n+1}|$ for $n\in\mathbb{N}$ and $|a_n|\to\infty$ as $n\to\infty$. Then, for a given sequence of polynomials $\{P_n(z)\}$ without constant term, there exists a meromorphic function f in $\mathbb{C}$ having poles at a_n with principal part $P_n(1/(z-a_n))$. Moreover, any other general meromorphic function F having the stated property will be of the form*

$$F(z)=\sum_n\left\{P_n\left(\frac{1}{z-a_n}\right)-Q_n(z)\right\}+h(z)$$

for some polynomial $Q_n(z)$ and for some entire function $h(z)$. The series converges absolutely and uniformly on any compact subset of $\mathbb{C}$ not containing the poles (This series is usually called a Mittag-Leffler expansion of F).

Proof. As $P_n(z)$ is a polynomial, we see that $\psi_n(z)$ defined by

$$\psi_n(z):=P_n\left(\frac{1}{z-a_n}\right)=P_n\left(-\frac{1}{a_n(1-z/a_n)}\right)$$

is analytic for $|z|<|a_n|$ and hence, we may expand it as a Taylor series

$$\psi_n(z)=\sum_{k=0}^{\infty}A_k^{(n)}z^k\ \text{ for } |z|<|a_n|.$$

By elementary facts about complex power series, the series on the right converges absolutely and uniformly to $\psi_n(z)$ for $|z| \leq \frac{1}{2}|a_n|$. Let

$$Q_n(z) = \sum_{k=0}^{\lambda_n} A_k^{(n)} z^k$$

be the partial sum of the series up to the degree λ_n, where λ_n has been chosen large enough to satisfy

$$|\psi_n(z) - Q_n(z)| < 2^{-n} \quad \text{for} \quad |z| \leq \frac{1}{2}|a_n|.$$

Since $\lim_{n\to\infty} |a_n| \to \infty$ and $|a_n| < |a_{n+1}|$, given any compact subset $K = \overline{\Delta}_R$ of $\mathbb{C}$, there exists an $N = N(K)$ such that $K \subset \Delta_{|a_n|/2}$ for $n \geq N$, i.e. $|a_n| > 2R$ for $n \geq N$. It follows that the series

$$\sum_{n=N}^{\infty} (\psi_n(z) - Q_n(z))$$

converges absolutely and uniformly on K, and therefore represents an analytic function on K. Since K is arbitrary, the full series

$$\sum_{n=1}^{\infty} (\psi_n(z) - Q_n(z)) = \left(\sum_{n=1}^{N-1} + \sum_{n=N}^{\infty} \right) (\psi_n(z) - Q_n(z)) \tag{11.6}$$

represents a meromorphic function in $\mathbb{C}$ which has a_n as its pole with principal part equal to $\psi_n(z) = P_n(1/(z - a_n))$. Note that the finite sum in (11.6) is a rational function with prescribed behavior of poles exactly at a_n with $|a_n| < R$. The rest of the theorem is trivial. ∎

For simplification purposes, we prove the following simple version that has wider applications and falls under the general theorem of the Mittag-Leffler (see Theorem 11.5).

11.7. Theorem. *Let f be meromorphic with only simple poles at $a_1, a_2, \ldots$ such that*

$$0 < |a_1| \leq |a_2| \leq \cdots \leq |a_k| \leq \cdots, \quad \text{and} \quad b_k = \operatorname{Res}[f(z); a_k].$$

Let $\{C_k\}$ be a nested sequence of positively oriented simple closed contours (which avoid these poles) such that each C_k includes only a finite number of poles. Suppose that

$$\begin{aligned} R_k &= \operatorname{dist}(0, C_k) \to \infty \quad \text{as} \quad k \to \infty, \\ L_k &= \text{length of } C_k = O(R_k), \\ |f(z)| &\leq M \quad \text{for each } k \text{ for } z \in C_k \end{aligned}$$

(e.g. C_k is a square with vertices $R_k(\pm 1 \pm i)$ such that $R_k \to \infty$ as $k \to \infty$). Then, for all z except at these poles, we have the Mittag-Leffler series expansion of f

$$f(z) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{z - a_k} + \frac{1}{a_k} \right).$$

Proof. If α is not a pole of f and $\alpha \notin C_k$, then

$$g(z) = \frac{f(z)}{z - \alpha}$$

has simple poles at α, and at each a_k with $\mathrm{Res}\,[g(z); \alpha] = f(\alpha)$ and

$$\mathrm{Res}\,[g(z); a_k] = \lim_{z \to a_k} (z - a_k) \frac{f(z)}{z - \alpha} = \frac{b_k}{a_k - \alpha}.$$

Then

$$\begin{aligned} F_k(\alpha) &:= \frac{1}{2\pi i} \int_{C_k} \frac{f(z)}{z - \alpha}\, dz \\ &= \sum \mathrm{Res}\,[g(z); C_k] \\ &= f(\alpha) + \sum_k \frac{b_k}{a_k - \alpha}, \end{aligned} \tag{11.8}$$

where $\sum_k$ is taken over all the poles of f inside C_k and k is chosen large enough so that α lies inside C_k. Letting $\alpha = 0$ gives

$$F_k(0) = \frac{1}{2\pi i} \int_{C_k} \frac{f(z)}{z}\, dz = f(0) + \sum_k \frac{b_k}{a_k}. \tag{11.9}$$

Subtracting (11.8) from (11.9) gives

$$\begin{aligned} F_k(0) - F_k(\alpha) &= -\frac{1}{2\pi i} \int_{C_k} \frac{f(z)}{z(z - \alpha)}\, dz \\ &= f(0) + \sum_k b_k \left(\frac{1}{\alpha - a_k} + \frac{1}{a_k} \right) - f(\alpha). \end{aligned} \tag{11.10}$$

Now, for $z \in C_k$, $|z| \geq R_k = \mathrm{dist}\,(0, C_k)$ and $|z - \alpha| \geq |z| - |\alpha| \geq R_k - |\alpha| > 0$ so that

$$|F_k(\alpha) - F_k(0)| = \left| \frac{-1}{2\pi i} \int_{C_k} \frac{f(z)}{z(z - \alpha)}\, dz \right| \leq \frac{1}{2\pi} \frac{ML(C_k)}{R_k(R_k - |\alpha|)} \to 0 \text{ as } k \to \infty$$

and therefore, the sequence $\{F_k(\alpha) - F_k(0)\}$ converges to zero uniformly on the compact set C_k. Allowing $k \to \infty$ in (11.10) gives

$$f(\alpha) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{\alpha - a_k} + \frac{1}{a_k} \right). \quad \blacksquare$$

Most often C_k is taken to be a circle or the boundary of a rectangle. In practice, verifying the boundedness condition, namely $|f(z)| \le M$ on C_k, is often a difficult job. Suppose we wish to expand a meromorphic function f which has a simple pole at the origin, then the theorem is not directly applicable. However, we can apply Theorem 11.5 or 11.7 by considering the function $f(z) - p_0(z)$, where $p_0(z)$ is the principal part of the Laurent series expansion of f about $z = 0$.

The partial fraction expansions of functions such as

$$\csc z,\ \sec z,\ \tan z,\ \cot z,\ \cos az,\ \sec z \cos az,\ \csc z \sin az,\ \sec z \sin az,$$

($|a| < 1$) and of their corresponding hyperbolic functions are well-known and may be found in some standard texts either as examples or as exercises.

For example, in order to apply Theorem 11.7 for the functions $\pi \csc \pi z$ and $\pi \cot \pi z$, we need to consider

$$\pi \csc \pi z - 1/z, \quad \text{and} \quad \pi \cot \pi z - 1/z,$$

respectively. Let us now discuss the situation in detail.

11.11. Example. Let us derive the Mittag-Leffler expansion of

$$g(z) = \pi \cot \pi z.$$

The poles of $g(z)$ are at n, $n \in \mathbb{Z}$, each pole being simple. As g has a simple pole at the origin, we need to compute

$$\mathrm{Res}\,[g(z); 0] = \lim_{z \to 0} \left(\frac{\pi z}{\sin \pi z} \right) \cos \pi z = 1$$

and consider the modified function

$$f(z) = \begin{cases} \pi \cot \pi z - \dfrac{1}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

which is now analytic at the origin and has simple poles only at n, $n \in \mathbb{Z} \setminus \{0\}$. It follows that, for each $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathrm{Res}\,[f(z); n] = \lim_{z \to n} (z - n) \left[\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z} \right] = \frac{\cos \pi n}{\cos \pi n} = 1.$$

Choose $C_k = \{R_k(\pm 1 \pm i) : k \in \mathbb{N}\}$ with $R_k = k + 1/2$ so that

$$L(C_k) = 8(k + 1/2) = 8\,\mathrm{dist}\,(0; C_k) \to \infty \quad \text{as} \quad k \to \infty$$

and all the other required conditions of Theorem 11.7 are satisfied. Recall that (see the proof of Theorem 9.65) $|\cot \pi z| < 2$ for all z on C_k. Clearly, for $z \in C_k$, we have

$$|z| \geq \operatorname{dist}(0; C_k) = R_k = k + 1/2 > 1$$

and therefore,

$$|f(z)| \leq \pi |\cot \pi z| + \frac{1}{|z|} < 2\pi + 1 \quad \text{for all } z \text{ on } C_k.$$

Finally, it now follows from Theorem 11.7 that

$$\pi \cot \pi z - \frac{1}{z} = \lim_{n\to\infty} \sum_{\substack{k=-n \\ k\neq 0}}^{n} \left(\frac{1}{z-k} + \frac{1}{k} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{z-k} + \frac{1}{z+k} \right)$$

which shows that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z}. \tag{11.12}$$

From this equation, the partial fraction decomposition of $\pi^2 / \sin^2 \pi z$ can be obtained by differentiating (11.12):

$$\frac{\pi^2}{\sin^2 \pi z} - \frac{1}{z^2} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{(z-k)^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z}.$$

Allowing $z \to 0$ gives that

$$\frac{\pi^2}{3} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2}, \quad \text{i.e.} \quad \frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Similarly, if we consider $g(z) = \pi \csc \pi z$ then it is easy to see that g has a simple pole at n ($n \in \mathbb{Z}$) with $\operatorname{Res}[g(z); n] = (-1)^n$ for $n \in \mathbb{Z}$. Now, we define $f(z) = \pi \csc \pi z - z^{-1}$. Then, applying Theorem 11.7 (since f is again uniformly bounded on C_k), it follows easily that

$$\pi \csc \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2(-1)^k z}{z^2 - k^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z}.$$ •

11.13. Example. We consider $f(z) = \pi \tan \pi z$. Then we see that

(i) f has simple poles at $(n + 1/2)$, $n \in \mathbb{Z}$ with $\operatorname{Res}[f(z); n + 1/2] = -1$

(ii) f is analytic at $z = 0$ with $f(0) = 0$

(iii) $|f(z)| \leq 2$ for each z in the square C_k with vertices $k(\pm 1 \pm i)$, $k \in \mathbb{N}$ and f is analytic on C_k, $k \in \mathbb{N}$.

By Theorem 11.7, we see that

$$\begin{aligned} f(z) &= \lim_{k\to\infty} \sum_{n=-k}^{k-1} (-1) \left(\frac{1}{z-(n+1/2)} + \frac{1}{n+1/2} \right) \\ &= -\lim_{k\to\infty} \left(\sum_{n=0}^{k-1} \frac{z}{(z-(n+1/2))(n+1/2)} \right. \\ &\qquad \left. + \sum_{n=-k}^{-1} \frac{z}{(z-(n+1/2))(n+1/2)} \right) \\ &= -\lim_{k\to\infty} \left(\sum_{n=0}^{k-1} \frac{z}{(z-(n+1/2))(n+1/2)} \right. \\ &\qquad \left. + \sum_{n=0}^{k-1} \frac{z}{z-(-n-1/2))(-n-1/2)} \right) \\ &= -\lim_{k\to\infty} \sum_{n=0}^{k-1} \frac{z}{n+1/2} \left(\frac{1}{z-(n+1/2)} - \frac{1}{z+(n+1/2)} \right) \\ &= -\lim_{k\to\infty} \sum_{n=0}^{k-1} \frac{2z}{z^2-(n+1/2)^2}. \end{aligned}$$

Thus, except at the simple poles of $\tan \pi z$, we have

$$\pi \tan \pi z = \sum_{n=0}^{\infty} \frac{2z}{(n+1/2)^2 - z^2}, \quad z \in \mathbb{C} \setminus \{(n+1/2)\pi : n \in \mathbb{Z}\}.$$

Letting $z \to 0$ gives

$$\pi^2 \lim_{z\to 0} \frac{\tan \pi z}{\pi z} = \sum_{n=0}^{\infty} \frac{2}{(n+1/2)^2}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Choosing $z = \frac{\pi}{4}$ shows that

$$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{1}{4(2n+1)^2 - 1}.$$

One can also obtain the partial fraction decomposition of $\pi \tan \pi z$ just by using the identity $\tan w = \cot w - 2\cot(2w)$. •

11.2 Infinite Product of Complex Numbers

Let us start by introducing the definition of infinite products. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers. We want to discuss the convergence of products of the form $\prod_{k=1}^{\infty} z_k$. To do this, we define the partial product P_n (or n-th partial product) to be $P_n = \prod_{k=1}^{n} z_k$. The infinite product is said to converge if the sequence $\{P_n\}_{n=1}^{\infty}$ of partial products converges to a non-zero (finite) limit P as $n \to \infty$. We symbolize the convergence of the product by writing $P = \prod_{k=1}^{\infty} z_k$, where $P \neq 0$. If $\lim_{n\to\infty} P_n$ fails to exist or if $\lim_{n\to\infty} P_n = 0$, then we say that the infinite product diverges.

Clearly, some technicalities are involved if $z_k = 0$ for a finite number of k's. But we would like to allow the infinite product to be zero yet able to discuss the convergence of such products by imposing some conditions. Suppose that $z_k = 0$ for finitely many values of k and let $z_k \neq 0$ for $k > m$. Then, for $k > m$, we can write

$$P_n = (z_1 z_2 \cdots z_m)[z_{m+1} z_{m+2} \cdots z_n] := (z_1 z_2 \cdots z_m) P_{m,n}$$

where $P_{m,n} = z_{m+1} z_{m+2} \cdots z_n$. In this case, we say that the infinite product P_n converges to zero provided $P_{m,n}$ converges to a non-zero limit Q as $n \to \infty$. Indeed, if $\lim_{n\to\infty} P_{m,n} = Q \quad (Q \neq 0, \infty)$ then

$$\lim_{n\to\infty} P_n = (z_1 z_2 \cdots z_m) Q = 0.$$

If $Q = 0$ or $Q = \infty$ or if $\{P_{m,n}\}$ has no limit in $\mathbb{C}$ as $n \to \infty$, then the infinite product $\prod_{k=1}^{\infty} z_k$ is said to be divergent.

More generally, we say that the infinite product converges to zero if $z_k = 0$ for a countable number of k's and that $\prod_{k=1, z_k \neq 0}^{\infty} z_k$ converges according to the earlier definition. For instance, if

$$c_k = \begin{cases} 1 + \dfrac{1}{2^k} & \text{if } k = 2m \\ 0 & \text{if } k = 2m-1 \end{cases}, \quad m \in \mathbb{N},$$

then $\prod_{k=1}^{\infty} c_k$ converges to 0. For instance, the product $\prod_{k=1}^{\infty} \left(1 + (-1)^k\right)$ diverges because infinitely many factors are zero, and that

$$\prod_{\substack{k=1 \\ k \in 2\mathbb{N}}}^{\infty} \left(1 + (-1)^k\right) = \lim_{k\to\infty} 2^k = \infty.$$

Sequences of this type do not occur in practice and hence, no importance is given for such sequences in the discussion of the convergence of the corresponding products. In conclusion, we formulate

11.14. Definition. The infinite product $\prod_{k=1}^{\infty} z_k$ is said to converge iff

(i) $z_k = 0$ for "at most" finitely many values of k, $k \in \mathbb{N}$

(ii) $\prod_{k=m}^{n} z_k = z_m z_{m+1} \cdots z_n$ (when $z_k \neq 0$ for $k \geq m$) converges to a non-zero (finite) limit as $n \to \infty$.

11.15. Examples of convergence and divergence of products.

(i) Consider the infinite product $\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)$. Set $z_k = 1 + 1/k \neq 0$ for $k \in \mathbb{N}$. Then

$$P_n = \frac{2}{1}\frac{3}{2}\frac{4}{3} \cdots \frac{n}{n-1}\frac{n+1}{n} = n + 1 \to \infty \quad \text{as } n \to \infty$$

showing that the infinite product diverges.

(ii) The infinite product $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k}\right)$ has no zero factor but it does diverge because

$$P_n = \frac{1}{2}\frac{2}{3}\frac{3}{4} \cdots \frac{n-1}{n}\frac{n}{n+1} = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Note that for the infinite product to be convergent we need the limit to be a non-zero number, which is not the case here. Even if we allow the product from $k = 1$, the product would still diverge. This example shows that just the existence of

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \prod_{k=1}^{n} (1 + a_k)$$

does not guarantee the convergence of $\prod_{k=1}^{\infty}(1+a_k)$ even if $1+a_k \neq 0$ for each $k \in \mathbb{N}$. Another simple example may be given by the case when all a_k's are equal to $-1/2$ (or $-1/3$ or $-1/4$).

(iii) Consider $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)$. Writing

$$1 - \frac{1}{k^2} = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right) = \frac{k-1}{k} \cdot \frac{k+1}{k}$$

the n-th partial product P_n is given by

$$\begin{aligned} P_n &= \left(\frac{1}{2}\frac{3}{2}\right)\left(\frac{2}{3}\frac{4}{3}\right) \cdots \left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right) \\ &= \frac{1}{2}\left(\frac{3}{2}\frac{2}{3}\right)\left(\frac{4}{3}\frac{3}{4}\right) \cdots \left(\frac{n+1}{n}\frac{n}{n+1}\right)\frac{n+2}{n+1} \\ &= \frac{1}{2}\left(\frac{n+2}{n+1}\right) \to \frac{1}{2} \text{ as } n \to \infty. \end{aligned}$$

Thus, the infinite product converges to $1/2$. If we had started the product with $k = 1$ to ∞ (instead of $k = 2$ to ∞), then the product would have converged to zero because $\prod_{k=2}^{\infty} z_k$, $z_k = 1 - 1/k^2$, converges.

(iv) Consider $\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k-1}}{k}\right)$. It follows that

$$1+\frac{(-1)^{k-1}}{k}=\begin{cases} 1-\dfrac{1}{2m}=\dfrac{2m-1}{2m} & \text{if } k=2m \\ 1+\dfrac{1}{2m-1}=\dfrac{2m}{2m-1} & \text{if } k=2m-1 \end{cases}, \quad m\in\mathbb{N},$$

and so, the infinite product converges to 1 because

$$\begin{aligned} P_n &= \frac{2}{1}\,\frac{1}{2}\,\frac{4}{3}\,\frac{3}{4}\cdots\left(1+\frac{(-1)^{n-1}}{n}\right) \\ &= \begin{cases} 1 & \text{if } n=2m \\ 1+\dfrac{1}{n} & \text{if } n=2m-1 \end{cases}, \quad m\in\mathbb{N}. \end{aligned}$$

In discussing the "convergence tests", without loss of generality, we may only consider the infinite product $\prod_{k=1}^{\infty} z_k$ with $z_k\neq 0$, $k\in\mathbb{N}$. Suppose that the infinite product $\prod_{k=1}^{\infty} z_k$ converges. Then, there exists $P\neq 0$ such that

$$\lim_{n\to\infty} P_n = P = \lim_{n\to\infty} P_{n-1}.$$

Further, as $P_{n-1}\neq 0$, we have

$$z_n = P_n/P_{n-1}$$

and therefore by the "Quotient Theorem" for limit, $z_n\to 1$; i.e. $z_n-1\to 0$. In view of this observation, it is customary to write $z_n := 1+a_n$ so that $a_n\to 0$. Thus, we express the infinite product $\prod_{k=1}^{\infty} z_k$ as $\prod_{k=1}^{\infty}(1+a_k)$. Using this notation, we formulate the following simple result which gives a necessary condition for the convergence of an infinite product.

11.16. Proposition. *If the infinite product $\prod_{k=1}^{\infty}(1+a_k)$ converges, then $a_k\to 0$ as $k\to\infty$.*

Clearly, Proposition 11.16 implies the following: *if $a_k\not\to 0$ as $k\to\infty$, then the infinite product diverges.* For instance, each of the products

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^k}{1+i}\right) \quad\text{and}\quad \prod_{k=1}^{\infty}\left(1+\frac{(-1)^k}{3}\right)$$

diverges. The converse of this proposition is false as Examples 11.15(i) and (ii) illustrate. Thus, there exists a divergent infinite product $\prod_{k=1}^{\infty}(1+a_k)$ with $a_k\to 0$ as $k\to\infty$.

Also, we observe that the necessary condition for the convergence of the infinite product $\prod_{k=1}^{\infty}(1+a_k)$ is similar to the necessary condition for the convergence of $\sum_{k=1}^{\infty} a_k$. It is then natural to investigate the connection between series and products. Before we present this connection in various general forms, since no factor $1 + a_k$ is zero, it is natural to relate the convergence of the product $\prod_{k=1}^{\infty}(1 + a_k)$ with the convergence of the series $\sum_{k=1}^{\infty} \operatorname{Log}(1 + a_k)$, where $\operatorname{Log} z$ denotes the principal branch of the logarithm, $-\pi < \operatorname{Arg} z = \operatorname{Im} \operatorname{Log} z \leq \pi$.

11.17. Theorem. *Suppose that $z_k \neq 0$ for $k \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} \operatorname{Log} z_k$ converges iff the product $\prod_{k=1}^{\infty} z_k$ converges, in which case $\prod_{k=1}^{\infty} z_k = \exp(\sum_{k=1}^{\infty} \operatorname{Log} z_k)$. Equivalently, the series $\sum_{k=1}^{\infty} \operatorname{Log} z_k$ and the product $\prod_{k=1}^{\infty} z_k$ either converge or diverge together.*

Proof. Let $P_n = \prod_{k=1}^{n} z_k$ and $S_n = \sum_{k=1}^{n} \operatorname{Log} z_k$. Then, because $e^{z+w} = e^z e^w$ for $z, w \in \mathbb{C}$, we have

$$e^{S_n} = e^{\operatorname{Log} z_1} e^{\operatorname{Log} z_2} \cdots e^{\operatorname{Log} z_n} = z_1 z_2 \cdots z_n = P_n.$$

$\Longrightarrow$: As e^z is continuous on $\mathbb{C}$, we get

$$S_n \to S \Rightarrow P_n = e^{S_n} \to e^S \neq 0.$$

Hence, if the series $\sum_{k=1}^{\infty} \operatorname{Log} z_k$ converges to S then the product $\prod_{k=1}^{\infty} z_k$ converges to e^S.

$\Longleftarrow$: Conversely, suppose that $P_n \to P \neq 0$. For each $k \in \mathbb{N}$, we write

$$\operatorname{Log} z_k = \ln |z_k| + i \operatorname{Arg} z_k.$$

Without loss of generality, we may assume that $P \notin (-\infty, 0]$. For, if $P \in (-\infty, 0]$ then we may simply consider in place of $\{z_k\}$ a new sequence $\{w_k\}$ with $w_1 = -z_1$ and $w_k = z_k$ for $k \geq 2$. Then,

$$\prod_{k=1}^{\infty} w_k = -\prod_{k=1}^{\infty} z_k = -P \notin (-\infty, 0],$$

so that the two series $\sum_{k=1}^{\infty} \operatorname{Log} z_k$ and $\sum_{k=1}^{\infty} \operatorname{Log} w_k$ differing only in their first term, either converge or diverge together. As $P_n \to P \notin (-\infty, 0]$, we have $P_n \in \mathbb{C} \backslash (-\infty, 0]$ for large n and, since $\operatorname{Log} z$ is continuous at P, $\operatorname{Log} P_n \to \operatorname{Log} P$ as $n \to \infty$. Since $e^{S_n} = P_n$, we can write S_n as a logarithm of P_n. Indeed (see Theorem 3.103) we have

$$S_n = \operatorname{Log} P_n + 2\pi k_n i \tag{11.18}$$

so that $\operatorname{Log} z_{n+1} = S_{n+1} - S_n = \operatorname{Log} P_{n+1} - \operatorname{Log} P_n + 2\pi i(k_{n+1} - k_n)$, for some k_{n+1} and $k_n \in \mathbb{Z}$. Equating the imaginary parts on both sides of this equation gives

$$\operatorname{Arg} z_{n+1} = \operatorname{Arg} P_{n+1} - \operatorname{Arg} P_n + 2\pi(k_{n+1} - k_n). \tag{11.19}$$

Now, we observe the following

- as the product $\prod_{n=1}^{\infty} z_n$ converges, we have $z_n \to 1$ and, because $\operatorname{Arg} z$ is continuous at 1, we obtain that $\operatorname{Arg} z_n \to \operatorname{Arg} 1 = 0$ as $n \to \infty$.
- since $P_n \to P$ and $\operatorname{Log} z$ is continuous at P, we have $\operatorname{Log} P_n \to \operatorname{Log} P$ as $n \to \infty$.

Now, allowing $n \to \infty$ in (11.19), it follows that $k_{n+1} - k_n \to 0$ as $n \to \infty$, but then, since k_n is an integer, there exists an N such that

$$k_{n+1} = k_n = m \quad \text{for } n \geq N,$$

m being an integer, independent of n. By (11.18), we conclude that

$$S_n \to \operatorname{Log} P + 2\pi m i \quad \text{as } n \to \infty$$

for some integer m, and we complete the proof. ■

11.20. Theorem. *The series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely iff the series* $\sum_{k=1}^{\infty} \operatorname{Log}(1 + a_k)$ *converges absolutely.*

Proof. As $\lim_{z\to 0} z^{-1} \operatorname{Log}(1+z) = 1$, for $\epsilon = 1/2$ (in fact any ϵ with $0 < \epsilon < 1$ would work), it follows that

$$\left| \frac{\operatorname{Log}(1+z)}{z} - 1 \right| < \epsilon \ \text{ whenever } \ z \to 0.$$

The triangle inequality shows that

$$(11.21) \quad (1-\epsilon)|z| < |\operatorname{Log}(1+z)| < (1+\epsilon)|z| \ \text{ whenever } \ z \to 0.$$

Similarly, $\lim_{\substack{t\to 0 \\ 0<t\leq 1}} t^{-1} \operatorname{Log}(1+t) = 1$, we have

$$(1-\epsilon)t < \operatorname{Log}(1+t) < (1+\epsilon)t \ \text{ whenever } \ t \to 0.$$

Now, assume $a_k \to 0$ as $k \to \infty$. Consequently, we have

$$(1-\epsilon)|a_k| < |\operatorname{Log}(1+a_k)| < (1+\epsilon)|a_k| \ \text{ as } k \to \infty$$

and

$$(1-\epsilon)|a_k| < \operatorname{Log}(1+|a_k|)| < (1+\epsilon)|a_k| \ \text{ as } k \to \infty.$$

By the comparison test, for any complex sequence $\{a_k\}$ convergent to 0, the three series

$$\sum_{k=1}^{\infty} |\operatorname{Log}(1+a_k)|, \quad \sum_{k=1}^{\infty} \operatorname{Log}(1+|a_k|), \quad \sum_{k=1}^{\infty} |a_k|$$

either converge or diverge together. ■

In analogy with series we introduce the following notion of absolute convergence of a product. *The infinite product* $\prod_{k=1}^{\infty}(1 + a_k)$ *is said to converge absolutely iff the product* $\prod_{k=1}^{\infty}(1 + |a_k|)$ *converges.* For instance, $\prod_{k=1}^{\infty} \exp(i^k/k^2)$ converges, since

$$\sum_{k=1}^{\infty} \left| \operatorname{Log}\left(\exp(i^k/k^2)\right) \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Our next result, which is a reformulation of Theorem 11.20, gives the connection between the absolute convergence of products and the absolute convergence of series. Here we present an alternate proof of this result.

11.22. Corollary. *Suppose that* $a_k \geq 0$ *for all* $k \in \mathbb{N}$. *Then the product* $\prod_{k=1}^{\infty}(1 + a_k)$ *converges iff the series* $\sum_{k=1}^{\infty} a_k$ *converges iff the series* $\sum_{k=1}^{\infty} \operatorname{Log}(1 + a_k)$ *converges.*

Proof. As $1 + x \leq e^x$ for all $x \geq 0$,

$$S_n = \sum_{k=1}^{n} a_k \leq P_n = \prod_{k=1}^{n}(1 + a_k) \leq \exp\left(\sum_{k=1}^{n} a_k\right) = e^{S_n}.$$

First we observe that $\{P_n\}_{n\geq 1}$ is an increasing sequence, since $a_k \geq 0$ for all $k \in \mathbb{N}$.

If $P_n \to P \neq 0$, as P_n is increasing, then it is bounded above by P. This observation shows that $\{S_n\}_{n\geq 1}$ is a sequence (increasing) bounded above by P. Therefore, by the monotonic-bounded principle, $\{S_n\}_{n\geq 1}$ converges, i.e. $\sum_{n=1}^{\infty} a_k$ converges. Conversely, if S_n converges to S, as $\{S_n\}_{n\geq 1}$ is an increasing sequence bounded above by S, the increasing sequence $\{P_n\}_{n\geq 1}$ is bounded above by e^S (as $e^{S_n} \to e^S$ by the continuity of the exponential function) and hence converges to a non-zero limit. ∎

The hypothesis that $a_k \geq 0$ in Corollary 11.22 is crucial as the Example 11.25 shows. If $a_k = 1/k$, then the corresponding product $\prod_{k=2}^{\infty}(1 - 1/k)$ diverges and $\sum_{k=2}^{\infty} \frac{1}{k} = \infty$. Moreover, if $a_k = O(1/k^p)$ as $k \to \infty$ then the series $\sum_{k=1}^{\infty} a_k$ is convergent if $p > 1$ and divergent if $p \leq 1$. From this, we can easily see that $\prod_{k=1}^{\infty}(1 + 1/k^p)$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Here is a simple result which is almost our *"standard convergence test"* for infinite products.

11.23. Corollary. *Let* $a_k \in \mathbb{C}$ *with* $1 + a_k \neq 0$ *for all* $k \in \mathbb{N}$. *If* $\sum_{k=1}^{\infty} |a_k|$ *converges, then* $\prod_{k=1}^{\infty}(1 + a_k)$ *converges.*

Proof. If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} \operatorname{Log}(1 + a_k)$ converges absolutely (and hence converges). By Theorem 11.17, the product $\prod_{k=1}^{\infty}(1 + a_k)$ converges. ∎

Corollary 11.22 shows that if $\prod_{k=1}^{\infty}(1+|a_k|)$ converges then $\sum_{k=1}^{\infty}|a_k|$ converges which, in turn, by Corollary 11.23, implies that $\prod_{k=1}^{\infty}(1+a_k)$ converges. Thus, as in the case of series where *"absolute convergence implies convergence"*, we also have

11.24. Theorem. *If an infinite product converges absolutely, then it converges but not conversely.*

11.25. Example. Next we show by an example that the convergence of the product $\prod_{k=1}^{\infty}(1+a_k)$ is neither sufficient nor necessary for the convergence of the series $\sum_{k=1}^{\infty}a_k$ unless $a_k\geq 0$. To do this, we define

$$a_{2k}=\frac{1}{\sqrt{k+1}}\ \text{ and }\ a_{2k-1}=-\frac{1}{\sqrt{k+1}}\ \text{ for } k\in\mathbb{N}.$$

Then the product $\prod_{k=1}^{\infty}(1+a_k)$ will be of the form

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\cdots$$

so that $P_{2n}=\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)$. Therefore, P_{2n} diverges. On the other hand, by the alternating series test, the corresponding series

$$\sum_{k=1}^{\infty}a_k=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$$

converges (but not absolutely). In fact

$$S_n=\sum_{k=1}^{n}a_k=\begin{cases}0 & \text{if } n=2m\\ -\dfrac{1}{\sqrt{m+1}} & \text{if } n=2m-1\end{cases},\quad m\in\mathbb{N}$$

and therefore, $S_n\to 0$ as $n\to\infty$. Also,

$$\sum_{k=1}^{2n}|a_k|=2\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}}\ \text{ and }\ \sum_{k=1}^{2n+1}|a_k|=2\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}}+\frac{1}{\sqrt{n+2}}$$

so that the series $\sum_{k=1}^{\infty}|a_k|$ is divergent.

Next, we define

$$a_{2k}=-\frac{1}{\sqrt{k+1}},\quad a_{2k-1}=\frac{1}{\sqrt{k+1}}+\frac{1}{k+1}\quad (k\in\mathbb{N})$$

and show that the series $\sum_{k=1}^{\infty}a_k$ diverges but the product $\prod_{k=1}^{\infty}(1+a_k)$ converges (but not absolutely). Now

$$S_{2k}\ =\ a_1+a_2+\cdots+a_{2k}$$

$$\begin{aligned} &= \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) - \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{3}} + \frac{1}{3}\right) - \frac{1}{\sqrt{3}} + \cdots \\ &\quad + \left(\frac{1}{\sqrt{k+1}} + \frac{1}{k+1}\right) - \frac{1}{\sqrt{k+1}} \\ &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} \end{aligned}$$

and

$$S_{2k-1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} + \frac{1}{\sqrt{k+1}} \geq S_{2k}$$

so that $\sum_{k=1}^{\infty} a_k$ is divergent. On the other hand, we find that

$$\begin{aligned} (1 + a_{2k-1})(1 + a_{2k}) &= \left(1 + \frac{1}{\sqrt{k+1}} + \frac{1}{k+1}\right)\left(1 - \frac{1}{\sqrt{k+1}}\right) \\ &= 1 - \frac{1}{(k+1)^{3/2}} \end{aligned}$$

so that $P_{2n} = \prod_{k=1}^{n}\left(1 - \frac{1}{(k+1)^{3/2}}\right) \to P \neq 0$ as $n \to \infty$ and

$$P_{2n+1} = P_{2n}\left(1 + \frac{1}{\sqrt{n+2}} + \frac{1}{n+2}\right) \to P \quad \text{as} \quad n \to \infty.$$

Thus, the product converges. By Corollary 11.22, the product does not converge absolutely since $\sum_{k=0}^{\infty} |a_k|$ diverges (in fact $\sum_{k=0}^{\infty} a_k$ diverges). •

11.3 Infinite Products of Analytic Functions

Until now we have discussed infinite products of complex numbers. Just as transferring from series of complex numbers to series of functions, we can discuss convergence of infinite products whose factors are functions of z rather than just complex numbers. Then the following problem occurs.

11.26. Problem. Given a sequence $\{f_k(z)\}_{k\geq 1}$ of functions defined on some set $\Omega \subseteq \mathbb{C}$, determine whether the infinite product

$$\prod_{k=1}^{\infty}(1 + f_k(z)) \tag{11.27}$$

(i) converges for all $z \in \Omega$

(ii) converges absolutely for all $z \in \Omega$

(iii) diverges for $z \in \Omega$

(iv) converges for a particular $z \in \Omega$.

We say that the infinite product (11.27) converges in Ω if, for each $a \in \Omega$,

$$\lim_{n\to\infty} P_n(a) = \lim_{n\to\infty} \prod_{k=1}^{n} (1 + f_k(a))$$

exists and is non-zero. The infinite product (11.27) is said to converge uniformly to a function $P(z)$ in Ω if the sequence $\{P_n(z)\}$ of partial products, defined by $P_n(z) = \prod_{k=1}^{n}(1+f_k(z))$, is uniformly convergent to the function $P(z)$ in Ω, with $P(z) \neq 0$ in Ω.

11.28. Example. Consider the product $\prod_{k=0}^{\infty}(1 + z^{2^k})$. We have

$$\sum_{k=0}^{\infty} \left| z^{2^k} \right| \le \sum_{k=0}^{\infty} |z^k| = \frac{1}{1-|z|} \quad \text{for } |z| < 1,$$

so that the series $\sum_{k=0}^{\infty} z^{2^k}$ converges absolutely for $|z| < 1$. It follows that the product converges absolutely (and hence converges by Theorem 11.24) for $|z| < 1$. Note also that $1 + z^{2^k} \neq 0$ for $|z| < 1$. Further,

$$P_0(z) = 1 + z, \;\; P_1(z) = (1 + z)(1 + z^2) = \sum_{k=0}^{2^2-1} z^k$$

and, by induction, it can be shown that for $|z| < 1$

$$P_n(z) = \sum_{k=0}^{2^{n+1}-1} z^k = \frac{1 - z^{2^{n+1}}}{1-z} \to \frac{1}{1-z} \quad \text{as } n \to \infty$$

so that $\prod_{k=0}^{\infty}(1 + z^{2^k}) = 1/(1 - z)$ for $|z| < 1$. ●

11.29. Lemma. *Define* $P_n = \prod_{k=1}^{n}(1+a_k)$ *and* $\tilde{P}_n = \prod_{k=1}^{n}(1+|a_k|)$. *Then*

(i) $|P_n| \le \tilde{P}_n$
(ii) $|P_n - 1| \le \tilde{P}_n - 1$
(iii) $|P_n - P_m| \le \tilde{P}_n - \tilde{P}_m$ *for* $n > m$.

Proof. Part (i) is clearly a consequence of the triangle inequality. For the proof of (ii), we write

$$\begin{aligned} P_n - 1 &= \sum_{1\le i_j \le n,\, 1\le k\le n} a_{i_1} a_{i_2} \cdots a_{i_k} \\ &= \text{monomial terms consisting of products of } a_k\text{'s} \end{aligned}$$

so that $|P_n - 1| \le \sum_{1\le i_j \le n,\, 1\le k\le n} |a_{i_1}|\,|a_{i_2}| \cdots |a_{i_k}| = \tilde{P}_n - 1.$

(iii) For $n > m$, we have

$$\begin{aligned} |P_n - P_m| &= |P_m \prod_{k=m+1}^{n} (1+a_k) - P_m| \\ &\leq |P_m| \left(\prod_{k=m+1}^{n} (1+|a_k|) - 1 \right), \quad \text{by (ii)}, \\ &\leq \prod_{k=1}^{m} (1+|a_k|) \left(\prod_{k=m+1}^{n} (1+|a_k|) - 1 \right), \quad \text{by (i)}, \\ &= \tilde{P}_n - \tilde{P}_m \end{aligned}$$

and we are done. ∎

11.30. Theorem. *Let $\{f_k(z)\}$ be a sequence of analytic functions in an open set $\Omega \subseteq \mathbb{C}$ such that $\sum_{k=1}^{\infty} |f_k(z)|$ converges uniformly on compacta (i.e. on compact subsets of Ω). Then the sequence of partial products $P_n(z) = \prod_{k=1}^{n}(1+f_k(z))$ converges uniformly on compacta to an analytic function $f(z)$ in Ω,*

$$f(z) = \prod_{k=1}^{\infty} (1+f_k(z)).$$

The order of the factors does not alter the limit function $f(z)$.[14] *Furthermore, $f(a) = 0$ at a point $a \in \Omega$ iff the factor $1 + f_k(z)$ vanishes at a for some k. The order of the zero of f at a is the sum of the multiplicities of the zeros at those factors having a zero at a.*

Proof. By virtue of Corollary 11.22, $\prod_{k=1}^{\infty}(1+f_k(z))$ converges absolutely (therefore converges) for each point in Ω and hence the product represents a well defined function. In view of this, it suffices to show that the sequence $\{P_n(z)\}$ forms a uniformly Cauchy sequence on every compact subset of Ω. To do this, we fix a compact subset $K \subset \Omega$. By hypothesis, $\sum_{k=1}^{\infty} |f_k(z)|$ converges uniformly on K and therefore, the partial sums

$$S_n(z) = \sum_{k=1}^{n} |f_k(z)|$$

of the series are uniformly bounded on K by a constant c. Since $1+x \leq e^x$ for $x \geq 0$, it follows that

$$|P_n(z)| \leq \tilde{P}_n(z) = \prod_{k=1}^{n} (1+|f_k(z)|) \leq \exp(S_n(z)) \leq e^c \quad \text{for each } n.$$

[14]We say that an absolutely convergent product (series) is unconditionally convergent iff the order of factors (the terms of the summation) does not alter the limit function.

Let $0 < \epsilon < \ln 2$. Choose N so large that if $n > m \geq N = N(\epsilon)$, then the Cauchy-criterion for convergence gives

$$S_n - S_m = \sum_{k=m+1}^{n} |f_k(z)| < \epsilon \quad \text{for} \quad z \in K.$$

For $n > m \geq N$ and $z \in K$, by Lemma 11.29(iii), we have

$$\begin{aligned} |P_n(z) - P_m(z)| &\leq |P_m(z)| \left(\prod_{k=m+1}^{n} (1 + |f_k(z)|) - 1 \right) \\ &\leq |P_m(z)|(e^{\epsilon} - 1) \end{aligned}$$

(Note that for $0 < \epsilon < \ln 2$, we have $e^{\epsilon} < e^{\ln 2} = 2$, i.e. $0 < e^{\epsilon} - 1 < 1$). Since $e^{\epsilon} - 1 \to 0^+$ as $\epsilon \to 0^+$ and $|P_m(z)| \leq \tilde{P}_m(z) \leq e^c$, it follows that the sequence of partial products $\{P_n(z)\}$ forms a uniformly Cauchy sequence and hence, converges uniformly on K. Letting $n \to \infty$, we obtain

$$|f(z) - P_m(z)| \leq |P_m(z)|(e^{\epsilon} - 1) \tag{11.31}$$

so that

$$-|f(z)| + |P_m(z)| \leq |f(z) - P_m(z)| \leq |P_m(z)|(e^{\epsilon} - 1).$$

If $0 < \epsilon < \ln 2$ (i.e. $2 - e^{\epsilon} > 0$), then, for any $m \geq N(\epsilon)$, and for $z \in K$

$$|f(z)| \geq |P_m(z)| - |P_m(z)|(e^{\epsilon} - 1) \geq |P_m(z)|(2 - e^{\epsilon}). \tag{11.32}$$

Therefore, if $P_m(z) = 0$ at $z = a$, that is, if one of the factors $1 + f_k(z)$ vanishes at $z = a$ when $1 \leq k \leq m$, then, by (11.31), we have $f(a) = 0$. Conversely if $f(z) = 0$ at some point $z = a$, then, by (11.32), $P_m(a) = 0$ and therefore, $1 + f_k(a) = 0$ for some k.

As in the case of the absolute convergence of the series, it is easy to see that the order of the factors in an infinite product is immaterial whenever it converges absolutely. Indeed if $\prod_{k=1}^{\infty}(1 + f_k(z))$ converges absolutely then, for every permutation $\sigma = (\sigma(1), \sigma(2), \ldots)$ of the list of positive integers $(1, 2, \ldots)$, Theorem 11.17 shows that

$$\begin{aligned} \prod_{k=1}^{\infty}(1 + f_k(z)) &= \exp\left(\sum_{k=1}^{\infty} \operatorname{Log}(1 + f_k(z)) \right) \\ &= \exp\left(\sum_{k=1}^{\infty} \operatorname{Log}(1 + f_{\sigma(k)}(z)) \right) \\ &= \prod_{k=1}^{\infty}(1 + f_{\sigma(k)}(z)). \end{aligned}$$

Thus, the sum of an absolutely convergent product is not affected by rearranging the order of the terms of the product. ∎

Some quick examples follow. For $|z| \leq \delta < 1$, we have

$$\sum_{k=1}^{\infty} |z|^k \leq \sum_{k=1}^{\infty} \delta^k = \frac{\delta}{1-\delta}$$

showing that $\sum_{k=1}^{\infty} |z|^k$ converges uniformly on every compact subset of Δ. Consequently, each of the products $\prod_{k=1}^{\infty}(1+z^k)$ and $\prod_{k=1}^{\infty}(1-z^k)$ converges uniformly on every compact subset of Δ and hence each of them represents an analytic function in the unit disk Δ.

It follows that if $\sum_{k=1}^{\infty} |a_k|$ converges (for instance for $a_k = k^{-p}$ with $p > 1$), then each of the products $\prod_{k=1}^{\infty}(1+a_k z)$ and $\prod_{k=1}^{\infty}(1+a_k z^j)$ ($j \in \mathbb{N}$ fixed) converges uniformly on every compact subsets of $\mathbb{C}$. Consequently, each of these products represents an entire function.

11.33. Example. Let $f_k(z) = e^{z^k} - 1$, $k \in \mathbb{N}$. Then $\{f_k(z)\}_{k\geq 1}$ is a sequence of entire functions and

$$\mathrm{Log}\,(1 + f_k(z)) = z^k \ \text{ for all } z.$$

Also, for $|z| \leq r < 1$, the series $\sum_{k=1}^{\infty} \mathrm{Log}\,(1 + f_k(z))$ converges uniformly to $z/(1-z)$ for $|z| \leq r$ $(r < 1)$. Therefore, $\prod_{k=1}^{\infty} e^{z^k}$ converges uniformly to $\exp(z/(1-z))$ for every compact subset of $|z| < 1$. Note that this can be verified directly. •

11.4 Factorization of Entire Functions

Every polynomial $p(z)$ of degree $n \geq 1$ can be factored as a product of linear factors. More precisely, if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ then $p(z)$ can be written in the form $p(z) = \prod_{k=1}^{n}(z - \alpha_k)$, or equivalently

$$p(z) = z^m \prod_{k=1}^{n-m} (z - \alpha_k)$$

whenever $p(z)$ has a zero of order $m \geq 0$ at the origin with $\alpha_k \neq 0$ for $k = 1, 2, \ldots, n-m$, counting multiplicities (take $m = 0$ when $p(z)$ has no zero at the origin). Then this expression takes the form

$$p(z) = \alpha z^m \prod_{k=1}^{n-m} \left(1 - \frac{z}{\alpha_k}\right)$$

with $\alpha = \alpha_1 \cdots \alpha_{n-m} \neq 0$. Note that if $\alpha_k = 0$ for all k, then $p(z) = z^n$ which is a trivial case. Since each polynomial of degree n has n roots and can be written as a product of the above form, it is therefore natural to know its generalization which we encounter for certain entire transcendental functions. Thus, our first task in this section is to undertake a solution to the following problem.

11.34. Problem. Find an infinite product representation of transcendental entire functions admitting an infinite number of zeros.

We recall that there are analytic functions having an infinite number of zeros (e.g. $z^{-1}\sin z$, $\cos z$) and therefore we wish to know whether each such function can be represented as an infinite product at least in some cases. So, the first thing that we should know is about the convergence of such a product representation.

We have already shown in Theorem 4.40 that an entire function f has no zeros iff $f(z) = e^{h(z)}$ for some entire function h. For example, e^z, e^{z^2}, e^{z+z^2}, $e^{\sin z}$ are all entire functions with no zeros in $\mathbb{C}$. So we need to deal with the following two sets of entire functions:

- entire functions which have a finite number of zeros in $\mathbb{C}$
- entire functions which have an infinite number of zeros in $\mathbb{C}$.

As a first step, we have the following result which concerns the first case while the second case will be done in the form of the Weierstrass factorization theorem.

11.35. Theorem. *Let f be an entire function with n zeros, say, $a_1, a_2, \ldots, a_n$ (multiple zeros being repeated according to their orders). Then there exists an entire function h such that $f(z) = p(z)\exp(h(z))$ where $p(z)$ is a polynomial of degree n with the same zeros as $f(z)$ (with the same multiplicities).*

Proof. Define

$$F(z) = \begin{cases} \dfrac{f(z)}{\prod_{k=1}^{n}(z-a_k)} & \text{for } z \neq a_k \\ \lim\limits_{z\to a_k} \dfrac{f(z)}{\prod_{k=1}^{n}(z-a_k)} & \text{for } z = a_k \end{cases}, \quad k \in \{1, 2, \ldots, n\}.$$

Then, F is a zero free entire function. By Theorem 4.40, there exists an entire function $h(z)$ such that $F(z) = e^{h(z)}$; i.e. $f(z) = e^{h(z)} \prod_{k=1}^{n}(z-a_k)$, as desired. ∎

11.36. Remark. Observe that if the function f considered in Theorem 11.35 is a polynomial, then $h(z)$ would be a constant function. Further, by Theorem 11.35, it follows that if f is an entire function with $(m+1)$

distinct zeros, say, $0 = a_0, a_1, a_2, \dots, a_m$ ($a_k \neq 0$ for $k = 1, 2, \dots, m$) of order $p_0, p_1, p_2, \dots, p_m$ ($p_k \geq 0$), then we can express $f(z)$ as

$$f(z) = z^{p_0} e^{h(z)} \prod_{k=1}^{m} \left(1 - \frac{z}{a_k}\right)^{p_k} \tag{11.37}$$

for some entire function $h(z)$. That is, $f(z)$ can be expressed as the product of a polynomial and a zero free entire function. The zeros of the polynomial are exactly the same as the zeros of $f(z)$. •

Our next step is to consider entire functions which have infinitely many zeros in $\mathbb{C}$. In principle, allowing $m \to \infty$ in (11.37), we get a statement about the infinite product provided the corresponding right hand side of (11.37) makes sense. Clearly, such a representation is valid if the infinite product of functions converges uniformly on every compact subset. Let us now start our discussion by raising some simple questions. *Are there entire functions with a limit point of a set of zeros in* $\mathbb{C}$? Notice that if f is entire with zeros at the points a_k, with $a_k \to a \in \mathbb{C}$ as $k \to \infty$, then, by the uniqueness theorem, f is identically zero in $\mathbb{C}$. Next we ask: *Is there a non-trivial entire function having a zero at* ∞? Again the answer is no, because, otherwise this would imply that the function is analytic at ∞, and so is a constant (since every analytic function in the extended complex plane is constant, by Liouville's theorem)-hence, identically zero in $\mathbb{C}_\infty$. Consequently, the set of zeros of an entire function which has infinitely many zeros in $\mathbb{C}$ must have ∞ as its only limit point. For example, each of the familiar entire functions $\cos z$, $\sin z$ and $e^z - 1$, has infinitely many zeros in every neighborhood of ∞, and the limit point of these zeros, in each case, is ∞.

Now, we are in a position to discuss in detail those entire functions which have infinitely many zeros in $\mathbb{C}$. Suppose that $\{a_k\}_{k \geq 1}$ is a sequence of complex numbers such that $a_k \to \infty$ as $k \to \infty$. We wish to construct an entire function with zeros precisely at each a_k (with desired multiplicity) and nowhere else. To do this, let us make an attempt to consider the product of the form

$$\prod_{k=1}^{\infty} (z - a_k).$$

Does this product then converge? We note that, as $a_k \to \infty$ as $k \to \infty$, for no fixed $z \in \mathbb{C}$ would we get $z - a_k \to 1$ as $k \to \infty$, which is a required necessary condition for the convergence of the product. Thus, the product of the above form diverges. In view of this observation, it is natural to modify the present form to a new form

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \tag{11.38}$$

assuming for the moment that $a_k \neq 0$ for each $k \in \mathbb{N}$. Although this new form is similar to the polynomial case above, the product unfortunately may diverge. For instance, if $a_k = k$ then the corresponding product $\prod_{k=1}^{\infty}(1 - z/k)$ does not represent an entire function (e.g. set $z = -1$ and recall that $\prod_{k=1}^{\infty}(1 + 1/k)$ is divergent). On the other hand, if the series $\sum_{k=1}^{\infty} 1/|a_k|$ converges (e.g. $a_k = (k+1)\ln(1+k)$, $(k+1)(\ln(1+k))^2$, k^2), then the series

$$\sum_{k=1}^{\infty}\left|\frac{z}{a_k}\right| = |z|\sum_{k=1}^{\infty}\frac{1}{|a_k|}$$

converges uniformly on every compact subset of $\mathbb{C}$ and so, by Theorem 11.30, the product given by (11.38) represents an entire function $f(z)$ with zeros at the same points a_k ($k \in \mathbb{N}$) with the same multiplicities as $f(z)$, and at no other points. Therefore, if the product (11.38) is not convergent then we must somehow modify the product form to ensure the convergence of the product which will enable us to construct an entire function which vanishes at a_k, $k \in \mathbb{N}$, and at no other points. This is exactly what Weierstrass demonstrated by means of '*primary factors*' which ensure the convergence of the infinite product.

To carry out further discussion on this topic, we let $E_0(z) = 1-z$. Then $0 \notin E_0(\Delta)$. Moreover, for $z \in \Delta$,

$$\int_0^z \frac{E_0'(\zeta)}{E_0(\zeta)}\,d\zeta = -\int_0^z \frac{1}{1-\zeta}\,d\zeta = -\sum_{k=1}^{\infty}\frac{z^k}{k} = \operatorname{Log}(1-z)$$

and exponentiating the last equality gives

$$1 - z = \exp\left(-\sum_{k=1}^{\infty}\frac{z^k}{k}\right), \quad \text{i.e.} \quad (1-z)\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\right) = 1.$$

This suggests to define

$$E_p(z) = \begin{cases} 1 - z & \text{if } p = 0 \\ (1-z)\exp\left(\displaystyle\sum_{k=1}^{p}\frac{z^k}{k}\right) & \text{if } p \in \mathbb{N}. \end{cases}$$

We call $E_p(z)$, the Weierstrass primary factors. We note that

$$\lim_{p\to\infty} E_p(z) = (1-z)\exp(-\operatorname{Log}(1-z)) = 1$$

and we have the following result which serves to estimate the error term in the above approximation.

11.39. Lemma. *For $p \in \mathbb{N} \cup \{0\}$, we have $|1 - E_p(z)| \leq |z|^{p+1}$ for all $|z| \leq 1$.*

Proof. For $p = 0$, the inequality is trivial. For $p \geq 1$, each E_p is analytic in $\mathbb{C}$ and is in fact an entire transcendental function admitting $z = 1$ as its only zero. Now, for $p \geq 1$, $E_p(0) = 1$ and a simple calculation gives

$$\begin{aligned} E_p'(z) &= \left(-1 + (1-z)\sum_{k=1}^{p} z^{k-1}\right)\exp\left(\sum_{k=1}^{p}\frac{z^k}{k}\right) \\ &= -z^p \exp\left(\sum_{k=1}^{p}\frac{z^k}{k}\right) \\ &= -z^p \sum_{k=0}^{\infty} a_k z^k \quad \text{(say)}. \end{aligned}$$

It follows that $E_p'(z)$ has a zero of order p at $z = 0$, $a_0 = 1$ and $a_k > 0$ for all $k \geq 1$ (using Cauchy's rule on multiplication of series). Moreover,

$$E_p(z) - 1 = \int_0^z E_p'(\zeta)\,d\zeta = -\sum_{k=0}^{\infty} a_k \frac{z^{k+p+1}}{k+p+1}$$

which, for $z = 1$, gives $1 = \sum_{k=0}^{\infty} \frac{a_k}{k+p+1}$. Using this, one obtains

$$|E_p(z) - 1| \leq \sum_{k=0}^{\infty} a_k \frac{|z|^{k+p+1}}{k+p+1} \leq |z|^{p+1} \sum_{k=0}^{\infty} \frac{a_k}{k+p+1} = |z|^{p+1}$$

and we are done. ∎

With the help of $E_p(z)$, we are now prepared to construct entire functions with prescribed zeros. We consider an arbitrary sequence of non-zero complex numbers $\{a_n\}_{n\geq 1}$ such that $|a_n| \to \infty$ as $n \to \infty$ and show the existence of a sequence of nonnegative integers $p_n \geq 0$ such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \tag{11.40}$$

for each $r > 0$. Such a sequence $\{p_n\}_{n\geq 1}$ exists, for example, any p_n with $p_n \geq n - 1$ will always work (implying the condition that is necessary for the uniform convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{z}{|a_n|}\right)^{p_n+1}$$

in every disk $|z| \leq r$). This fact is clear because, if $|a_n| \to \infty$ as $n \to \infty$ then for each given $R > 0$ only a finite number of a_n's lies in $|z| < R$. Further, as $|a_n| \to \infty$, for the given $R = 2r > 0$, there exists an $N = N(r)$ such that $|a_n| > 2r$ for $n \geq N$. (In particular, if $p_n \geq n - 1$ then

$$\left(\frac{r}{|a_n|}\right)^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{n} < \frac{1}{2^n} \quad \text{for} \quad n \geq N$$

so that for each $r > 0$

$$\sum_{n=N}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \sum_{n=N}^{\infty} \frac{1}{2^n} = \frac{1}{2^{N-1}} < \infty\,).$$

Note that for $|z| \leq R = 2r$,

$$\left|1 - \frac{z}{a_n}\right| \geq 1 - \left|\frac{z}{a_n}\right| \geq 1 - \frac{R}{|a_n|} > 0 \quad \text{for } n \geq N$$

and therefore, in the disk $|z| \leq R$, only a finite number of factors $(1 - z/a_n)$ vanish. In particular, for all $n \geq N = N(r)$ and $|z| \leq r$, Lemma 11.39 yields that

$$\left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \leq \left(\frac{|z|}{|a_n|}\right)^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}$$

which, by (11.40), assures that the series $\sum_{n=1}^{\infty} |1 - E_{p_n}(z/a_n)|$ converges uniformly for $|z| \leq r$ (see Weierstrass M-test for the uniform convergence of the series for compact subset of $\mathbb{C}$). Since r is arbitrary, this series converges uniformly on every compact subsets of $\mathbb{C}$. Thus, by Theorem 11.30, the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) = \prod_{n=1}^{\infty} \left[1 + \left\{E_{p_n}\left(\frac{z}{a_n}\right) - 1\right\}\right]$$

converges absolutely and uniformly on every compact subset of $\mathbb{C}$. Letting $P_n(z)$ denote the n-th partial product for $P(z)$, we have for $|z| < r$

$$P_n(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{P_n(\zeta)}{\zeta - z}\, d\zeta,$$

by the Cauchy integral formula. Since $P_n \to P$ uniformly on compact subsets of $\mathbb{C}$ and each P_n is entire, P is entire (e.g. see Theorem 12.4 and recall that analyticity is preserved under uniform limits). Therefore, the infinite product defines an entire function. Moreover, $P(z)$ has zeros only at the points a_n with prescribed multiplicities. Indeed, by Theorem 11.30, the zeros of $P(z)$ occur exactly at the points z, where $E_{p_n}(z/a_n) = 0$. Thus, we need $E_{p_n}(z/a_n) = 0$ for some n. From the definition of $E_p(z)$, we know that this happens exactly at $z/a_n = 1$ so that the zeros of P are the points $\{a_n : n \in \mathbb{N}\}$. In conclusion, the desired entire function which vanishes at a_n ($n \in \mathbb{N}$), and at no other points is given by

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

and therefore, we have the following result due to Weierstrass.

11.41. Theorem. *Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero complex numbers (not necessarily distinct) such that $|a_n| \to \infty$ as $n \to \infty$. Suppose that there exists a sequence of non-negative integers $\{p_n\}$ such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty$$

for every $r > 0$ (Recall that any $p_n \geq n-1$ will always work). Then the infinite product $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ represents an entire function $P(z)$ and has zeros at each point a_n and not elsewhere. If a zero a_j occurs m_j times, then P has a zero of order m_j at a_j.

First, we should make it a point that it is possible that the series $\sum_{n=1}^{\infty} 1/|a_n|$ may diverge but $\sum_{n=1}^{\infty} 1/|a_n|^2$ converges. Moreover, there are sequences $\{a_n\}$ (e.g. $a_n = \ln(n+1)$) such that $|a_n| \to \infty$ and yet $\sum_{n=1}^{\infty} 1/|a_n|^p$ diverges for each p. Secondly, the Weierstrass factorization theorem actually *constructs* entire functions with prescribed zeros. In this context, the Weierstrass factorization theorem may be considered to be "nicer than" the Riemann mapping theorem which demonstrates *existence* of conformal maps between certain simply connected domains.

11.42. Corollary. *Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero complex numbers such that $\sum_{n=1}^{\infty} |a_n|^{-2} < \infty$, and $m \in \mathbb{N}$. Then the product*

$$z^m \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right) = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

defines an entire function, and it has zeros precisely at a_n ($n \in \mathbb{N}$) with prescribed multiplicities and has a zero of order m at $z = 0$, but has no other zeros.

Theorem 11.41 is often stated in the following simplified form.

11.43. Theorem. (Weierstrass Factorization Theorem) *Let f be an entire function satisfying the following conditions:*

(i) *f has a zero of order m at $z = 0$ (by convention take $m = 0$ if $f(0) \neq 0$)*
(ii) *the remaining zeros of f are $\{a_n\}_{n\geq 1}$ listed with multiplicities.*

Then there is an entire function h such that

$$f(z) = z^m e^{h(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \tag{11.44}$$

for some sequence of non-negative integers $\{p_n\}_{n\geq 1}$.

Proof. As mentioned already, we may construct an entire function $g(z)$ by defining

$$g(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Then, g and f have exactly the same zeros with the same multiplicities and therefore, the quotient $f(z)/g(z)$ is a nowhere vanishing entire function. Consequently (see Theorem 4.40), there exists an entire function $h(z)$ such that $f(z) = g(z)\exp(h(z))$ as desired. ∎

Note that the infinite product representation given by (11.44) is not unique since the sequence $\{p_n\}$ can be chosen in various ways. We know that $p_n = n-1$ always works but we do not know whether this is the only choice that might work. In Examples 11.45, we deal with two cases where $p_n = 0$ and $p_n = 1$, respectively. The formula (11.44) constitutes the Weierstrass infinite product representation of an entire function in terms of its zeros.

11.45. Examples. Suppose that we wish to construct an entire function f with simple zeros at the points $a_0 = 0$ and $a_n = n^p$ ($n \in \mathbb{N}$), where $p > 1$ is fixed. We may choose $p_n = 0$, since the series

$$\sum_{n=1}^{\infty} \left|\frac{z}{n^p}\right|^{p_n+1} = |z| \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for each z, and the desired entire function is then given by

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^p}\right).$$

Similarly, if we want to construct an entire function f with simple zeros at $a_0 = 0$ and at $a_n = n$, $n \in \mathbb{Z}$ (note that these are precisely the zeros of $\sin \pi z$). We may therefore choose $p_n = 1$, since the series

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\frac{z}{a_n}\right|^{p_n+1} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\frac{z}{a_n}\right|^{2} = 2|z|^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges for each z. Again, Weierstrass's theorem gives

$$\begin{aligned} f(z) &= z \prod_{n=1}^{\infty} E_1\left(1 - \frac{z}{n}\right) \prod_{n=1}^{\infty} E_1\left(1 + \frac{z}{n}\right) \\ &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \end{aligned}$$

which is an entire function with the desired properties. Note that the rearrangement of the factors in the last expression is justified by the absolute convergence of the infinite product. •

As in Examples 11.45, for certain sequences $\{a_n\}_{n\geq 1}$ of points, the non-negative integers in $\{p_n\}_{n\geq 1}$ can be chosen independent of n, say $p_n = p$ for all n and for some $p \geq 0$. In this situation, the product in Theorem 11.43 takes the form $\prod_{n=1}^{\infty} E_p(z/a_n)$ for some non-negative integer p, and the product converges uniformly on every compact subset of $\mathbb{C}$, see Section 11.8 for further discussion. The Weierstrass factorization theorem has the following important corollary.

11.46. Corollary. *Every meromorphic function in $\mathbb{C}$ can be represented as a quotient of two entire functions.*

Proof. Assume that f is analytic on $\mathbb{C}$ except for poles say at the points $a_1, a_2, \ldots$. Let g be an entire function with zeros precisely at a_n such that the order of the zero of g at a_n equals the order of the pole of f at a_n (such a function g exists by Theorem 11.41). Then, fg has only removable singularities at the points $a_1, a_2, \ldots$ and thus can be extended to an entire function G with no zeros in $\mathbb{C}$. Hence

$$G(z) = f(z)g(z) \quad \text{or} \quad f(z) = \frac{G(z)}{g(z)},$$

where g and G are entire. ■

We have the following interpolation result for analytic functions.

11.47. Corollary. *Suppose $\{a_n\}_{n\geq 1}$ is a sequence of distinct complex numbers having no finite limit point. Then for a given sequence $\{b_n\}_{n\geq 1}$ of complex numbers, there exists an entire function G such that $G(a_n) = b_n$ for every n.*

Proof. Assume that f is analytic on $\mathbb{C}$ except for simple poles at $z = a_n$ with $\text{Res}\,[f(z); a_n] = b_n/g'(a_n)$, where g is an entire function with simple zeros precisely at a_n. Note that such functions exist by Mittag-Leffler's theorem and Weierstrass's theorem (In case b_n is zero for some n, then assume that f is analytic at a_n). Then fg has only removable singularities at the points $a_1, a_2, \ldots$ and thus can be extended to an entire function G. Further, we have the expansions for g and f

$$g(z) = \sum_{k=1}^{\infty} \frac{g^{(k)}(a_n)}{k!}(z - a_n)^k \quad \text{and} \quad f(z) = \frac{b_n/g'(a_n)}{z - a_n} + f_1(z),$$

where $f_1(z)$ is analytic in some neighborhood of a_n. Finally,

$$G(a_n) = \lim_{z\to a_n} f(z)g(z) = b_n$$

and we complete the proof. ∎

11.48. Example. We know that $\sin \pi z$ has simple zeros at $n \in \mathbb{Z}$. Arrange the non-zero zeros of $\sin \pi z$ in a such a way that they form a sequence with non-decreasing moduli, and observe that the series $\sum_{n=1}^{\infty} 1/n^2$ converges whereas $\sum_{n=1}^{\infty} 1/n$ diverges. Therefore, it is convenient to choose the number p_n figuring in Theorem 11.43 equal to 1. Choosing $p_n = 1$, by Theorem 11.43, it follows that

$$\begin{aligned} \sin \pi z &= e^{h(z)} z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right) e^{z/n} \\ &= z e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \left(1 + \frac{z}{n}\right) e^{-z/n} \\ &= z e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \end{aligned} \tag{11.49}$$

where $h(z)$ is some entire function. Next we need to show that $h(z)$ is a constant such that $e^{h(z)} = \pi$. Once this is done, then we get the product representation for $\sin \pi z$ in the form

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{11.50}$$

Now, to find $h(z)$, we proceed as follows: Let

$$P_n(z) = e^{h(z)} z \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right). \tag{11.51}$$

Then we know that $P_n \to \sin \pi z$ (uniformly on discs) as $n \to \infty$ so that $P_n'(z) \to \pi \cos \pi z$ as $n \to \infty$. Thus, as $n \to \infty$,

$$\frac{P_n'(z)}{P_n(z)} \to \pi \cot \pi z \quad \text{for} \quad z \in \mathbb{C} \setminus \{m : m \in \mathbb{Z}\}.$$

Using the local uniform convergence of $\sum_{k=1}^{\infty} |z/k|^2$ in $\mathbb{C}$, we may form the logarithmic derivative on both sides of (11.51) and find that

$$\frac{P_n'(z)}{P_n(z)} = h'(z) + \frac{1}{z} + \sum_{k=1}^{n} \frac{2z}{z^2 - k^2} \to \pi \cot \pi z \quad \text{as } n \to \infty.$$

A comparison of this with (11.12) shows that $h'(z) = 0$, so $h(z)$ is a constant, say, c. Therefore, (11.49) becomes

$$\frac{\sin \pi z}{z} = e^c \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Since $\lim_{z\to 0}(\sin \pi z)/z = \pi$ and the right hand side approaches e^c as $z \to 0$, we have $e^c = \pi$ and we obtain the desired formula (11.50). ●

11.52. Example. Using Example 11.11, we can provide an alternate proof of (11.50). To do this, we start by observing that the series $\sum_{k=1}^{\infty} z^k/k^2$ converges uniformly on every compact subset of $\mathbb{C}$. Thus,

$$g(z) = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$$

represents an entire function. Observe that for $x \in (0,1)$, $g(x) > 0$ and

$$\ln g(x) = \ln x + \sum_{k=1}^{\infty} \ln\left(1 - \frac{x^2}{k^2}\right),$$

by the uniform convergence of the derived series. Finally, for $x \in (0,1)$,

$$\begin{aligned} \frac{d}{dx}(\ln g(x)) &= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x}{x^2 - k^2} \\ &= \pi \cot \pi x, \quad \text{by Example 11.12,} \\ &= \frac{d}{dx}(\ln \sin \pi x), \end{aligned}$$

which shows that

$$\frac{d}{dx}\left(\ln\left(\frac{g(x)}{\sin \pi x}\right)\right) = 0, \quad \text{i.e.} \quad \ln\left(\frac{g(x)}{\sin \pi x}\right) = \text{constant}.$$

Therefore, there exists a real constant c such that

$$g(x) = c\sin \pi x, \quad \text{i.e.} \quad \frac{\sin \pi x}{\pi x} = \frac{1}{c\pi}\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right), \quad x \in (0,1).$$

Taking the limit $x \to 0$, we get $c\pi = 1$. Thus, the formula (11.50) follows from the uniqueness theorem. Now, some special cases follow.

(i) Substituting $z = i$ in (11.50) gives

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right) = \frac{e^{\pi} - e^{-\pi}}{2\pi} = \frac{\sinh \pi}{\pi}.$$

(ii) The product representation for the cosine function may be obtained directly from (11.50). Indeed, by (11.50), we see that

$$\frac{\sin 2\pi z}{2\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2}\right)$$

which, by the identity $2\sin \pi z \cos \pi z = \sin 2\pi z$, gives

$$\begin{aligned}\left(\frac{\sin \pi z}{\pi z}\right)\cos \pi z &= \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{n^2}\right)\\ &= \prod_{n=1}^{\infty}\left(1-\left(\frac{2z}{2n}\right)^2\right)\left(1-\left(\frac{2z}{2n-1}\right)^2\right)\\ &= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2}\right)\end{aligned}$$

so that

$$\cos \pi z = \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2}\right), \quad z \in \mathbb{C}.$$

Using the formulae

$$e^{iz}+1 = 2e^{iz/2}\cos(z/2) \quad \text{and} \quad e^{iz}-1 = 2ie^{iz/2}\sin(z/2),$$

the product representation for the entire functions $e^{iz}+1$ and $e^{iz}-1$ can be achieved with the help of the corresponding representations for $\cos z$ and $\sin z$. Consequently, if we replace z by $-iz$ then we can quickly establish the formula

$$\text{(11.53)} \qquad e^z - 1 = ze^{z/2}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2 n^2}\right), \quad z \in \mathbb{C}.$$

(iii) A comparison of the product expansion of an entire function with its power series expansion often leads to interesting conclusions. For example, by equating the product expansion (11.50) of $\sin \pi z$ with its Taylor's series expansion about 0, we see that

$$\text{(11.54)} \qquad \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots.$$

Because of the uniqueness of the Taylor coefficients, a comparison of the coefficients of z^3 on both sides shows that

$$-\pi\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) = -\frac{\pi^3}{3!}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

(iv) If $z = 1/2$, then (11.50) gives the 'Wallis product formula'

$$1 = \frac{\pi}{2}\prod_{k=1}^{\infty}\left(1-\frac{1}{4k^2}\right) = \frac{\pi}{2}\prod_{k=1}^{\infty}\left(\frac{4k^2-1}{4k^2}\right).$$

Consequently,

$$\begin{aligned}\frac{\pi}{2} &= \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots \\ &= \lim_{n\to\infty}\left\{\left(\frac{2\cdot 2}{1\cdot 1}\right)\left(\frac{4\cdot 4}{3\cdot 3}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)\cdot(2n-1)}\right)\frac{1}{2n+1}\right\}\end{aligned}$$

or equivalently

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\left\{\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\frac{1}{\sqrt{2n+1}}\right\}$$

which is often called the "Wallis identity". This formula can be written in the form

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\left\{\frac{(2\cdot 4\cdot 6\cdots 2n)^2}{(1\cdot 3\cdots(2n-1))(2\cdot 4\cdots 2n)}\frac{1}{\sqrt{2n}\sqrt{1+1/2n}}\right\}$$

which gives $\sqrt{\pi} = \lim_{n\to\infty} \dfrac{2^{2n}(n!)^2}{(2n)!n^{1/2}}$. ●

11.5 The Gamma Function

The study of the Γ-function that involves the integral form

$$\Gamma(x) := \int_0^1 \left(\log\frac{1}{t}\right)^{x-1} dt = \int_0^\infty e^{-t}t^{x-1}\,dt \quad (x>0) \tag{11.55}$$

was first introduced by Euler in 1729. Differentiating the integral (11.55) gives

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}t^{x-1}(\log t)^n\,dt, \quad n \in \mathbb{N}.$$

Clearly $\Gamma(1) = 1$ and for $x > 0$, integration by parts yields the functional equation

$$\Gamma(x+1) = x\Gamma(x) \tag{11.56}$$

because

$$\Gamma(x+1) = \int_0^\infty t^x\,d(-e^{-t}) = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt = x\Gamma(x).$$

For $x = n \in \mathbb{N}$, the functional equation becomes $\Gamma(n+1) = n!$ and therefore, the Γ-function is seen as an extension of the factorial function for x belonging to $\mathbb{R}\backslash\{0,-1,-2,\dots\}$. Indeed, in view of the functional equation (11.56), we have

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+2)}{x(x+1)} = \cdots = \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.$$

This observation suggests that it is possible to extend the Γ-function on the whole real axis except on the negative integers $\{0, -1, -2, \dots\}$. We are interested in discussing the following questions:

- Does there exist an analog of (11.55) for $\operatorname{Re} z > 0$?
- Is it possible to extend $\Gamma(x)$ defined by (11.55) to $\mathbb{C} \backslash \{0, -1, -2, \dots\}$?
- What can we say about the points in $\{0, -1, -2, \dots\}$?
- What is the counterpart of (11.56) when x takes complex values?
- What are the basic consequences of the extended gamma function?

There are two simple approaches through which one can answer these questions. Let us first allow the integral (11.55) to extend the "factorial function" to the complex plane. Now,

$$|t^z| = |e^{z \log t}| = e^{\operatorname{Re} z \ln t} = t^{\operatorname{Re} z} \quad \text{for } t > 0$$

so that if we replace x in (11.55) by the complex variable z, the resulting function (called the classical gamma function) given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \tag{11.57}$$

is uniformly convergent for $\operatorname{Re} z > 0$. For this function, the point $z = 0$ is a singularity because

$$\Gamma(\epsilon) = \int_0^\infty e^{-t} t^{\epsilon - 1}\, dt = \int_0^\infty \frac{e^{-t}}{t^{1-\epsilon}}\, dt \to \infty \quad \text{as } \epsilon \to 0.$$

11.58. Gamma function via product representation. We recall that

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}, \quad G(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right]. \tag{11.59}$$

Here $G(z)$ is entire and it has simple zeros at all negative integers. Clearly, the zeros of G and $G(z-1)$ are the same, except that $G(z-1)$ has a zero at $z = 0$. It follows from the Weierstrass factorization theorem that

$$G(z-1) = z e^{\gamma(z)} G(z)$$

or equivalently,

$$\prod_{n=1}^{\infty} \left(\frac{z+n-1}{n} \right) e^{-(z-1)/n} = z e^{\gamma(z)} \prod_{n=1}^{\infty} \left(\frac{z+n}{n} \right) e^{-z/n},$$

for some entire function $\gamma(z)$ which is to be determined. In order to find $\gamma(z)$, we take the logarithmic derivative of the last equation and obtain

$$\sum_{n=1}^{\infty} \left(\frac{1}{z+n-1} - \frac{1}{n} \right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right).$$

Replace n by $n+1$ on the L.H.S and note that $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right) = -1$. This gives that $\gamma'(z) = 0$ so that $\gamma(z)$ is a constant which we denote by γ. Therefore,

$$G(z-1) = ze^{\gamma}G(z).$$

To find the value of the constant γ, we observe that $G(0) = 1$ and so we may let $z = 1$. This gives

$$1 = e^{\gamma}G(1), \quad \text{i.e.} \quad e^{-\gamma} = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}.$$

Taking the logarithm on both sides

$$\begin{aligned}
-\gamma &= \sum_{n=1}^{\infty}\left\{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}\right\} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\ln\left(1+\frac{1}{k}\right)-\frac{1}{k}\right\} \\
&= \lim_{n\to\infty}\left\{\ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n}{n-1}\right)+\ln\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right\} \\
&= \lim_{n\to\infty}\left\{\ln n-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right\}, \quad \text{since } \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) = 0.
\end{aligned}$$

Thus,

$$\gamma = \lim_{n\to\infty}\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right\} = 0.5772\cdots. \tag{11.60}$$

The constant γ is called Euler's constant. The question of whether γ is rational or irrational seems to be still open. In conclusion, the product representation of the gamma function is defined by

$$\Gamma(z) = \frac{1}{ze^{\gamma z}G(z)} = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1}e^{z/n}\right]. \tag{11.61}$$

We have the following consequences of the definition:

- In view of (11.61) and taking into account (11.60), we deduce the formula of Gauss

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty}\frac{e^{-\gamma z}}{z}\prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)^{-1}e^{z/k}\right] \\
&= \lim_{n\to\infty}\frac{e^{z\ln n}}{z}\prod_{k=1}^{n}\left(\frac{k}{z+k}\right) \\
&= \lim_{n\to\infty}\frac{n!n^{z}}{z(z+1)\cdots(z+n)}
\end{aligned}$$

  which is valid for all $z \neq 0, -1, -2, \ldots$.

- In view of (11.61), we have $\Gamma(z+1) = z\Gamma(z)$, which is called *Riemann's Functional Relation* for the gamma function.
- By (11.59), we have the identity

$$\Gamma(1-z)\Gamma(z) = -z\Gamma(-z)\Gamma(z) = \frac{1}{zG(z)G(-z)} = \frac{\pi}{\sin \pi z}.$$

  In particular, $\Gamma(1/2) = \sqrt{\pi}$.
- Repeated application of $\Gamma(z+1) = z\Gamma(z)$ gives $\Gamma(n+1) = n!$.
- $\Gamma(z)$ never vanishes in $\mathbb{C}$ as, for $z \notin \{0, -1, -2, \cdots\}$, the gamma function is given by a convergent infinite product of non-zero factors. Moreover, the representation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}G(z) = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right], \quad z \in \mathbb{C},$$

  makes explicit that the gamma function never vanishes and that it has simple poles precisely at $0, -1, -2, \ldots$. The functional equation

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \cdots = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$$

  shows that $\text{Res}\,[\Gamma(z); 0] = \lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = \Gamma(1) = 1$ and for $n \in \mathbb{N}$, we have

$$\begin{aligned}\text{Res}\,[\Gamma(z); -n] &= \lim_{z\to -n}(z+n)\Gamma(z) \\ &= \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)} \\ &= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!}.\end{aligned}$$

  So, $\Gamma(z)$ is meromorphic in $\mathbb{C}$.
- Take the logarithmic derivative of (11.61) to get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(-\frac{1}{z+n} + \frac{1}{n}\right)$$

  so that

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}. \tag{11.62}$$

  As $\Gamma(z)\Gamma(z+1/2)$ and $\Gamma(2z)$ have the same set of poles, we can write

$$\Gamma(z)\Gamma(z+1/2) = e^{\alpha(z)}\Gamma(2z)$$

for some entire function $\alpha(z)$. In fact, by (11.62), we see that

$$\begin{aligned}\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) &= \sum_{n=0}^{\infty}\frac{4}{(2z+2n)^2}+\sum_{n=0}^{\infty}\frac{4}{(2z+2n+1)^2}\\ &= 4\sum_{n=0}^{\infty}\frac{1}{(2z+n)^2}\\ &= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)\end{aligned}$$

which, upon integration, shows that

$$\Gamma(z)\Gamma(z+1/2)=e^{az+b}\Gamma(2z)$$

for some constants a and b. To find the values of a and b, we substitute $z = 1/2, 1$ and obtain (note that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \sqrt{\pi}/2$)

$$\sqrt{\pi}=e^{(a/2)+b}, \qquad \frac{\sqrt{\pi}}{2}=e^{a+b}$$

which, by dividing one by the other, gives $e^{a/2} = 1/2$ and $e^b = 2\sqrt{\pi}$. It follows that $a = -2\ln 2$ and thus, we obtain the so called *Legendre's duplication formula*

$$\sqrt{\pi}\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(z+1/2). \tag{11.63}$$

11.6 The Zeta Function

The zeta function was first introduced by Euler. There are two important formulas which define the zeta function in two different ways either as series form or as an Euler product. By the series formula of Riemann, this is expressed as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \quad s=\sigma+it, \tag{11.64}$$

where we use the traditional notation to denote the complex variable by $s = \sigma + it$. The function ζ represented by the series (11.64) is called *the Riemann zeta function or simply the zeta function.* This function, which is of special interest, provides a link between number theory and function theory. Also, it is of central importance in number theory particularly in the study of the distribution of prime numbers. First we shall show that the series (11.64) converges for $\sigma > 1$. To do this we let $\sigma \geq \sigma_0 > 1$. Then, by the definition of n^s,

$$|n^s|=|e^{(\sigma+it)\log n}|=e^{\sigma\ln n}=n^\sigma\geq n^{\sigma_0}, \text{ i.e. } \left|\frac{1}{n^{\sigma+it}}\right|\leq\frac{1}{n^{\sigma_0}}.$$

Further, since $\frac{1}{x^{\sigma_0}}$ is decreasing for $x > 0$,

$$\int_n^{n+1} \frac{dx}{x^{\sigma_0}} < \frac{1}{n^{\sigma_0}} < \int_{n-1}^{n} \frac{dx}{x^{\sigma_0}}$$

where the inequality on the left holds for $n \geq 1$ and that on the right holds for $n \geq 2$. Hence, for any natural number $N \geq 2$, we have

$$\sum_{n=2}^{N} \frac{1}{n^{\sigma_0}} < \int_1^N \frac{dx}{x^{\sigma_0}} = \frac{1}{-\sigma_0 + 1}\left[\frac{1}{N^{\sigma_0 - 1}} - 1\right] \to -\frac{1}{1-\sigma_0} \quad \text{as } N \to \infty$$

and so, as the partial sum is an increasing sequence bounded above, the series converges. Indeed,

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^{\sigma+it}}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} < 1 + \int_1^{\infty} \frac{dx}{x^{\sigma_0}} = 1 + \frac{1}{\sigma_0 - 1} = \frac{\sigma_0}{\sigma_0 - 1}.$$

Therefore, the series on the right of (11.64) converges uniformly and absolutely for $\sigma \geq \sigma_0 > 1$. As $f(s) = n^{-s} = e^{-s \ln n}$, each term in the series (11.64) is an analytic function of s and therefore, $f'(s) = -\frac{\ln n}{n^s}$. Moreover, the Weierstrass M-test shows that the ζ function is analytic for $\operatorname{Re} s > 1$ with derivatives

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s} \quad \text{and} \quad \zeta^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{(\ln n)^k}{n^s}, \quad k \in \mathbb{N}.$$

The ζ function so defined is related to the study of prime numbers by the following

11.65. Theorem. (Euler's Product Formula) *For $s = \sigma + it$, $\sigma > 1$, we have*

$$(11.66) \qquad \zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{or} \quad \frac{1}{\zeta(s)} = \prod \left(1 - \frac{1}{p^s}\right),$$

where the product is taken over all prime numbers p. In particular, $\zeta \neq 0$ for $\operatorname{Re} s > 1$.

Proof. By Corollary 11.22, the infinite product $\prod_{p,\,\text{prime}} (1 - p^{-s})$ converges uniformly for $\sigma \geq \sigma_0 > 1$ as the series $\sum_{p-\text{prime}} p^{-s}$ is obtained by omitting terms of $\sum_{n=1}^{\infty} n^{-s}$ which converges uniformly for $\sigma \geq \sigma_0 > 1$. Now consider the series (11.64) for $\sigma > 1$, and

$$\zeta(s)\frac{1}{2^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots.$$

Subtracting this equation from (11.64), we obtain that

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots.$$

Similarly, one can find that

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots.$$

More generally,

$$\zeta(s)\left(1-2^{-s}\right)\left(1-3^{-s}\right)\cdots\left(1-p_N^{-s}\right)=\sum m^{-s}=1+p_{N+1}^{-s}+\cdots$$

where the sum on the right being over all positive integers that contain none of the prime factors $2,3,\ldots,p_N$. Therefore, allowing $N\to\infty$, it follows that $\zeta(s)\prod_{p,\,\mathrm{prime}}(1-p^{-s})=1$ and the conclusion follows. ■

The two definitions of the Riemann zeta function, namely the series form (11.64) and the product form (11.66), are equivalent. Either of them may be taken as a definition of $\zeta(s)$ for $\mathrm{Re}\,s>1$. The following relationship between the Γ-function and the ζ-function may be studied in several forms.

11.67. Theorem. (Integral representation of zeta function) *For* $\mathrm{Re}\,s>1$, *we have*

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx. \tag{11.68}$$

Proof. The integral definition of the Γ-function given by (11.57) together with the change of variable $t=nx$ imply that for $\mathrm{Re}\,s>1$

$$\frac{\Gamma(s)}{n^s}=\int_0^\infty e^{-nx}x^{s-1}\,dx.$$

Summation over all positive integer n in the equation gives

$$\Gamma(s)\zeta(s)=\int_0^\infty x^{s-1}\sum_{n=1}^\infty(e^{-x})^n\,dx=\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx$$

where we have used the fact that the partial sums of the geometric series $\sum_{n=1}^\infty(e^{-x})^n$ form an increasing sequence of functions that converges uniformly on each interval $[\epsilon,\infty)$, $\epsilon>0$. This observation justifies the interchange of the summation and the integration. ■

11.69. Global representation of $\zeta(s)$. The definition of the ζ function given by (11.64) and (11.66) are valid only for $\mathrm{Re}\,s>1$. We wish to extend the ζ function to $\mathbb{C}\setminus\{1\}$ and show that ζ is meromorphic in $\mathbb{C}$ and

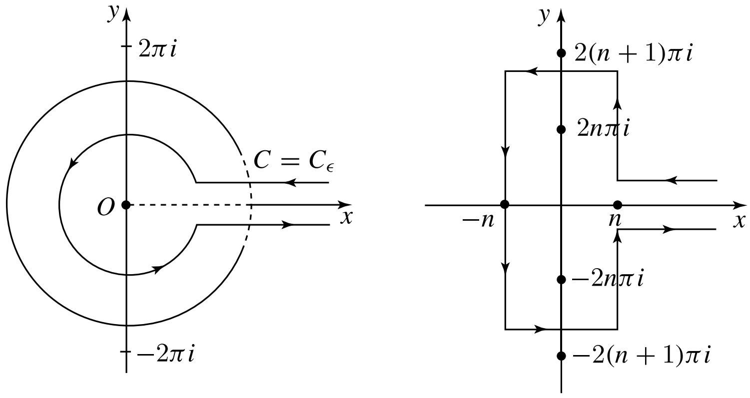


Figure 11.1: Contour for the continuation of ζ function.

has a simple pole only at $s = 1$ with residue 1. According to (11.68), if we can show that the integral in (11.68) is a meromorphic function of s in $\mathbb{C}$ then the ζ-representation given by this integral can be used as the definition of the ζ function on $\mathbb{C}$. Unfortunately, in the present form, the integral in (11.68) is improper and is divergent for $\mathrm{Re}\, s < 0$, because as x nears zero,

$$\left|\frac{x^{s-1}}{e^x - 1}\right| \sim x^{\mathrm{Re}\, s - 2}.$$

Riemann overcame this difficulty, through a trick. To discuss this, we need to represent $\zeta(s)$ as a contour integral by choosing a suitable contour that avoids the origin so that the resultant form becomes entire. For $s \in \mathbb{C}$ fixed, consider

$$f(z) = \frac{(-z)^{s-1}}{e^z - 1} = \frac{e^{(s-1)\,\mathrm{Log}\,(-z)}}{e^z - 1}.$$

This function is analytic in $\mathbb{C} \setminus [\{x \in \mathbb{R} : x \geq 0\} \cup \{2k\pi i : k \in \mathbb{Z}\}]$. Define

$$\phi(s) = \frac{1}{2\pi i} \int_C f(z)\, dz, \tag{11.70}$$

where $C = C_\epsilon = C_\epsilon(\delta)$ is the "Hankel contour" shown in Figure 11.1 with $0 < \epsilon < 2\pi$. This integral converges, and it represents an entire function of s. Indeed, the integrand is an entire function of s implying that $\phi(s)$ is entire. Since the region bounded by the contours C_ϵ and C_η contains no poles of $f(z)$ for $0 < \epsilon < \eta < 2\pi$, by Cauchy's theorem, we see that the value of the integral in (11.70) is independent of ϵ.

11.71. Theorem. *For* $\mathrm{Re}\, s > 1$,

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz. \tag{11.72}$$

Proof. Since the integral in (11.70) is independent of the shape of C as long as C does not enclose any integer multiple of $2\pi i$, we are free to allow $\epsilon \to 0$. We also note that, for $z = x + i\delta := \epsilon e^{i\theta}$ with $x > 0$,

$$\mathrm{Log}\,(-z) = \mathrm{Log}\,(-x - i\delta) \to \ln\, x - i\pi \quad \text{as } 0 < \delta \to 0$$

and, for $z = x - i\delta$,

$$\mathrm{Log}\,(-z) = \mathrm{Log}\,(-x + i\delta) \to \ln\, x + i\pi \quad \text{as } 0 > \delta \to 0.$$

First, as $\delta \to 0$, we express the integral (11.70) as

$$\begin{aligned}
2\pi i\phi(s) &= \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) f(z)\,dz, \quad f(z) = \frac{(-z)^{s-1}}{e^z - 1}, \\
&= \int_\infty^\epsilon \frac{e^{(s-1)(\ln x - i\pi)}}{e^x - 1}\,dx + \int_{|z|=\epsilon} f(z)\,dz + \int_\epsilon^\infty \frac{e^{(s-1)(\ln x + i\pi)}}{e^x - 1}\,dx \\
&= -2i\sin(\pi s)\int_\epsilon^\infty \frac{x^{s-1}}{e^x - 1}\,dx + \int_{|z|=\epsilon} f(z)\,dz
\end{aligned}$$

because

$$\begin{aligned}
e^{(s-1)(\ln x + i\pi)} - e^{(s-1)(\ln x - i\pi)} &= x^{s-1}\left[e^{(s-1)i\pi} - e^{-(s-1)i\pi}\right] \\
&= -x^{s-1} 2i\sin(\pi s).
\end{aligned}$$

Suppose, for the moment, that $\mathrm{Re}\, s > 1$. As $e^z - 1$ has a simple zero at the origin, by the continuity of $e^z - 1$, we see that for $|z|$ sufficiently small, $|e^z - 1| \geq |z|/2$ and so there exists an $M > 0$ such that

$$\left|\int_{|z|=\epsilon} \frac{(-z)^{s-1}}{e^z - 1}\,dz\right| \leq M \int_{|z|=\epsilon} \frac{2\epsilon^{\mathrm{Re}\, s-1}}{\epsilon}\,|dz| = 4\pi M\epsilon^{\mathrm{Re}\, s-1}.$$

Consequently, the integral tends to zero as $\epsilon \to 0$ and because $\mathrm{Re}\, s > 1$. Finally, we may evaluate $\phi(s)$ by letting $\epsilon \to 0$ in the last equation. This gives

$$\begin{aligned}
\phi(s) &= -\frac{\sin \pi s}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx \\
&= -\frac{\sin \pi s}{\pi}\Gamma(s)\zeta(s), \quad \text{by Lemma 11.67,} \\
&= -\frac{\zeta(s)}{\Gamma(1-s)}, \qquad \text{since } \Gamma(1-s)\Gamma(s) = \tfrac{\pi}{\sin \pi s}.
\end{aligned}$$

Thus, (11.72) follows. ■

Note that $\phi(s)$ is an entire function of s and the factor $\Gamma(1-s)$ has simple poles at $s = 1, 2, \ldots$. The left hand side of (11.72), on the other hand, is defined at all such points except for $s = 1$. Thus, the equation (11.72) defines an analytic function except possibly with a simple pole at $s = 1$. Now, $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 and so,

$$\Gamma(1-s) \sim \frac{1}{1-s} = -\frac{1}{s-1} \quad \text{for } s \text{ near } 1.$$

Thus,

$$\begin{aligned} \lim_{s\to 1}(s-1)\zeta(s) &= -\lim_{s\to 1}(s-1)\Gamma(1-s)\left(\frac{1}{2\pi i}\int_\gamma \frac{(-z)^{s-1}}{e^z-1}\,dz\right) \\ &= -(-1)\left[\frac{1}{2\pi i}\int_\gamma \frac{dz}{e^z-1}\right] \\ &= \text{Res}\left[\frac{1}{e^z-1};0\right] = 1 \end{aligned}$$

and we have established the following

11.73. Theorem. *The* ζ *function is analytic in* $\mathbb{C}\setminus\{1\}$ *and has a simple pole at* $s = 1$ *with residue* 1.

Riemann used the definition of the ζ function defined by (11.72) to derive many interesting relations (such as the functional equation of the ζ function and the function $\xi(s)$, below) which actually help to verify the so-called Riemann hypothesis. Let us first prove the functional equation which provides more explicit information about the analytic continuation of ζ to $\mathbb{C}\setminus\{1\}$.

11.74. Theorem. *For all* $s \in \mathbb{C}$, *the* ζ*-function satisfies Riemann's functional equation*

$$\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s). \tag{11.75}$$

In particular (as $\zeta(1-s) \neq 0$ *for* $\text{Re}\, s < 0$*),* $\zeta(-2k) = 0$ *for* $k \in \mathbb{N}$.

Proof. It suffices to prove this theorem for $s < 0$. Define

$$\phi_n(s) = \frac{1}{2\pi i}\int_{C_n} f(z)\,dz, \quad f(z) = \frac{(-z)^{s-1}}{e^z-1},$$

where C_n is as shown in Figure 11.1. Note that the rectangle has vertices at $\pm n \pm (2n+1/2)\pi i$. The idea is to relate $\phi(s)$ to $\phi_n(s)$ using the calculus of residues and then to let $n \to \infty$.

For z on the sides of the rectangle, we have $|e^z - 1| > 1/2$ and so for $s < 0$,

$$|f(z)| = \left|\frac{e^{(s-1)\,\text{Log}\,(-z)}}{e^z-1}\right| \leq 2n^{s-1}.$$

By the ML-inequality (see Theorem 4.9(iii)), $|\phi_n(s)| \leq Kn^s \to 0$ as $n \to \infty$. Consequently,

$$\phi_n(s) - \phi(s) = 2\pi i \times (\text{sum of the residues of } f(z) \text{ inside the rectangle}).$$

The poles of $f(z)$ are simple and they occur at $z_k = \pm 2k\pi i$, $1 \leq k \leq n$, with residue

$$\begin{aligned} \lim_{z \to z_k} (z - z_k) \frac{(-z)^{s-1}}{e^z - 1} &= \lim_{z \to z_k} \frac{1}{e^z} \lim_{z \to z_k} (-z)^{s-1} \\ &= (\mp 2\pi k i)^{s-1} \\ &= e^{(s-1)[\ln|2\pi k| \mp i\pi/2]} \\ &= (2\pi k)^{s-1} e^{\mp(s-1)i\pi/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_n \setminus C} \frac{(-z)^{s-1}}{e^z - 1}\, dz &= \sum_{k=1}^{n} (2\pi k)^{s-1} \left(e^{-(s-1)i\pi/2} + e^{(s-1)i\pi/2} \right) \\ &= 2 \sum_{k=1}^{n} (2\pi k)^{s-1} \cos[(s-1)\pi/2] \\ &= 2^s \pi^{s-1} \sin(\pi s/2) \sum_{k=1}^{n} \frac{1}{k^{1-s}}. \end{aligned}$$

Allowing $n \to \infty$, it follows that (since $\phi_n(s) \to 0$ as $n \to \infty$)

$$-\frac{1}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz = 2^s \pi^{s-1} \sin(\pi s/2) \zeta(1-s).$$

If we combine this with (11.72) and follow the discussion presented after its proof, we obtain the functional equation for $s < 0$. Since both sides of the functional equation are meromorphic and agree on a non-empty open set, it holds for all s by the uniqueness theorem. ■

There are equivalent forms of the functional equation (see for example, Exercise 11.125). If we replace z by $(1-s)/2$, then Legendre's duplication formula for the gamma function (11.63) becomes

$$\pi^{1/2} \Gamma(1-s) = 2^{-s} \Gamma((1-s)/2) \Gamma(1-s/2) = \frac{2^{-s} \pi \Gamma((1-s)/2)}{\Gamma(s/2) \sin(\pi s/2)}$$

so that

$$\Gamma(1-s) \sin(\pi s/2) = \frac{2^{-s} \pi^{1/2} \Gamma((1-s)/2)}{\Gamma(s/2)}.$$

In view of this equation, (11.75) is equivalent to

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s).$$

Thus, the functional equation takes the form

$$\Phi(s) = \Phi(1-s), \quad \Phi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Clearly, Φ has simple poles at $s = 0, 1$. If we multiply it by $s(1-s)/2$, we see that the function

$$\xi(s) = \frac{s(1-s)}{2}\Phi(s) = \frac{s(1-s)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire and satisfies the relation $\xi(s) = \xi(1-s)$.

11.76. Riemann Hypothesis. The functional equation of the ζ function enables us to locate the zeros of $\zeta(s)$. Because of the product representation (11.66), the ζ function has no zeros if $\operatorname{Re} s > 1$. We know that $\Gamma(s)$ never vanishes in $\mathbb{C}$ and is analytic except at $s = 0, -1, -2, \ldots$. Consequently, both $\Gamma(1-s)$ and $\zeta(1-s)$ are analytic and non-zero for $\operatorname{Re} s < 0$. The functional equation then says that the only zeros of $\zeta(s)$ for $\operatorname{Re} s < 0$ are the zeros of $\sin(\pi s/2)$, that is only at $s = -2, -4, \ldots$. These are known as the *trivial zeros* of the ζ function. It is not difficult to show that there are no zeros on $\operatorname{Re} s = 0$ and $\operatorname{Re} s = 1$. We conclude that all the *non-trivial* zeros of the ζ function lies in the strip $\{s : 0 < \operatorname{Re} s < 1\}$, which is called the *critical strip*. It is known that there are infinitely many zeros on the line $s = 1/2 + it,\ t \in \mathbb{R}$. This line in the s-plane is called *the critical line*. Again, since

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots > 0$$

for $0 < s < 1$ and $\zeta(0) \neq 0$, $\zeta(s)$ has no zeros on the real interval $(0, 1)$. This observation implies that all possible zeros of $\zeta(s)$ in the critical strip are complex numbers. The Riemann hypothesis asserts that *"All the nontrivial zeros of ζ function lie on the critical line* $\operatorname{Re} s = 1/2$".

Although this has been shown to be true for more than one billion non-trivial zeros, the conjecture remains open, despite the efforts of some of the greatest analysts since Riemann's time. It is the most famous unsolved problem confronting 21-th century mathematicians, especially after the proof of Fermat's last theorem. No calculation had ever contradicted the hypothesis. The Clay Mathematics Institute of Cambridge, Massachusetts, offers one million US dollars for a proof of the Riemann hypothesis.

11.7 Jensen's Formula

Suppose that $g \in \mathcal{H}(\overline{\Delta})$ and that it is zero-free on $\overline{\Delta}$. Then g is a nowhere vanishing analytic function on Δ_R for some $R > 1$, and so there exists an $h \in \mathcal{H}(\overline{\Delta})$ such that $g(z) = e^{h(z)}$. In particular, g admits an analytic logarithm on Δ_R. The Cauchy integral formula applied to $\log g(z)$ yields that

$$\log g(0) = \frac{1}{2\pi}\int_0^{2\pi} \log g(e^{i\theta})\, d\theta.$$

Equating the real parts gives

$$\ln|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(e^{i\theta})|\,d\theta \tag{11.77}$$

(we observe that the right hand side of (11.77) is an improper integral if g has a zero on the circle $|z| = 1$). For example, if $|a| < 1$ and one considers $g(z) = 1 - \overline{a}z$ then $g \in \mathcal{H}(\overline{\Delta})$ and is zero-free there so that (11.77) gives

$$0 = \ln 1 = \frac{1}{2\pi}\int_0^{2\pi} \ln|1 - \overline{a}e^{i\theta}|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln|1 - ae^{-i\theta}|\,d\theta. \tag{11.78}$$

More generally, we prove

11.79. Lemma. *For $a \in \mathbb{C}$ with $|a| < r$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|r - ae^{-i\theta}|\,d\theta = \ln r. \tag{11.80}$$

(*Note that $r = 1$ gives* (11.78)).

Proof. We may rewrite $|r - ae^{-i\theta}|$ as $r|1 - (a/r)e^{-i\theta}| = r|1 - (\overline{a}/r)e^{i\theta}|$ so that $\ln|r - ae^{-i\theta}| = \ln r + \ln|1 - (\overline{a}/r)e^{i\theta}|$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|r - ae^{-i\theta}|\,d\theta = \ln r + \frac{1}{2\pi}\int_0^{2\pi} \ln|1 - (\overline{a}/r)e^{i\theta}|\,d\theta.$$

Note that the integral on the right vanishes with the same reasoning as above (by considering $g(z) = 1 - (\overline{a}/r)z$). Therefore, (11.80) follows.

Alternatively, for $|b| < 1$, we obtain that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \ln|1 - be^{i\theta}|\,d\theta &= \operatorname{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Log}(1 - be^{i\theta})\,d\theta\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\int_{|z|=b} \frac{\operatorname{Log}(1-z)}{z}\,dz\right] \\
&= \ln 1 = 0, \quad \text{by the Cauchy integral formula,}
\end{aligned}$$

and therefore, (11.80) holds. ■

11.81. Lemma. *If $I = \frac{1}{2\pi}\int_0^{2\pi} \ln|1 - e^{i\theta}|\,d\theta$, then $I = 0$.*

Proof. Since $\sin\varphi \geq 0$ on $[0, \pi]$, we write

$$|1 - e^{i\theta}| = |e^{-i\theta/2} - e^{i\theta/2}| = 2\sin(\theta/2)$$

for $\theta \in [0, 2\pi]$. Therefore, $\ln|1 - e^{i\theta}| = \ln 2 + \ln\sin(\theta/2)$ so that

$$\begin{aligned} I = \int_0^{2\pi} \ln|1 - e^{i\theta}|\, d\theta &= 2\pi \ln 2 + \int_0^{2\pi} \ln\sin(\theta/2)\, d\theta \\ &= 2\pi \ln 2 + 2\int_0^{\pi} \ln\sin t\, dt \quad (\theta = 2t). \end{aligned}$$

To complete the proof it suffices to show that $\int_0^\pi \ln\sin t\, dt = -\pi \ln 2$. Notice that this integral is improper, but its convergence is obvious. Now, we find that

$$J = \int_0^{\pi} \ln\sin t\, dt = 2\int_0^{\pi/2} \ln\sin t\, dt = 2\int_0^{\pi/2} \ln\cos s\, ds$$

(use the change of variable $t = \pi/2 - s$ in the second integral) and we also see that

$$\begin{aligned} J &= \int_0^{\pi} \ln[2\sin(t/2)\cos(t/2)]\, dt \\ &= \int_0^{\pi} \ln 2\, dt + \int_0^{\pi} \ln\sin(t/2)\, dt + \int_0^{\pi} \ln\cos(t/2)\, dt \\ &= \pi \ln 2 + 2\int_0^{\pi/2} \ln\sin(t)\, dt + 2\int_0^{\pi/2} \ln\cos(t)\, dt \\ &= \pi \ln 2 + 2J. \end{aligned}$$

Thus, $J = -\pi \ln 2$ and so, $I = 0$. ∎

We will now investigate what happens to (11.77) in the presence of zeros as well as poles in Δ. Also, we remark that there is a similar generalization of the Poisson integral formula (see Exercise 11.119).

11.82. Theorem. (Jensen's Formula for the closed unit disk) *Let f be meromorphic in $\overline{\Delta}$. Suppose that*

(i) *0 is neither a zero nor a pole of f*

(ii) *a_i $(1 \le i \le m)$ and b_j $(1 \le j \le n)$ denote the zeros and poles of f in $\overline{\Delta}$ (counted as many times as its order of multiplicities), respectively.*

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\theta})|\, d\theta = \ln|f(0)| + \ln\left(\frac{\prod_{j=1}^n |b_j|}{\prod_{i=1}^m |a_i|}\right). \tag{11.83}$$

11.84. Observations. There are a number of interesting observations we can make from (11.83).

(i) Equation (11.83) is called the Jensen formula for meromorphic function in the closed unit disk $\overline{\Delta}$. First we shall obtain a general result (see Corollary 11.85) as a consequence of this result.

(ii) The second sum on the right hand side of (11.83) will not appear when f has neither zeros nor poles on $|z| \leq 1$ (in this case, f is actually in $\mathcal{H}(\overline{\Delta})$ with $f(0) \neq 0$; compare with (11.77)). This fact will be clear in the proof.

(iii) Equation (11.83) yields Jensen's inequality:

$$\ln \left[|f(0)| \left(\frac{\prod_{j=1}^{n} |b_j|}{\prod_{i=1}^{m} |a_i|} \right) \right] \leq \sup_{0 \leq \theta < 2\pi} \ln |f(e^{i\theta})|.$$

(iv) Jensen's formula (11.83) may be equivalently written as

$$|f(0)| \left(\frac{\prod_{j=1}^{n} |b_j|}{\prod_{i=1}^{m} |a_i|} \right) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \ln |f(e^{i\theta})| \, d\theta \right).$$

(v) Formula (11.83) shows that the integral on the right exists even if the function f has a zero or pole on the unit circle $|z| = 1$. In case f in Theorem 11.82 has a zero of order $p \geq 1$ at $z = 0$ but has no poles at $z = 0$, then consider the function

$$G(z) = \begin{cases} z^{-p} f(z) & \text{if } z \neq 0 \\ \lim\limits_{z \to 0} z^{-p} f(z) & \text{if } z = 0 \end{cases}$$

which guarantees that $G(0) = f^{(p)}(0)/p! \neq 0$. Now, apply Theorem 11.82 to $G(z)$ and obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(e^{i\theta})| \, d\theta = \ln \left| \frac{f^{(p)}(0)}{p!} \right| + \ln \left(\frac{\prod_{j=1}^{n} |b_j|}{\prod_{i=1}^{m} |a_i|} \right).$$

Do the same trick if f has a pole of order q at 0. This means that one has to multiply by z^q and apply Theorem 11.82 for $z^q f(z)$.

Proof. Case (i): Suppose that none of the zeros or poles of $f(z)$ are situated on $|z| = 1$. We need to define a function $F(z)$ which is free from all the zeros and the poles of $f(z)$ in $\overline{\Delta}$ and such that $F \in \mathcal{H}(\overline{\Delta})$ with $|f(z)| = |F(z)|$ on $|z| = 1$. By recalling

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z} \quad (\phi_a \in \mathcal{H}(\overline{\Delta}),\ \phi_a(0) = a,\ \phi_a(\Delta) = \Delta,\ \phi_a(\partial\Delta) = \partial\Delta),$$

such a function is clearly given by

$$F(z) = f(z) \left(\prod_{i=1}^{m} \frac{1}{\phi_{a_i}(z)} \right) \left(\prod_{j=1}^{n} \phi_{b_j}(z) \right).$$

Since $F \in \mathcal{H}(\overline{\Delta})$ and $F(z) \neq 0$ for $|z| \leq 1$, there exists a $g \in \mathcal{H}(\overline{\Delta})$ such that $F(z) = e^{g(z)}$. In particular, $g(z) = \log F(z)$ and its real part is $\ln |F(z)|$ so that

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |F(e^{i\theta})| \, d\theta = \ln |F(0)|$$

which is same as (11.83), since $|f(e^{i\theta})| = |F(e^{i\theta})|$ and

$$F(0) = f(0) \left(\prod_{i=1}^{m} \frac{1}{a_i} \right) \left(\prod_{j=1}^{n} b_j \right).$$

Case (ii): Suppose that f does have zeros on $|z| = 1$. Then these zeros may be enumerated so that

$$|a_1| \leq \cdots \leq |a_p| < 1, \quad |a_{p+1}| = \cdots = |a_m| = 1.$$

In this case $F(z)$ defined in Case (i) should be replaced by

$$F(z) = f(z) \left(\prod_{i=1}^{p} \frac{1}{\phi_{a_i}(z)} \right) \left(\prod_{i=p+1}^{m} \frac{a_i}{a_i - z} \right) \left(\prod_{j=1}^{n} \phi_{b_j}(z) \right)$$

provided none of the poles lies on $|z| = 1$. In case some of the poles also lie on $|z| = 1$, then enumerate these poles as above, say

$$|b_1| \leq \cdots \leq |b_q| < 1, \quad |b_{q+1}| = \cdots = |b_m| = 1,$$

and set $F(z)$ by

$$F(z) = f(z) \left(\prod_{i=1}^{p} \frac{1}{\phi_{a_i}(z)} \right) \left(\prod_{i=p+1}^{m} \frac{a_i}{a_i - z} \right) \left(\prod_{j=1}^{q} \phi_{b_j}(z) \right) \left(\prod_{j=q+1}^{n} \frac{b_j - z}{b_j} \right).$$

Note that (as $|\phi_a(e^{i\theta})| = 1$ for each $a \in \Delta$)

$$|F(e^{i\theta})| = \left| f(e^{i\theta}) \left(\prod_{i=p+1}^{m} \frac{a_i}{a_i - e^{i\theta}} \right) \left(\prod_{j=q+1}^{n} \frac{b_j - e^{i\theta}}{b_j} \right) \right|$$

and

$$F(0) = f(0) \left(\prod_{i=1}^{p} \frac{1}{a_i} \right) \left(\prod_{j=1}^{q} b_j \right).$$

In either situation, $F \in \mathcal{H}(\overline{\Delta})$ and $F(z) \neq 0$ for $|z| \leq 1$. Thus,

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |F(e^{i\theta})| \, d\theta = \ln |F(0)|$$

and it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=p+1}^{m}\ln|a_i - e^{i\theta}| - \sum_{i=q+1}^{n}\ln|b_j - e^{i\theta}|\right)d\theta$$

$$= \ln\left|f(0)\left(\prod_{i=1}^{p}\frac{1}{a_i}\right)\left(\prod_{j=1}^{q} b_j\right)\right|.$$

Recall that if $|a| = 1$ then $\ln|a - e^{i\theta}| = \ln|1 - \overline{a}e^{i\theta}|$ and so, by Lemma 11.81, $\int_0^{2\pi}\ln|a - e^{i\theta}|\,d\theta = 0$ whenever $|a| = 1$. In view of this observation, the last equation becomes

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|f(e^{i\theta})|\,d\theta = \ln\left|f(0)\left(\prod_{i=1}^{p}\frac{1}{a_i}\right)\left(\prod_{j=1}^{q} b_j\right)\right|$$

which is clearly equivalent to (11.83). ■

11.85. Corollary. *Let f be meromorphic in $\overline{\Delta}_R$. Suppose that*

(i) 0 *is neither a zero nor a pole of* f

(ii) a_i $(1 \le i \le m)$ *and* b_j $(1 \le j \le n)$ *denote the zeros and poles of f in $\overline{\Delta}_R$ (counted as many times as its order of multiplicities), respectively.*

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|f(Re^{i\theta})|\,d\theta = \ln|f(0)| + \sum_{i=1}^{m}\ln\left(\frac{R}{|a_i|}\right) - \sum_{j=1}^{n}\ln\left(\frac{R}{|b_j|}\right).$$

Proof. It is sufficient to note that $f(z)$ is meromorphic in $\overline{\Delta}_R$ if and only if $g(z)$ defined by $g(z) = f(Rz)$ is meromorphic in $\overline{\Delta}$. Note also that $g(e^{i\theta}) = f(Re^{i\theta})$ and $a \in \overline{\Delta}_R$ iff $a/R \in \overline{\Delta}$. The result follows if we apply Theorem 11.82 for $g(z)$ and replace a_i and b_j by a_i/R and b_j/R, respectively. This gives

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|f(Re^{i\theta})|\,d\theta = \ln|f(0)| + \ln\left(\frac{\prod_{j=1}^{n}|b_j/R|}{\prod_{i=1}^{m}|a_i/R|}\right)$$

which is the desired formula. ■

Actually, rather than Theorem 11.82, Corollary 11.85 is often referred to as the Jensen formula for meromorphic functions. We may need to remember that $\ln(|a_i|/R) = 0$ if $|a_i| = R$ for some i. The same applies when $|b_j| = R$ for some j. Moreover, if f is analytic in $\overline{\Delta}_R$ then f is free from poles so that the Jensen formula in Corollary 11.85 takes a simple form

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|\,d\theta = \ln|f(0)| + \sum_{i=1}^{m} \ln\left(\frac{R}{|a_i|}\right).$$

There are several types of interesting functions in the theory of entire functions. The simplest of these is the function $\gamma(t)$, denoting the number of zeros of an analytic function $f(z)$ in $|z| \le t$. Now we draw an important corollary dealing with $\gamma(t)$.

11.86. Corollary. *Let $f \in \mathcal{H}(\overline{\Delta}_{2\rho})$ and $f(0) \neq 0$. Then*

$$\gamma(\rho)\ln 2 \le \frac{1}{2\pi}\int_0^{2\pi} \ln|f(2\rho e^{i\theta})|\,d\theta - \ln|f(0)| \le \ln\left(\frac{M(2\rho)}{|f(0)|}\right), \tag{11.87}$$

where $\gamma(t)$ denotes the number of zeros of $f(z)$ in the closed disk $|z| \le t$.

Proof. Jensen's formula applied to $f \in \mathcal{H}(\overline{\Delta}_R)$ tells us that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|\,d\theta - \ln|f(0)| = \sum_{|a_i|<R} \ln\left(\frac{R}{|a_i|}\right),$$

where $a_1, \dots, a_{\gamma(R)}$ are the zeros of f in $|z| \le R$, arranged in the nondecreasing order, $|a_1| \le |a_2| \le \cdots$ so that each zero being counted with multiplicity. If we consider only these zeros on the closed disk $|z| \le r$ $(r < R)$, then the sum on the right is nonincreasing, because we omit those zeros (if there is any) a_i with $r < |a_i| \le R$. Thus,

$$\begin{aligned}\sum_{|a_i|<R} \ln\left(\frac{R}{|a_i|}\right) &\ge \sum_{|a_i|\le r} \ln\left(\frac{R}{|a_i|}\right)\\ &\ge \sum_{|a_i|\le r} \ln(R/r) \quad (\because\ |a_i| \le r < R \ \to\ R/|a_i| \ge R/r)\\ &= \gamma(r)\ln(R/r).\end{aligned}$$

The first inequality in (11.87) follows if we replace R by 2ρ and r by ρ. The second inequality is clear because $|f(2\rho e^{i\theta})| \le M(2\rho)$. ■

Suppose that we know a bound for the maximum modulus of $f(z)$ on some circle $|z| = R$, and the value of the $f(z)$ at the origin. Then using Corollary 11.86, one can find an upper bound for $\gamma(r)$. From the proof of this corollary, it is also clear that if $r < R$, then one has

$$\gamma(r) \le \frac{1}{\ln(R/r)} \ln\left(\frac{M(R)}{|f(0)|}\right). \tag{11.88}$$

11.89. Example. Suppose that $f \in \mathcal{H}(\overline{\Delta}_R)$, $f(0) = 1 + i\sqrt{3}$ and $M(R) \leq 256$. Then, for $r < R$, (11.88) implies that

$$\ln\left(\frac{R}{r}\right) \cdot \gamma(r) \leq \ln\left(\frac{M(R)}{|f(0)|}\right) \leq \ln\left(\frac{256}{2}\right) = \ln(2^7) = 7\ln 2.$$

In particular, if $r = R/2$, then the last inequality gives that $\gamma(R/2) \leq 7$ and so f cannot have more than 7 zeros inside and on the circle of radius $R/2$. ●

11.8 The Order and the Genus of Entire Functions

The maximum modulus theorem plays an important role in the theory of entire functions. It asserts that if f is a non-constant entire function, then $M(r, f)$ defined by

$$M(r,f) := M(r) = \max_{|z|=r} |f(z)|$$

is an increasing function of r ($0 \leq r < \infty$). Also, by Liouville's theorem, f is unbounded. Moreover, $\lim_{r\to\infty} M(r) = \infty$. Otherwise, there exists a sequence $\{r_n\}$ with $r_n \to \infty$ such that $\{M(r_n)\}$ is bounded, say by M, for all large r_n. But then by the Cauchy integral formula applied to f for all points on $|z| \leq r_n/2$ gives

$$|f(z)| = \left|\frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right| \leq \frac{M}{2\pi}\left(\frac{1}{r_n - r_n/2}\right) 2\pi r_n = 2M$$

for large r_n, so that f is bounded for $|z| \leq r_n/2$. Taking r_n sufficiently large, it follows that $f(z)$ is bounded on $\mathbb{C}$, a contradiction. Recall that if $f_1(z) = e^z$, $f_2(z) = \cos z$ and $f_3(z) = \sin z$, then we have

$$M(r, f_1) = e^r, \quad M(r, f_2) = \frac{e^r + e^{-r}}{2}, \quad M(r, f_3) = \frac{e^r - e^{-r}}{2}$$

and, for each $k = 1, 2, 3$, we see that $M(r, f_k) \to \infty$ as $r \to \infty$. In view of this, it is interesting to know how fast $M(r)$ approaches infinity. Thus, the modulus of every transcendental entire function f grows faster than any fixed positive power of r.

The rate of growth of an entire function is of great importance and is characterized by comparing $M(r)$ with usual functions tending to infinity as $r \to \infty$. Therefore, it is natural to utilize exponential functions to measure the rate of growth of f, i.e. the growth of $M(r)$.

11.90. Theorem. *Let f be a non-constant entire function. Define*

$$\lambda_1(f) = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln r} \tag{11.91}$$

and

(11.92) $$\lambda_2(f) = \inf\{A \geq 0 : |f(z)| \leq e^{|z|^A} \text{ for sufficiently large } |z|\}.$$

Then $\lambda_1(f) = \lambda_2(f)$.

Proof. Let $0 \leq \lambda_1, \lambda_2 < \infty$. According to (11.92), for every $\epsilon > 0$, there exists an r_0 such that

$$|f(z)| \leq \exp\left(|z|^{\lambda_2+\epsilon}\right) \quad \text{for } |z| \geq r_0 \ .$$

This implies that

$$M(r) = \max_{|z|=r} |f(z)| \leq \exp\left(r^{\lambda_2+\epsilon}\right) \quad \text{for } |z| = r \geq r_0$$

so that for large r, we have

$$\ln\ln M(r) \leq (\lambda_2 + \epsilon)\ln r, \quad \text{i.e.} \quad \frac{\ln\ln M(r)}{\ln r} \leq \lambda_2 + \epsilon.$$

On the other hand, by definition,

$$M(r) > \exp\left(r^{\lambda_2-\epsilon}\right), \quad \text{i.e} \quad \frac{\ln\ln M(r)}{\ln r} > \lambda_2 - \epsilon$$

for an infinite number of r's, $r \to \infty$. Thus,

$$\lambda_2 = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln r}$$

and hence, $\lambda_1 = \lambda_2$.

Finally, $\lambda_1 = \infty$ iff $\lambda_2 = \infty$. Indeed, if $\lambda_2 = \infty$ then for any given $A > 0$,

$$M(r) > \exp(r^A) \quad \text{or} \quad \frac{\ln\ln M(r)}{\ln r} > A$$

for some values of r sufficiently large. Thus, $\lambda_2 = \infty$ iff $\lambda_1 = \infty$. ∎

We have the following

(i) the number $\lambda(f) = \lambda_1 = \lambda_2$ is called the order of $f(z)$. When there is no confusion, we may simply use the notation λ instead of $\lambda(f)$ as we did in the proof of the above theorem. So, $0 \leq \lambda < \infty$ as we work here only with functions of finite order.

(ii) Clearly, if $\lambda_2 < 0$, then $f(z)$ is bounded on $\mathbb{C}$ and so reduces to a constant, by Liouville's theorem. This is another reason why we have defined $\lambda_2(f)$ in (11.92) through nonnegative real numbers A satisfying the growth condition.

11.93. Example. For $z = re^{i\theta}$

(i) $|\exp(e^z)| = |\exp(e^{r\cos\theta}e^{ir\sin\theta})| = |\exp\left(e^{r\cos\theta}\cos(r\sin\theta)\right)|$ so that $M(r) = \exp(e^r)$ and

$$\frac{\ln\ln M(r)}{\ln r} = \frac{r}{\ln r} \to \infty \quad \text{as} \quad r \to \infty.$$

Therefore, $\lambda = \infty$ and we say that $f(z) = \exp(e^z)$ is of infinite order. Alternatively, as $x \to \infty$ along the positive real axis, we have

$$\exp(e^x) > \exp(x^k)$$

so that $\lambda \geq k$ for any $k \geq 0$. Consequently, by (11.92), $f(z) = \exp(e^z)$ is of infinite order.

(ii) If $f(z) = \exp(z^m)$ for some $m \in \mathbb{N}$, then $|f(z)| = |\exp(r^m\cos(m\theta))|$ so that $M(r) = \exp(r^m)$ which gives easily that $\lambda = m$, by (11.91). In particular, the order of e^z is 1 while the order of e^{z^2} is 2. More generally, if $p(z)$ is a polynomial of degree m (≥ 1) then the order of $\exp(p(z))$ is m.

(iii) If $f(z) = \cos z$, then

$$|f(z)| \leq |(e^{iz} + e^{-iz})/2| \leq (e^y + e^{-y})/2 \leq (e^{|z|} + e^{|z|})/2 = e^{|z|}$$

so that $\lambda \leq 1$, according to (11.92). But, for $z = iy$,

$$|f(iy)| = \left|(e^{-y} + e^y)/2\right| \geq e^{|y|}/2$$

which shows that λ is at least 1. Thus, the order of $\cos z$ is 1.

(iv) If $f(z) = \sin z$, then for $|z| = r$

$$|f(z)| = \left|\frac{e^{iz} - e^{-iz}}{2i}\right| \leq \frac{e^y + e^{-y}}{2} \leq e^{|z|}$$

so that $\lambda \leq 1$, according to (11.92). But, for $z = iy$

$$|f(iy)| = \left|(e^{-y} - e^y)/2\right| \geq (e^{|y|} - e^{-|y|})/2 \geq e^{|y|}/3 \quad \text{as} \quad y \to \infty$$

which shows that λ is at least 1. Thus, $\lambda(\sin z) = 1$.

(v) If $f(z) = a_0 + a_1z + \cdots + a_nz^n$, then, for $|z| = r > \max\{1, \sum_{k=0}^n |a_k|\}$,

$$\begin{aligned} |f(z)| &\leq (|a_0| + |a_1|r + \cdots + |a_n|r^n) \\ &< (|a_0| + |a_1| + \cdots + |a_n|)\, r^n \end{aligned}$$

and so, $|f(z)| < r^{n+1}$, i.e. $M(r) \leq r^{n+1}$. Thus,

$$\frac{\ln\ln M(r)}{\ln r} \leq \frac{\ln(n+1) + \ln\ln r}{\ln r} \to 0 \quad \text{as } r \to \infty$$

showing that every polynomial is of order 0. ●

Before we turn to the discussion on Hadamard's factorization theorem for entire functions, we recall the Weierstrass factorization theorem (see Theorem 11.43) which states that every entire function f can be factored in the form

$$f(z) = z^m e^{h(z)} \prod_{1 \le n \le \omega} E_{p_n}(z/a_n) \tag{11.94}$$

where

(i) h is entire

(ii) $\{a_n\}_{n=1}^{\omega}$ forms a sequence of zeros of f distinct from $z = 0$, each of them listed according to its multiplicity

(iii) $\omega \in \mathbb{N}$ if the sequence is finite and $\omega = \infty$ otherwise

(iv) $m = 0$ is allowed if $f(0) \neq 0$; otherwise m is the multiplicity of the zero of f at the origin

(v) $\{p_n\}_{n=1}^{\omega}$ is a sequence of non-negative integers such that

$$\sum_{1 \le n \le \omega} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty \quad \text{for every } r > 0.$$

Our main task is to deal with the situation $\omega = \infty$.

Let f be entire with zeros $\{a_n\}_{n \ge 1}$, listed according to multiplicity and arranged such that $0 < |a_1| \le |a_2| \le \cdots$. If p is *the smallest* nonnegative integer such that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$, then f is said to be of *rank* p. It is trivial to see that $p = 0$ whenever f has only a finite number of zeros. An entire function f is of *infinite rank*, i.e. if there exists no p for which $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$. This is possible whenever $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} = \infty$ for all $p \ge 0$.

If f is of finite rank p, i.e., there exists a nonnegative integer $p \ge 0$ such that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$, then, by (11.94), $f(z)$ can be rewritten as

$$f(z) = z^m e^{h(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n} \right) =: z^m e^{h(z)} P(z). \tag{11.95}$$

Here the product $P(z) := \prod_{n=1}^{\infty} E_p(z/a_n)$ is called the Weierstrass product or the *canonical product* associated with the sequence $\{a_n\}_{n \ge 1}$. Further, if f is of finite rank p, and p' is any integer with $p' > p$, then

$$|a_n|^{-(p'+1)} \le |a_n|^{-(p+1)}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} |a_n|^{-(p'+1)} < \infty$$

and so there is another product $\prod_{n=1}^{\infty} E_{p'}(z/a_n)$ showing that the factorization for f is not unique.

If p happens to be the rank of f, then p is called the *genus* of the canonical product $P(z)$. Moreover, the product $P(z)$ so defined is said to be in *standard form* for f. It is important to remember that $\sum_{n=1}^{\infty}|a_n|^{-p}=\infty$ whereas $\sum_{n=1}^{\infty}|a_n|^{-(p+1)}<\infty$. Further (as in Examples 11.45), in the representation (11.95), the factorization is unique except that $h(z)$ may be replaced by $h(z)+2m\pi i$ for any $m\in\mathbb{Z}$.

If p is the genus of the canonical product and $h(z)$ is a polynomial in the representation (11.95), then the function $f(z)$ is said to be of *finite genus* and, in addition, if q is the degree of the polynomial $h(z)$ then the $\mu:=\max\{p,q\}$ is defined to be the genus of $f(z)$. The number q is referred to as the exponential degree of $f(z)$. If $P(z)$ is not of finite rank or $h(z)$ is not a polynomial, then f is said to be of *infinite genus.* In the sequel, we are interested only on functions of finite genus. For example, an entire function of genus zero is of the form

$$cz^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

with $\sum_{n=1}^{\infty}|a_n|^{-1}<\infty$. By definition, the canonical representation of an entire function of genus 1 is either of the form

$$cz^me^{cz}\prod_{n=1}^{\infty}E_1\left(\frac{z}{a_n}\right),\quad c\in\mathbb{C},$$

with $\sum_{n=1}^{\infty}|a_n|^{-2}<\infty$, $\sum_{n=1}^{\infty}|a_n|^{-1}=\infty$, or of the form

$$cz^me^{cz}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

with $\sum_{n=1}^{\infty}|a_n|^{-2}<\infty$ and $c\neq 0$.

11.96. Examples.

(i) From the representation of $\sin\pi z$ shown in (11.50), we see that $\sin\pi z$ is an entire function of genus 1.

(ii) e^z is of genus 1 whereas e^{z^2} is of genus 2.

(iii) A polynomial $p(z)=a_0+a_1z+\cdots+a_nz^n$ is of genus 0. •

11.97. Lemma. *Let f be a nonconstant entire function of order λ, with zeros at $a_1,a_2,\ldots$, with counting multiplicities. Suppose that $0<|a_1|\leq|a_2|\leq\cdots$. If p is an integer with $p+1>\lambda$, then for $z\in\mathbb{C}\setminus\{a_1,a_2,\ldots\}$,*

$$\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right)=-p!\sum_{j=1}^{\infty}\frac{1}{(a_j-z)^{p+1}}.$$

Proof. First we assume $f(0) \not\equiv 0$. Let $a_1, a_2, \ldots, a_n$ be the zeros of f in Δ_R so that $f(z) \neq 0$ on $|z| = R$. Define

$$F(z) = f(z) \prod_{j=1}^{\gamma(R)} \frac{1}{\phi_{a_j/R}(z/R)}, \quad \phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Then F is a non-vanishing analytic function in $\overline{\Delta}_R$ with $|F(z)| = |f(z)|$ on $|z| = R$ and so there exists a $g \in \mathcal{H}(\Delta_R)$ such that $F(z) = \exp(g(z))$, $z \in \Delta_R$. This gives

$$f(z) \prod_{j=1}^{\gamma(R)} \frac{R^2 - \overline{a}_j z}{R(a_j - z)} = \exp(g(z)).$$

Taking the logarithmic derivative with respect to z, we obtain

$$\frac{f'(z)}{f(z)} - \sum_{j=1}^{\gamma(R)} \left[-\frac{1}{a_j - z} + \frac{\overline{a}_j}{R^2 - \overline{a}_j z} \right] = g'(z)$$

and so differentiating p-times yields that

$$\frac{d^p}{dz^p} \left(\frac{f'(z)}{f(z)} \right) = p! \sum_{j=1}^{\gamma(R)} \left[\frac{-1}{(a_j - z)^{p+1}} + \frac{\overline{a}_j^{p+1}}{(R^2 - \overline{a}_j z)^{p+1}} \right] + g^{(p+1)}(z),$$
(11.98)

$p = 0, 1, \ldots$. As for the second term on the right in (11.98), we first note that for $|z| < R/2$ and $|a_n| \leq R$,

$$|R^2 - \overline{a}_j z| \geq R^2 - |\overline{a}_j|\,|z| > R^2 - R^2/2 = R^2/2$$

and so

$$\left| \sum_{j=1}^{\gamma(R)} \frac{\overline{a}_j^{p+1}}{(R^2 - \overline{a}_j z)^{p+1}} \right| \leq \frac{R^{p+1}}{(R^2/2)^{p+1}} \gamma(R) \leq \frac{2^{p+1}}{R^{(p+1)}} (AR^{\lambda+\epsilon} + B)$$

for some constants A and B. Here, as f is of order λ, we have used the estimate (see (11.88))

$$\gamma(R) \leq AR^{\lambda+\epsilon} + B \quad \text{for } R \text{ sufficiently large.}$$

As $p+1-\lambda > 0$, if we choose ϵ so that $\lambda + \epsilon < p+1$ (e.g. $\epsilon = (p+1-\lambda)/2$), $2^{p+1}\gamma(R)R^{-(p+1)}$ approaches zero as $R \to \infty$; that is the second sum in (11.98) converges to zero.

As for the last term, we note that $\operatorname{Re} g(z) = \ln |F(z)|$ for $z \in \Delta_R$ and $\operatorname{Re} g(z) = \ln |f(z)|$ for $|z| = R$. By Theorem 10.34, g has the form

$$g(z) = i\operatorname{Im} g(0) + \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) \ln |f(Re^{i\theta})|\, d\theta.$$

Thus, for $|z| < R$

$$g^{(p+1)}(z) = \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \ln|f(Re^{i\theta})|\, d\theta. \tag{11.99}$$

For each fixed z with $|z| < R/2$, the Cauchy integral formula applied to the function $\psi(w) = (w - z)^{-p-2}$ shows that

$$0 = \frac{(p+1)!}{2\pi i} \int_{|w|=R} \frac{dw}{(w-z)^{p+2}} = \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}}\, d\theta.$$

In view of this, for $|z| < R/2$

$$\begin{aligned} |g^{(p+1)}(z)| &= \left| \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \big(\ln|f(Re^{i\theta})| - \ln M(R)\big)\, d\theta \right| \\ &\leq \frac{(p+1)!R}{\pi(R - |z|)^{p+2}} \int_0^{2\pi} \big(\ln M(R) - \ln|f(Re^{i\theta})|\big)\, d\theta \\ &\leq \frac{(p+1)!R}{\pi} \left(\frac{2}{R}\right)^{p+2} \int_0^{2\pi} (\ln M(R) - \ln|f(0)|)\, d\theta \end{aligned}$$

on account of $R - |z| > R - R/2 = R/2$ and the Jensen inequality

$$\frac{1}{2\pi} \int_0^{1\pi} \ln|f(Re^{i\theta})|\, d\theta \geq \ln|f(0)|.$$

Therefore, since $\ln M(R) < R^{\lambda+\epsilon}$ for large R and for a given $\epsilon > 0$, we have

$$\begin{aligned} |g^{(p+1)}(z)| &\leq \frac{(p+1)!}{\pi} \left(\frac{2}{R}\right)^{p+2} (R^{\lambda+\epsilon} - \ln|f(0)|)2\pi R \\ &= (p+1)!2^{p+3}(R^{\lambda+\epsilon} - \ln|f(0)|)R^{-(p+1)} \\ &\to 0 \ \text{ as } R \to \infty \ (\text{ since } (\lambda+\epsilon)/(p+1) < 1). \end{aligned}$$

Finally, as $R \to \infty$, (11.99) gives the desired formula whenever $f(0) \neq 0$. If $f(0) = 0$ and zero is of multiplicity m, then we write

$$f(z) = z^m G(z), \quad G(0) \neq 0.$$

Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \frac{G'(z)}{G(z)}.$$

Differentiate p times to complete the argument. ∎

11.100. Theorem. (Hadamard's Factorization Theorem) *Let f be an entire function of finite order λ. Suppose that $a_1, a_2, \ldots,$ are the zeros* *of $f(z)$ listed with multiplicities and $0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots$. Then f has a finite genus μ satisfying the inequality $\mu \le \lambda$.*

Proof. Let f have an order λ and $p = [\lambda]$. Then, $p \le \lambda < p+1$ and for any small $\epsilon > 0$

$$\ln M(r) \le r^{\lambda+\epsilon/2} \quad \text{for } r \text{ sufficiently large.}$$

If f has a zero of order m at the origin, then for $|z| = r$

$$\begin{aligned} \ln |f(z)/z^m| &\le \ln(M(r)r^{-m}) \\ &\le r^{\lambda+\epsilon/2} - m \ln r \\ &\le r^{\lambda+\epsilon} \quad \text{for } r \text{ sufficiently large.} \end{aligned}$$

So $F(z) = f(z)z^{-m}$ is an entire function of order λ with no zero at the origin and $F(0) = \lim_{z\to 0} f(z)z^{-m} = f^{(m)}(0)/m!$. Thus, by Corollary 11.86,

$$\gamma(r) \le \frac{1}{\ln 2}((2r)^{\lambda+\epsilon} - \ln |F(0)|) = Ar^{\lambda+\epsilon} + B, \tag{11.101}$$

where $A = 2^{\lambda+\epsilon}/\ln 2$ and $B = -\ln |F(0)|/\ln 2$, independent of r. Therefore, taking $r = |a_n|$, we obtain

$$n \le \gamma(|a_n|) \le A|a_n|^{\lambda+\epsilon} + B \quad \text{for large } n.$$

In particular,

$$\frac{1}{|a_n|^{p+1}} \le \left(\frac{A}{n-B}\right)^{(p+1)/(\lambda+\epsilon)} \quad \text{for large } n \tag{11.102}$$

and if we choose ϵ (e.g. $\epsilon = (p+1-\lambda)/2$) so that $p+1 > \lambda+\epsilon$ (recall that $p+1-\lambda > 0$), the series $\sum |a_n|^{-p-1}$ is dominated by a convergent series. Therefore,

$$\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$$

and so, $f(z)$ can be written in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(z/a_n) =: z^m e^{g(z)} P(z), \quad p = [\lambda], \tag{11.103}$$

where $m = 0$ when $f(0) \ne 0$. Next, we claim that $g(z)$ is a polynomial of degree $\le \lambda$.

First we let $P_N = \prod_{j=1}^{N} q_j(z)$ be the Nth partial product of $P(z)$. Note that $P_N(z) \to P(z)$ locally uniformly on $\mathbb{C}$ and so, $P_N'(z) \to P'(z)$ locally uniformly on $\mathbb{C}$. Consequently,

$$\frac{P_N'(z)}{P_N(z)} = \sum_{j=1}^{N} \frac{q_j'(z)}{q_j(z)} \to \frac{P'(z)}{P(z)} = \sum_{j=1}^{\infty} \frac{q_j'(z)}{q_j(z)}$$

where the series on the right converges uniformly on every compact subset of $\mathbb{C}$ not containing the zeros of $P(z)$. Also, we note that

$$q_j(z) = E_p(z/a_j) = (1 - z/a_j)\exp\left(\frac{z}{a_j} + \frac{z^2}{2a_j^2} + \cdots + \frac{z^p}{pa_j^p}\right)$$

so that

$$\frac{q_j'(z)}{q_j(z)} = -\frac{1}{a_j - z} + \frac{1}{a_j} + \frac{z}{a_j^2} + \cdots + \frac{z^{p-1}}{a_j^p}$$

and

$$\frac{d^p}{dz^p}\left(\frac{q_j'(z)}{q_j(z)}\right) = -\frac{p!}{(a_j - z)^{p+1}}.$$

Thus

$$\frac{d^p}{dz^p}\left(\frac{P'(z)}{P(z)}\right) = -p!\sum_{j=1}^{\infty}\frac{1}{(a_j - z)^{p+1}}, \quad z \neq a_1, a_2, \ldots . \tag{11.104}$$

Finally, by (11.103), we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \frac{P'(z)}{P(z)} + g'(z).$$

Differentiating both sides p times and applying Lemma 11.97 and (11.104) gives $g^{(p+1)}(z) = 0$ and g is a polynomial of degree at most p. In particular, the genus μ of f is less than p. Since $p \leq \lambda$, we have $\mu \leq \lambda$. ∎

The Hadamard factorization theorem is often stated in the following equivalent form.

11.105. Theorem. (Hadamard's Factorization Theorem) *Let f be an entire function of finite order λ. Suppose that $a_1, a_2, \ldots,$ are the zeros of $f(z)$ listed with multiplicities and $0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots$. Then there exists a polynomial $g(z)$ of degree not greater than λ, and a nonnegative integer p ($p \leq \lambda$) such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(z/a_n).$$

The converse of Theorem 11.100 is also true. In fact, we have

11.106. Theorem. *If f is an entire function of finite genus μ, then f is of finite order λ and $\lambda \leq \mu + 1$.*

Proof. If f is of finite genus μ, then

$$f(z) = z^m e^{h(z)} P(z), \quad P(z) = \prod_{n=1}^{\infty} E_\mu(z/a_n)$$

where the degree q of the polynomial $h(z)$ is $\leq \mu$. We note that the order of the product of the two entire functions cannot exceed the order of the each of the factors. In view of this observation,

$$\lambda(f) \leq \max\{\lambda(e^{h(z)}), \lambda(P(z))\}.$$

Since f is of finite genus μ, the order of $\exp(h(z))$ is q which is $\leq \mu$. Hence, to complete the proof, it suffices to show that the order of the canonical product $P(z)$ is $\leq \mu + 1$. Since $\mu = \max\{q, p\} \geq p$ and the convergence of the product $P(z)$ implies $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$, it follows that $\sum_{n=1}^{\infty} |a_n|^{-(\mu+1)} < \infty$. First we claim that

$$\ln |E_\mu(z)| \leq (1+\mu)|z|^{\mu+1}, \quad z \in \mathbb{C}. \tag{11.107}$$

Note that $1 + |z| \leq e^{|z|}$, i.e. $\ln(1 + |z|) \leq |z|$ for $z \in \mathbb{C}$. Consequently

$$\ln |E_0(z)| = \ln |1 - z| \leq \ln(1 + |z|) \leq |z|, \quad z \in \mathbb{C},$$

showing that the estimate (11.107) holds when $\mu = 0$. Next we prove (11.107) for $\mu \geq 1$. To do this, we recall Lemma 11.39:

$$-1 + |E_\mu(z)| \leq |E_\mu(z) - 1| \leq |z|^{\mu+1} \ \text{ for } |z| \leq 1$$

so that $|E_\mu(z)| \leq 1 + |z|^{\mu+1}$ for $|z| \leq 1$. Consequently, for each $\mu \geq 1$

$$\ln |E_\mu(z)| \leq \ln(1 + |z|^{\mu+1}) \leq |z|^{\mu+1} \leq (\mu+1)|z|^{\mu+1} \ \text{ for } |z| \leq 1. \tag{11.108}$$

For arbitrary $z \in \mathbb{C}$ with $|z| \geq 1$, and $\mu \geq 1$

$$|E_\mu(z)| = |1 - z|\, |e^z|\, |e^{z^2/2}| \cdots |e^{z^\mu/\mu}| \leq (1 + |z|) e^{|z|} e^{|z|^2/2} \cdots e^{|z|^\mu/\mu}$$

so that, for all $|z| \geq 1$,

$$\begin{aligned}
\ln |E_\mu(z)| &\leq \ln(1 + |z|) + \sum_{k=1}^{\mu} \frac{|z|^k}{k} \\
&\leq |z| + \sum_{k=1}^{\mu} \frac{|z|^\mu}{k} \quad (\text{because } |z|^k \leq |z|^\mu \text{ for } k = 1, 2, \ldots, \mu) \\
&\leq \left(1 + \sum_{k=1}^{\mu} \frac{1}{k}\right) |z|^\mu \leq (1+\mu)|z|^\mu \\
&\leq (1+\mu)|z|^{\mu+1}.
\end{aligned}$$

Combining the last inequality with (11.108) proves (11.107). The estimate (11.107) gives at once

$$\ln |P(z)| = \sum_{k=1}^{\infty} \ln |E_\mu(z/a_n)| \leq (1+\mu)|z|^{\mu+1} \sum_{n=1}^{\infty} |a_n|^{-(\mu+1)}$$

and so, there exists an $\alpha > 0$ with $\ln|P(z)| \leq \alpha|z|^{\mu+1}$ for all $z \in \mathbb{C}$. It follows that $P(z)$ is at most of order $\mu + 1$, i.e. $\lambda \leq \mu + 1$. ∎

Finally, we obtain the following result which exhibits the strength of Hadamard's factorization Theorem.

11.109. Corollary. *An entire function of fractional order assumes every complex value infinitely often.*

Proof. Let f be an entire function of order λ, where $\lambda \notin \mathbb{N}$. Clearly, $\lambda(f) = \lambda(f - a)$ for any constant $a \in \mathbb{C}$. Therefore, it suffices to show that f has infinitely many roots. Suppose not. Then f has only finitely many zeros, namely $a_1, a_2, \dots, a_n$. Then f is of the form

$$f(z) = e^{h(z)} \prod_{k=1}^{n} (z - a_k) =: e^{h(z)} P(z).$$

By Hadamard's factorization theorem, h is a polynomial of degree $m \leq \lambda(f)$. But then $\lambda(f) = \lambda(e^{h(z)}) = m$. This contradiction proves the corollary. ∎

Thus, every entire function of finite order has either infinitely many zeros or else $f(z) = e^{h(z)} P(z)$, where $h(z)$ and $P(z)$ are some polynomials.

11.110. Convergence exponent. Let $\{a_n\}_{n\geq 1}$ be a sequence of nonzero complex numbers, listed according to increasing moduli such that $|a_n| \to \infty$ as $n \to \infty$. Then the convergence exponent of $\{a_n\}_{n\geq 1}$ is defined by

$$\sigma = \inf \left\{ \alpha > 0 : \sum_{n=1}^{\infty} |a_n|^{-\alpha} < \infty \right\}.$$

Thus, for each $\epsilon > 0$, one has

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\alpha+\epsilon}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^{\alpha-\epsilon}} = \infty.$$

For example, if $a_n = n$ for all $n \in \mathbb{N}$, then we have $\sigma = 1$. Also, we have

(i) If $\sum_{n=1}^{\infty} |a_n|^{-\alpha} = \infty$ for all $\alpha > 0$, then we set $\sigma = +\infty$ as the infimum of an empty set.
(ii) If $\sum_{n=1}^{\infty} |a_n|^{-\alpha} < \infty$ for all $\alpha > 0$, then $\sigma = 0$.
(iii) From the proof of Theorem 11.100, we have the following: if f is an entire function of order λ and $a_1, a_2, \dots,$ are the zeros of $f(z)$ listed with multiplicities and $0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots$, then

$$\sum_{n=1}^{\infty} |a_n|^{-\alpha} < \infty \quad \text{for } \alpha > \lambda.$$

Indeed, let β be a number such that $\lambda < \beta < \alpha$. Then, for small $\epsilon > 0$ and for large n, we have

$$n \leq \gamma(|a_n|) \leq A|a_n|^{\beta+\epsilon} + B, \text{ i.e. } \frac{1}{|a_n|^\alpha} \leq \left(\frac{A}{n-B}\right)^{\alpha/(\beta+\epsilon)}.$$

The conclusion follows, as $\alpha/\beta > 1$.

If f is an entire function having $\{a_n\}_{n\geq 1}$ as its non-zero zeros, then $\sigma := \sigma(f)$ defined as above is called the *convergence exponent* for the zero-sequence $\{a_n\}_{n\geq 1}$ of f.

11.111. Examples.

(i) For $c > 0$, set $a_n = n^{1/c}$ for all $n \in \mathbb{N}$. Then, as

$$\sum_{n=1}^{\infty} |a_n|^{-\alpha} = \sum_{n=1}^{\infty} n^{-\alpha/c} < \infty \text{ if } \alpha > c,$$

the exponent of convergence σ of $\{n^{1/c}\}$ is c. Note that

$$\sum_{n=1}^{\infty} |a_n|^{-\sigma} = \sum_{n=1}^{\infty} |a_n|^{-c} = \sum_{n=1}^{\infty} n^{-1} = \infty.$$

(ii) If $f(z) = \sin z$, then the zeros of $f(z)$ are $a_n = n\pi$ $(n \in \mathbb{Z})$ and so

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} |a_n|^{-\alpha} = 2\sum_{n=1}^{\infty} |a_n|^{-\alpha} = \frac{2}{\pi^\alpha}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$$

showing that $\sigma(f) = 1 = \lambda(f)$. If $g(z) = \cos z$, then $\sigma(g) = \lambda(g) = 1$.

(iii) Since e^z has no zero in $\mathbb{C}$, we set that $\sigma(e^z) = 0$. •

11.9 Exercises

11.112. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The meromorphic functions $\sin z/[e^{2iz} + 1]$ and $1/(2i \cos z)$ differ by an entire function.

(b) If f is meromorphic and has an infinite number of poles, then every closed disk $|z| \leq R$ $(0 < R < \infty)$ contains only a finite number of poles inside it.

(c) There can be no meromorphic function in $\mathbb{C}$ with poles at $1/n$, $n \in \mathbb{N}$.

(d) If f is entire and $f(z) \not\equiv 0$, then either f has a finite number of zeros or has an infinite number of zeros a_n with $a_n \to \infty$ as $n \to \infty$.

(e) An infinite product $\prod_{k=1}^{\infty}(1+a_k)$ is divergent iff the sequence of partial products either diverges to infinity or converges to zero or oscillates.

(f) If f is a non-constant entire function such that $f(z) \neq 0$ in $\mathbb{C}$, then $f'(z)$ may or may not have a zero in $\mathbb{C}$.

(g) The infinite product $\prod_{k=1}^{\infty}(1+a_k)$ converges iff the series $\sum_{k=1}^{\infty} a_k$ converges.

(h) For $-1 < a_k \leq 1$ for $k \in \mathbb{N}$, the product $\prod_{k=1}^{\infty}(1+a_k)$ converges iff the series $\sum_{k=1}^{\infty} a_k$ converges.

(i) The convergence of the product $\prod_{k=1}^{\infty}(1+a_k)$ is necessary and sufficient for the absolute convergence of the series $\sum_{k=1}^{\infty} a_k$.

(j) Each of the products $\prod_{k=1}^{\infty}\left(1+\frac{1}{2^{k+1}-1}\right)$ and $\prod_{k=2}^{\infty}\left(1+\frac{1}{k^2-1}\right)$ converges.

(k) For $0 \neq \alpha \in \mathbb{C}$, the infinite product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^{\alpha}}\right)$ converges absolutely provided $\operatorname{Re}\alpha > 1$.

(l) The infinite product $\prod_{n=1}^{\infty}(1+a^n z)$, $|a| < 1$, defines an entire function.

(m) If R is the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$, then the product $\prod_{n=1}^{\infty}(1+a_n z^n)$ converges for $|z| < R$.
Note: As the series $\sum_{n=1}^{\infty} |z^{3^n}|$ converges for $|z| < 1$, the product $\prod_{n=1}^{\infty}(1+z^{3^n})$ converges for $|z| < 1$.

(n) For $z \in \mathbb{C}\backslash\{x+i0 : x \leq 0\}$, $\prod_{n=1}^{\infty} z^{2^{-n}}$ converges to z whereas $\prod_{n=1}^{\infty} z^{3^{-n}}$ does not converge to z.

(o) Let $p(z)$ be a non-constant polynomial in z (e.g. z, z^2). If the series $\sum_{k=1}^{\infty} |a_k|$ is convergent (e.g. $a_k = 1/k^{\lambda}$ with $\lambda > 1$), then the product $\prod_{k=1}^{\infty}(1+p(z)a_k)$ is convergent for all $z \in \mathbb{C}$ and represents an entire function.

(p) The product $\prod_{k=1}^{\infty}\left(1+\frac{2}{k^z+k^3-1}\right)$ represents an analytic function for $\operatorname{Re} z > 3$.

(q) The region of absolute convergence of each of $\prod_{n=1}^{\infty}\left(1+z^{n^2}\right)$ and $\prod_{n=1}^{\infty}\left(1+z/n^2\right)$ is different.

(r) The product $\prod_{n=1}^{\infty}(1-z/n^2)$ represents an entire function whereas the product $\prod_{n=1}^{\infty}(1-z/n)$ does not although $\prod_{n=1}^{\infty}(1-z/n)e^{z/n}$ is an entire function.

(s) There exists an entire function which has zeros of multiplicity n at $z = n$ $(n \in \mathbb{N})$, and no other zeros.

(t) The product $\prod_{n=1}^{\infty}(1+n^{-z})$ converges uniformly on every compact subset $\operatorname{Re} z \geq 1+\delta$ $(\delta > 0)$ and represents an analytic function in the half-plane $\operatorname{Re} z > 1$.

(u) The product $\prod_{n=1}^{\infty}\left(1-\frac{2z}{n^2+z}\right)$ represents a bounded analytic function on the right half-plane $\operatorname{Re} z > 0$.

(v) The terms of an absolutely convergent product can be rearranged without affecting the convergence or the value of the product.

(w) There exist entire functions with simple zeros at n^2, $n \in \mathbb{N}$.

(x) There exists no non-constant entire function having zeros at $z = 1/n$ (or $1/n^2$ or $1/n^3$), $n \in \mathbb{N}$.

(y) An entire function, which has simple zeros at 0, $\pm n^{1/4}$ ($n \in \mathbb{N}$) and no other zeros, is given by

$$z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\sqrt{n}}\right)\exp\left(\frac{z^2}{\sqrt{n}}+\frac{z^4}{2n}\right).$$

(z) There exists an analytic function f in the unit disk such that $f(z) = 0$ iff $z = 1 - 1/n$, $n \in \mathbb{N}$.

11.113. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) The order of entire functions $f(z)$ and $cf(z)$ ($c \neq 0$) are the same.

(b) The entire functions $f(z)$ and $z^{-n}f(z)$ ($n \in \mathbb{N}$) have the same order, where the point $z = 0$ is a zero of multiplicity n for $f(z)$.

(c) The order of the entire function $f(z) = ze^z$ is 1.

(d) For any two entire functions f and g, $\lambda(f+g) \leq \max\{\lambda(f), \lambda(g)\}$.

(e) For any two entire functions f and g, $\lambda(f+g) = \max\{\lambda(f), \lambda(g)\}$ whenever $\lambda(f) \neq \lambda(g)$.

(f) If f and g are two entire functions such that $\lambda(f) > \lambda(g)$, then $\lambda(f+g) = \lambda(f)$.

(g) For any two entire functions f and g, $\lambda(fg) \leq \max\{\lambda(f), \lambda(g)\}$.

(h) If f is an entire function and $h(z) = f(az)$, then $\lambda(h) = \lambda(f)$.

(i) If f is an entire function and $h(z) = f(z^n)$, then $\lambda(h) = n\lambda(f)$.

(j) If f is an entire function and $h(z) = z^n f(z)$, then $\lambda(h) = \lambda(f)$.

(k) The exponent of convergence of $\prod_{n=1}^{\infty} E_1(z/n)$ is 1.

(l) The order of $z^m \prod_{n=1}^{\infty} E_1(z/n)$ ($m \in \mathbb{N}$) is 1.

(m) If $\lambda(f) \neq 0$ and $\lambda(g) \neq 0$, then $\lambda(f+g)$ is not necessarily a non-zero number.

(n) There are infinitely many prime numbers.

(o) If R is the rectangle with vertices at $\pm n \pm (2n+1/2)\pi i$, then one has $|e^z - 1| > 1/2$ for $z \in \partial R$.

11.114. Construct meromorphic functions f in $\mathbb{C}$ with the following properties:

(i) simple poles only at $a_n = n \in \mathbb{N}$ with $\text{Res}\,[f(z); a_n] = n$

(ii) simple poles only at $a_n = \sqrt{n}$, $n \in \mathbb{N}$, with $\text{Res}\,[f(z); a_n] = 1$

(iii) simple poles only at $a_n = n$, $n \in \mathbb{Z}$, with $\text{Res}\,[f(z); a_n] = 1$

(iv) simple poles only at $a_n = n(1+i)$, $n \in \mathbb{Z} \setminus \{0\}$, with $\text{Res}\,[f(z); a_n] = 1$

(v) poles only at $a_n = n \in \mathbb{N}$ of order n.

11.115. Set $a_n = (-1)^{n+1}/\sqrt{n}$. Show that $\prod_{n=1}^{\infty}(1 + a_n)$ diverges even though $\sum_{n=1}^{\infty} a_n$ converges. Does this provide an example of a series that is convergent but not absolutely?

11.116. Construct entire functions with the following properties

(i) simple zeros at $a_n = -n$, $n \in \mathbb{N}$ and no other zeros

(ii) simple zeros at $a_n = n$, $n \in \mathbb{N}$ and no other zeros

(iii) simple zeros at $a_n = \sqrt{n}$, $n \in \mathbb{N}$ and double zeros at $b_n = \pm i\sqrt{n}$, $n \in \mathbb{N}$, and no other zeros

(iv) simple zeros at $a_n = n^{5/4}$, $n \in \mathbb{N}$ and no other zeros

(v) simple zeros at $a_n = n^{4/5}$, $n \in \mathbb{N}$ and no other zeros

(vi) simple zeros at $a_n = n^{1/2}$, $n \in \mathbb{N}$ and no other zeros.

11.117. Suppose that the product $\prod_{k=1}^{\infty}(1+a_k)$ converges whereas the product $\prod_{k=1}^{\infty}(1 + |a_k|)$ diverges. What can we say about the sequence?

11.118. Show that product $\prod_{k=1}^{\infty}(1 + f_k(z))$ is uniformly convergent in a domain Ω if the series $\sum_{k=1}^{\infty} |f_k(z)|$ converges uniformly in Ω. Using this result, discuss the convergence of the product $\prod_{k=1}^{\infty}(1 + f_k(z))$ when

(i) $f_k(z) = (k/(k+1))^k\, z^k$, $z \in \Delta$

(ii) $f_k(z) = (1-z)/(1-z^{2k})$, $z \in \Delta$

(iii) $f_k(z) = (1-z^{2k})/(1-z)$, $z \in \Delta$

(iv) $f_k(z) = z^k/k!$, $z \in \mathbb{C}$

(v) $f_k(z) = z/[k(\ln k)^2]$, $z \in \mathbb{C}$.

11.119. Let f be meromorphic in $\overline{\Delta}_R$. Suppose that

(i) a is neither a zero nor a pole of f

(ii) a_i $(1 \le i \le m)$ and b_j $(1 \le j \le n)$ denote, respectively, the zeros and poles of f in $\overline{\Delta}_R$ (counted as many times as its order of multiplicities).

Then show the Poisson-Jensen Formula of the form

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{R+ae^{-i\theta}}{R-ae^{-i\theta}}\right)\ln|f(Re^{i\theta})|\,d\theta$$

$$= \ln|f(a)| + \sum_{i=1}^{m}\ln\left|\frac{R^2-\overline{a}_i a}{R(a_i-a)}\right| - \sum_{j=1}^{n}\ln\left|\frac{R^2-\overline{b}_j a}{R(b_j-a)}\right|$$

$\phi_a(z) = (a-z)/(1-\overline{a}z)$ (Note that if f is free from poles, then the last term on the right does not appear in the formula). Also, discuss what modification is required if a is a zero or a pole of $f(z)$.

11.120. Give an example of an entire function $f(z)$ whose order is not an integral number.

11.121. Prove that, whenever $\alpha \notin \mathbb{Z}$,

$$\sin\pi(z+\alpha) = e^{\pi z\cot\pi\alpha}\sin\pi\alpha \prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}.$$

11.122. Find the order and genus of the entire function

$$f(z) = \prod_{n=2}^{\infty}\left(1-\frac{z}{n(\ln n)^2}\right).$$

11.123. For $a>0$, consider the Hurwitz zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}.$$

Show that $\zeta(s,a)$ is analytic for $\mathrm{Re}\, s>1$. Also show that

$$\zeta(s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-(a-1)sx}x^{s-1}}{e^x-1}\,dx.$$

11.124. Prove the Gauss multiplication formula

$$(2\pi)^{(n-1)/2}\Gamma(nz) = n^{nz-1/2}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right).$$

11.125. For $s\in\mathbb{C}$, prove the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s).$$

Chapter 12

Mapping Theorems

We shall discuss a number of interesting results concerning certain mapping problems between domains. Section 12.1 begins with the open mapping theorem which is a celebrated result about analytic mappings. In Section 12.2, we study some basic results on univalent functions. Section 12.3 is devoted to a preliminary discussion on normal families. In addition, we also prove Montel's theorem for normal families of analytic functions. The main result in Section 12.4 is the Riemann mapping theorem which asserts that every simply connected domain of the complex plane having at least two boundary points can be mapped conformally onto the open unit disk. In Section 12.5, we state the celebrated conjecture due to Bieberbach which led to the development of a great number of different and deep methods that have solved a large number of problems in function theory. Since the confirmation of the Bieberbach conjecture by de Branges, one of the outstanding open problem in complex analysis is that of finding the exact value of the Bloch constant. Our final section discusses this constant along with the long awaited Picard's little theorem and Schottky's theorem.

12.1 Open Mapping Theorem and Hurwitz' Theorem

In the subject of topology, continuity of f on $\Omega \subseteq \mathbb{C}$ is equivalent to saying that the inverse image of every open set in $f(\Omega)$ under f is open. We are interested now in these functions for which the direct image of any open set is open. A function defined on an open set D is said to be an *"open mapping"* if for every open subset Ω of D, the image $f(\Omega)$ is open. Thus if there exists an open set in D whose image under f is not open, then f is not an open mapping in D. Consider $f_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, 3)$ defined by

$$f_1(x) = x^2, \; f_2(x) = \sin x \;\text{ and }\; f_3(x) = \frac{e^x + e^{-x}}{2},$$

respectively. Clearly,

$$f_1(\mathbb{R}) = [0, \infty), \; f_2((0, \pi)) = (0, 1] \text{ and } f_3(\mathbb{R}) = [1, \infty)$$

showing that for each j, each of the real-valued functions f_j of a real variable f_j is not an open mapping. Our emphasize will be on plane domains in $\mathbb{C}$, and the above examples show that the following theorem has no analog in $\mathbb{R}$.

12.1. Theorem. (Open Mapping Theorem) *If f is a non-constant analytic function on a domain D, then f is an open mapping, i.e. $f(D)$ is an open set in $\mathbb{C}$.*

Clearly, the name of this theorem is derived from the property of "openness". What does this theorem convey? If D is a domain in $\mathbb{C}$ and $f \in \mathcal{H}(D)$, then $f(D)$ is either a domain or a single point; i.e. $f(D)$ is a domain or else f is a constant function. For example, this theorem prohibits a non-constant C^∞-analytic mapping of a disk onto a portion of a line (see Corollary 12.3). Is there an analogous result for real-valued C^∞ functions defined on $D \subset \mathbb{C}$? Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$. Then

$$f(\mathbb{C}) = \{x + i0 \in \mathbb{C} : \, x \geq 0\}$$

which is not an open subset of $\mathbb{C}$. Note that f is nowhere analytic but is real-differentiable because $f(x, y) = x^2 + y^2$ and the partial derivatives of all orders exist and are continuous on $\mathbb{R}^2$. On the other hand, $f(z) = z^2$ is an open mapping on $\mathbb{C}$. Is there a non-constant complex-valued function (need not be analytic) defined on a domain that is an open map? How about $f(z) = \overline{z}$, $z \in \mathbb{C}$?

Proof of the open mapping theorem. We shall show that if f is a non-constant analytic function in D and Ω is an open subset of D containing a, then $f(\Omega)$ contains an open disk about $f(a)$. Since zeros of the non-vanishing analytic function $f(z) - f(a)$ are isolated, there exists a disk $\Delta(a; r)$ with $\overline{\Delta}(a; r) \subset \Omega$ such that

$$f(z) - f(a) \neq 0 \text{ in } 0 < |z - a| < r.$$

In particular, $f(\zeta) - f(a) \neq 0$ for $\zeta \in C = \partial\Delta(a; \rho)$ where $0 < \rho < r$. Let

$$2m = \min_{\zeta \in C} |f(\zeta) - f(a)|.$$

Then $m > 0$. Further, for every $w \in \Delta(f(a); m)$, we note that

$$|f(\zeta) - w| \geq |f(\zeta) - f(a)| - |f(a) - w| > 2m - m = m > |f(a) - w|$$

for all $\zeta \in C$. Rewrite the last inequality as

$$|f(a) - w| = |(f(\zeta) - w) - (f(\zeta) - f(a))| < |f(\zeta) - w| \quad \text{for all } \zeta \in C.$$

It follows from Rouché's theorem that the functions $f(\zeta)-w$ and $f(\zeta)-f(a)$ have the same number of zeros inside the circle C. But as $f(\zeta)-f(a)$ has at least one zero inside C, $f(\zeta)-w$ has at least one zero inside C. Hence there exists $z' \in \Delta(a;\rho)$ such that $f(z')=w$ and so w is in the range of f, w being an arbitrary element of $\Delta(f(a);m)$ the assertion is true. ■

As the connectedness of D implies the connectedness of $f(D)$ (see Theorem 2.24), the open mapping theorem is often formulated in the following form:

12.2. Theorem. *A non-constant analytic function maps the domain D onto the domain $f(D)$.*

Proof. By the open mapping theorem $f(D)$ is open; so we only need to show that $f(D)$ is connected. We provide a direct proof to show that $f(D)$ is connected. Let $w_1, w_2 \in f(D)$. Then there exist $z_1, z_2 \in D$ such that $f(z_1)=w_1$ and $f(z_2)=w_2$. Because D is connected, z_1 and z_2 can be connected by a finite number of line segments that lie entirely within D. The image of each line segment under f is always a curve in $f(D)$, since f is differentiable in D. It follows that w_1 and w_2 can be connected by a curve in $f(D)$. Note that this curve can be approximated by line segments in $f(D)$. ■

12.3. Corollary. *Theorem* 3.31(ii) *follows from Theorem* 12.2.

Proof. The hypothesis of Theorem 3.31(ii) shows that, in the w-plane, $f(D)$ is either a subset of a circle or $u=$ constant, or $v=$ constant, or $\tan^{-1}(v/u)=$ constant, respectively. However, we note that none of these takes open sets into open sets. It follows from Theorem 12.2 that in each case, f must be a constant. This completes the proof. ■

Here is another application of Rouché's Theorem.

12.4. Theorem. (Hurwitz' Theorem) *Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a domain D which converges to f uniformly on every compact subset of D. Then either $f(z)\equiv 0$ or f has no zeros.*

Proof. Since, by assumption, $\{f_n\}$ converges uniformly on D, f is analytic on D. Suppose $f(z_0)=0$ but $f(z)\not\equiv 0$. Then (as zeros are isolated) there exists a small circle $C=\partial\Delta(z_0;\delta)$ such that $\overline{\Delta}(z_0;\delta)\subset D$ and

$$\overline{\Delta}(z_0;\delta)\setminus\{z_0\}\cap\{z:\, f(z)=0\}=\emptyset.$$

In particular, $f(z)\neq 0$ on C. Let f have k zeros in $\Delta(z_0;\delta)$. Since f is analytic on $\overline{\Delta}(z_0;\delta)$, $|f|$ attains a minimum value m on C. As $f(z)\neq 0$

on C, $m > 0$. Since $f_n \to f$ uniformly on the compact set C, for a given $m > 0$, there exists an N such that

$$|f_n(z) - f(z)| < m/2 < m \leq |f(z)| \text{ for } n \geq N \text{ and for all } z \in C.$$

By Rouché's Theorem, it follows that f_n and f have the same number of zeros on $\Delta(z_0; \delta)$, and for some $n \geq N$. Thus, f_n has a zero in $\Delta(z_0; \delta)$; this is a contradiction. Hence we must have $f(z) \equiv 0$. ■

We note that for each $n \in \mathbb{N}$, $f_n(z) = e^z/n$ has no zeros in $\mathbb{C}$, but the limit function f is identically zero in $\mathbb{C}$.

12.5. Corollary. *Let $\{f_n\}$ be a sequence of analytic and univalent functions in a domain D which converges to f uniformly on every compact subset of D. Then either f is constant or univalent on D.*

Proof. Assume that f is a non-constant analytic function that is not univalent. Then there exist two distinct points z_1 and z_2 in D such that $f(z_1) = f(z_2) = \alpha$. Choose r so small that

$$\Delta(z_1; r) \subset D, \;\; \Delta(z_2; r) \subset D \;\text{ and }\; \Delta(z_1; r) \cap \Delta(z_2; r) = \emptyset.$$

If $f(z) \not\equiv \alpha$, then, since $f_n - \alpha \to f - \alpha$ uniformly on compact subsets of D, Hurwitz' Theorem applied to $f_n(z) - \alpha$ shows that there is an n such that $f_n(z) - \alpha$ has a zero in $\Delta(z_1; r)$ and a zero in $\Delta(z_2; r)$. That is,

$$f_n(z_1') = f_n(z_2') \;\text{ for some } z_1' \in \Delta(z_1; r) \text{ and } z_2' \in \Delta(z_2; r).$$

This is a contradiction to the hypothesis that f_n is univalent in D. So $f(z) \equiv \alpha$. ■

For example, we note that $f_n(z) = z/n$, $n \geq 1$, is univalent in $\mathbb{C}$, and the limit function f is identically zero in $\mathbb{C}$.

12.2 Basic Results on Univalent Functions

We have already encounted a large number of examples of univalent functions. Now start with

12.6. Theorem. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is such that $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is univalent in the unit disk Δ.*

Proof. Suppose that $\sum_{n\geq 2} n|a_n| \leq 1$. Then, we have $|a_n| \leq 1$ for all $n \geq 2$, and hence $\sum_{n\geq 2} |a_n z^n| \leq \sum_{n\geq 2} |z|^n$. Since $\sum_{n\geq 2} |z|^n$ converges for $|z| < 1$, by the comparison test, we see that the series represented by f converges for $|z| < 1$ so that f is analytic in Δ. Let $|z_0| < 1$. We have

$$\begin{aligned}
(f(z) - f(z_0)) - (z - z_0) \\
&= \sum_{n\geq 2} a_n(z^n - z_0^n) \\
&= (z - z_0)\sum_{n\geq 2} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}).
\end{aligned}$$

As $|z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}| < n$ for $|z| < 1$, we have

$$|(f(z) - f(z_0)) - (z - z_0)| < |z - z_0| \sum_{n\geq 2} n|a_n| \leq |z - z_0|.$$

According to Rouché's Theorem, $f(z) - f(z_0)$ and $z - z_0$ have the same number of zeros in Δ, that is $f(z) = f(z_0)$ has exactly one solution. ■

12.7. Theorem. *Let f be analytic at a. Then f is one-to-one in some neighborhood of a iff $f'(a) \neq 0$.*

Proof. Clearly, f is analytic at a iff g defined by $g(z) = f(z+a) - f(a)$ is analytic at 0. Note that $f'(a) = g'(0)$. In view of this observation, it suffices to prove the theorem with $a = 0$ and $f(a) = 0$.

$\Longleftarrow$: Let $f'(0) \neq 0$, $f(0) = 0$ and $h(z) = f(z)/f'(0)$. Then $h'(z) = f'(z)/f'(0)$ which is analytic at 0, and $h'(0) = 1$. The continuity of $h'(z)$ at $z = 0$ shows that there exists an open disk $|z| < \delta$ such that

$$|h'(z) - 1| < 1/4 \quad \text{for } |z| < \delta.$$

(The univalency of h in $|z| < \delta$ may be obtained quickly from Theorem 12.18). In particular, for any two distinct points z_1 and z_2 in this disk, and for $\gamma = [z_1, z_2]$, the line segment joining z_1 and z_2, we have

$$\begin{aligned}
|h(z_2) - h(z_1) - (z_2 - z_1)| &= \left|\int_{z_1}^{z_2} (h'(z) - 1)\, dz\right| \\
&\leq \int_{[z_1,z_2]} |h'(z) - 1|\,|dz| \\
&\leq (1/4)|z_1 - z_2|
\end{aligned}$$

which, by the triangle inequality, implies that

$$|h(z_2) - h(z_1)| \geq |z_2 - z_1| - (1/4)|z_1 - z_2| > 0.$$

Thus, h and (hence f) is one-to-one in a neighborhood of 0.

$\Longrightarrow$: Let f be analytic, $f(0) = 0$ and one-to-one in a neighborhood Δ_R of 0 . Assume the contrary that $f'(0) = 0$. Then by the Maclaurin series of f around 0, there exists $k \geq 2$ such that $f(z) = z^k\phi(z)$ where ϕ is analytic at 0 and $\phi(0) \neq 0$. By hypothesis, f is univalent (and so f is not a constant). As zeros are isolated (and $f'(z)$ is analytic in Δ_R with $f'(0) = 0$), we have

$$f(z) \neq 0 \text{ and } f'(z) \neq 0 \text{ for } 0 < |z| \leq \delta.$$

Now, $m = \min_{|z|=\delta<R} |f(z)| > 0$. Pick any complex number c such that $0 < |c| < m$. Then

$$|f(z) - (f(z) - c)| = |c| < m \leq |f(z)| \text{ for } |z| = \delta.$$

Therefore, by Rouché's Theorem, $f(z) - c$ has the same number of zeros inside $|z| = \delta$ as that of $f(z)$. But, since $f(z)$ has a zero of order k ($k \geq 2$) at the origin, we must have $f(z) = c$ at two or more points. This contradicts the hypothesis that f is univalent in Δ_R. Hence we must have $f'(0) \neq 0$, which proves the assertion. ∎

The vanishing of a derivative does not preclude the possibility of a real-valued function of a real variable being one-to-one. For example, if $f(x) = x^3$ then $f'(0) = 0$ but the function is still one-to-one on $\mathbb{R}$. This situation cannot occur for complex-valued functions. This fact is indeed a special case of Theorem 12.7 and is important enough to merit a special mention.

12.8. Corollary. (Local Mapping Theorem) *If f is analytic and univalent in a domain D, then $f'(z) \neq 0$ in D.*

12.9. Remark. Note that there exists an analytic function having non-vanishing derivative on a domain D without being univalent on D. Thus, the converse of Corollary 12.8 is not necessarily true. For example, the function $f(z) = e^z$ is not univalent in $|z| < R$ if $R > \pi$ even though $f'(z) = e^z \neq 0$ throughout $\mathbb{C}$. As another example we note that the function $g(z) = z^2$ is not univalent in the punctured plane $\mathbb{C}\setminus\{0\}$ even though $g'(z) = 2z \neq 0$ on $\mathbb{C}\setminus\{0\}$. Here we bring an interesting comparison between the derivatives of real and complex functions. For real differentiable functions, the non-vanishing of the derivative on an open interval is sufficient to guarantee that the function is one-to-one on that interval. As we have seen above, this is not the case for complex-valued functions defined on a domain. ●

12.10. Example. Consider $f(z) = z/(1-z)^{-3}$. Then

$$f'(z) = (1+2z)/(1-z)^{-4}$$

so that $f'(-1/2) = 0$. This means that f cannot be univalent in $|z| < r$ if $r > 1/2$. Next we show that f is actually univalent if $r = 1/2$. Indeed, if

$z_1, z_2 \in \Delta_{1/2}$, then

$$\begin{aligned} f(z_1) = f(z_2) &\Rightarrow z_1(1-z_2)^3 = z_2(1-z_1)^3 \\ &\Rightarrow z_1[1 - z_2^3 - 3z_2(1-z_2)] = z_2[1 - z_1^3 - 3z_1(1-z_1)] \\ &\Rightarrow (z_1 - z_2) - z_1z_2(z_2^2 - z_1^2) - 3z_1z_2(z_1 - z_2) = 0 \\ &\Rightarrow (z_1 - z_2)[1 + z_1z_2(z_1+z_2) - 3z_1z_2] = 0 \\ &\Rightarrow z_1 - z_2 = 0 \end{aligned}$$

as $|z_1z_2(z_1+z_2) - 3z_1z_2| < \frac{1}{2}\frac{1}{2}(\frac{1}{2}+\frac{1}{2}) + 3(\frac{1}{2})(\frac{1}{2}) = 1$. ●

The next result is a direct consequence of Corollary 12.8 and Theorem 7.43.

12.11. Theorem. *A function f is a conformal mapping from $\mathbb{C}$ onto $\mathbb{C}$ (i.e. $f \in \mathrm{Aut}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C})$) iff $f(z) = a_0 + a_1 z$ $(z \in \mathbb{C})$, a_0, a_1 constants, $a_1 \neq 0$.*

An analog of Theorem 12.7 for meromorphic functions follows.

12.12. Theorem. *Let g be meromorphic with a pole at a. Then g is one-to-one in some neighborhood of a iff a is a simple pole (of multiplicity one).*

Proof. Just look at $1/g(z)$ (see also Corollary 12.14). ■

12.13. Theorem. *Let f be analytic at $z = \infty$. Then f is univalent at $z = \infty$ (i.e. univalent in the neighborhood of ∞) if and only if* $\mathrm{Res}\,[f(z);\infty] \neq 0$.

Proof. Suppose f is analytic at $z = \infty$. Then f can be expanded in a power series which converges absolutely for $|z| > R$:

$$f(z) = \sum_{n \geq 0} a_n z^{-n}.$$

Since $w = 1/z$ is bianalytic which maps $\{z : |z| > R\} \setminus \{\infty\}$ onto $\Delta_r \setminus \{0\}$, $r = 1/R$, $f(z)$ is univalent at $z = \infty$ iff $g(w) = \sum_{n \geq 0} a_n w^n$ is univalent at $w = 0$. By Theorem 12.7 we note that g is univalent at 0 iff $a_1 = g'(0) \neq 0$. The result now follows from the fact that $a_1 = -\mathrm{Res}\,[f(z);\infty]$. ■

12.14. Corollary. *Let D be an open set in $\mathbb{C}$, and $f : D \to \mathbb{C}_\infty$ be meromorphic and univalent in D. Then f has only a simple pole in D.*

Proof. The result follows immediately from Theorem 12.13. An alternate proof of this corollary is as follows:

It is clear that a meromorphic univalent function $f : D \to \mathbb{C}_\infty$ cannot have two distinct poles in D because otherwise ∞ would be assumed more than once.

Let $z_0 \neq \infty$ be a pole of order n, $n \geq 1$. Then in a deleted neighborhood of z_0 we have $f(z) = (z-z_0)^{-n} g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$. As $g(z) \neq 0$ on a neighborhood of z_0, the reciprocal

$$\frac{1}{f(z)} = \frac{(z-z_0)^n}{g(z)}$$

is clearly analytic at z_0 and has a zero of order n at z_0 and is univalent on a neighborhood of z_0. Hence we must have $n = 1$.

If $z_0 = \infty$ is a pole of order m $(m \geq 1)$, then 0 is a pole of order n for $f(1/z) = g(z)$. Applying the preceding argument to $g = f \circ h$ $(h(z) = 1/z)$, we obtain $m = 1$. This completes the proof. ■

12.15. Example. Consider the function $f(z) = (1/2)(z + z^{-1})$. Evidently, f is meromorphic and has two simple poles at 0 and ∞. This function is often referred to as (a special case of) a Joukowski mapping.

(i) As $f(z_1) = f(z_2) \implies (z_1 - z_2)(1 - 1/z_1 z_2) = 0$, f is univalent on D if the domain D has the property that no two distinct points z_1, z_2 in D satisfy the condition $z_1 z_2 = 1$, i.e. $z_2 = \overline{z}_1/|z_1|^2$. For example, f is univalent in D whenever D is $\Delta\backslash\{0\}$, or $\mathbb{C}\backslash\overline{\Delta}$ or $\mathbb{H}^+$ or $\mathbb{H}^-$ respectively.

(ii) Clearly, $f'(z) = 0$ at $z = \pm 1$ and so, f is not conformal at $z = \pm 1$.

(iii) Setting $z = re^{i\theta}$ and $w = f(z) = u + iv$, we find that

$$u = a\cos\theta,\ v = b\sin\theta \quad \big(a = (1/2)(r + r^{-1}),\ b = (1/2)(r - r^{-1})\big)$$

and therefore, the image of the circle $|z| = r$ $(r \neq 1)$ is the ellipse defined by

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad (a^2 - b^2 = 1).$$

The semi-major and the semi-minor axes of the ellipse are seen to be a and b with facii at $w = \pm 1$. The ellipse degenerates into the line segment $[-1, 1]$ on the u-axis if $r \to 1$. Since $f(z) = f(1/z)$, the circle $|z| = r$ and $|z| = 1/r$ $(r \neq 1)$ are mapped into the same ellipse. Consequently, $f(\Delta) = f(\mathbb{C}\backslash\overline{\Delta}) = \mathbb{C}\backslash[-1, 1]$.

(iv) The image of the ray $\operatorname{Arg}\theta$ is a branch of the hyperbola

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1,$$

whose foci are also at ± 1, because $\cos^2\theta + \sin^2\theta = 1$.

(v) Rewrite $w = k(z) = z/(1-z)^2$ as

$$w = \frac{1}{\zeta - 2}, \quad \zeta = g(z) = z + \frac{1}{z}.$$

Recall that the unit disk Δ in the z-plane is mapped under g onto the ζ-plane minus the real interval $[-2, 2]$. Under $w = (\zeta - 2)^{-1}$, the domain $\mathbb{C}\backslash[-2, 2]$ is mapped onto the w-plane minus the negative real axis from ∞ to $-1/4$. ●

12.16. Theorem. *Let $f(z)$ be analytic and one-to-one on an open set D. Then the inverse map $g(w) = f^{-1}(w)$ is analytic on $f(D)$.*

Proof. Let f be one-to-one and analytic on D. Then $f'(z) \neq 0$ on D. Also, f^{-1} is single-valued and univalent in $f(D)$. We know that $f(D)$ is open (why?). By hypothesis, $g = f^{-1}$ is continuous in $f(D)$. Let z_0 be arbitrary, and set $w_0 = f(z_0)$. Suppose that w_n is a sequence such that $w_n \to w_0$, but never equals w_0. Setting $z_n = g(w_n)$, we see that $z_n \to z_0$ (as g is continuous), and therefore, taking the limit $w_n \to w_0$, we have

$$\frac{g(w_n) - g(w_0)}{w_n - w_0} = \frac{z_n - z_0}{f(z_n) - f(z_0)} \to \frac{1}{f'(z_0)} \quad \text{as } n \to \infty$$

so that $g'(w_0) = 1/f'(z_0)$. Therefore g is also analytic and conformal, since z_0 is arbitrary. ■

12.17. Corollary. **(Inverse function theorem)** *Let $f(z)$ be analytic in a neighborhood of z_0, and $f'(z_0) \neq 0$ in D. Then the relation $w = f(z)$ defines z as analytic function $f^{-1}(w)$ in some neighborhood of the point $w_0 = f(z_0)$.*

12.18. Theorem. *If $f(z)$ is analytic on a convex domain D, and* Re $f'(z) > 0$ *in D, then $f(z)$ is univalent in D.*

Proof. We pick two distinct points $z_1, z_2 \in D$. Then the straight line segment $z(t) = (1-t)z_1 + tz_2$ $(0 \leq t \leq 1)$ must lie in D. Now,

$$f(z_1) - f(z_2) = \int_{z_1}^{z_2} f'(z)\, dz = \int_0^1 f'(z(t))\,(z_2 - z_1)\, dt$$

so that

$$\left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| \geq \text{Re}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) = \text{Re}\left(\int_0^1 f'(z(t))\, dt\right) > 0.$$

Thus, $f(z_2) \neq f(z_1)$ and so f is univalent in D. ■

According to Theorem 12.18, the functions

$$f_1(z) = -z - 2\log(1-z) \quad \text{and} \quad f_2(z) = -\log(1-z)$$

are univalent in Δ.

12.19. Corollary. *Let $f \in \mathcal{H}(\Omega)$, where Ω is an open set in $\mathbb{C}$ and $f'(a) \neq 0$ for some $a \in \Omega$. If further, there exists a disk $\Delta(a;\delta) \subseteq \Omega$ such that*

(12.20) $$|f'(z) - f'(a)| < |f'(a)| \quad \textit{for } z \in \Delta(a;\,\delta),$$

then f is one-to-one on $\Delta(a;\delta)$.

12.21. Examples.

(i) Consider $f(z) = z + z^n$, $n \geq 2$. Then $f'(0) = 1$ and $f'(z) - 1 = nz^{n-1}$. Further, $f'(-\delta_n) = 0$ for $\delta_n = (-1/n)^{1/(n-1)}$ so that f cannot be univalent on $\Delta(0;\delta)$ if $\delta \geq n^{-1/(n-1)}$. Finally, for $|z| < n^{-1/(n-1)}$, we have
$$|f'(z) - 1| = n|z|^{n-1} < 1$$
implying that f is univalent on $|z| < n^{-1/(n-1)}$. In particular, $f(z) = z + z^2$ is univalent for $|z| < 1/2$ but not on any larger disk around 0. Similarly, we see that $g(z) = z + z^n/n$ is univalent in the unit disk Δ but not in any larger disk about 0.

(ii) Consider the polynomial $f(z) = \sum_{k=1}^{n} a_k z^k$ of degree $n \geq 2$ such that $\sum_{k=2}^{n} k|a_k| \leq |a_1|$ and $a_1 \neq 0$. Then, for $z \in \Delta$,
$$|f'(z) - a_1| \leq \sum_{k=2}^{n} k|a_k|\,|z|^{k-1} < \sum_{k=2}^{n} k|a_k| \leq |a_1| = |f'(0)|$$
so that f is one-to-one on Δ, by Corollary 12.19. Note that if $a_1 = 0$, then f cannot be one-to-one in any neighborhood of 0 (prove!). ●

12.3 Normal Families

The proof of the Riemann mapping theorem which we are going to deal with in the next section relies on the idea of a normal family which is a basic concept in function theory. We know that every bounded sequence of complex numbers $\{z_n\}$ possesses a limit point and so, $\{z_n\}$ has a convergent subsequence. Analogous situation for sequences of functions $\{f_n\}$ leads to the definition of a normal family.

Let Ω be a domain in $\mathbb{C}$, and $\mathcal{F} \subseteq \mathcal{H}(\Omega)$. We say that $\mathcal{F}$ is a *normal family* if every sequence of functions contains a subsequence which converges uniformly on compact subsets of Ω (i.e. locally uniformly on Ω). The limit of such a subsequence must be analytic on Ω.

A family $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ is said to be *locally uniformly bounded* in Ω if for any compact set $K \subset \Omega$, there exists a constant $M = M(K)$ such that

$$|f(z)| \le M \quad \text{for all } f \in \mathcal{F}$$

and for all $z \in K$. We say that the family $\mathcal{F}$ is *uniformly bounded* on Ω if there exists a constant $M > 0$ such that

$$|f(z)| \le M \quad \text{for all } z \in \Omega \text{ and all } f \in \mathcal{F}.$$

Clearly, uniformly boundedness of a family implies that each member of the family is bounded. However the converse is not true. For instance, consider

$$\mathcal{F}_1 = \{nz : n \in \mathbb{N}\}, \quad \mathcal{F}_2 = \{1/(z-e^{in}) : n \in \mathbb{N}\}, \quad \mathcal{F}_3 = \{1/(1-z^n) : n \in \mathbb{N}\}.$$

Then each $\mathcal{F}_k\,(k = 1, 2, 3)$ is a locally uniformly bounded family in Δ but none of them is uniformly bounded in Δ.

Our next result shows that every compact subset K of a domain Ω is contained in a finite union of closed disks contained in Ω.

12.22. Lemma. *Let Ω be a domain in $\mathbb{C}$. Then there exists a sequence $\{K_n\}$ of compact sets such that*

(i) $\Delta(z;\delta_n) \subseteq K_{n+1}$ *with* $\delta_n = \frac{1}{n(n+1)}$ *and for* $z \in K_n$

(ii) $K_n \subseteq \operatorname{int}(K_{n+1})$, $\operatorname{int} K$ *denotes the interior of the set* K

(iii) $\Omega = \cup_{n\in\mathbb{N}} K_n$

(iv) *If $K \subset \Omega$ is any compact set, then $K \subset K_n$ for some $n \in \mathbb{N}$.*

Proof. If $\Omega = \mathbb{C}$, then we let $K_n = \overline{\Delta}(0;n)$. If $\Omega \neq \mathbb{C}$, then we let

$$K_n = \{z \in \Omega : \operatorname{dist}(z, \Omega^c) \ge 1/n\} \cap \overline{\Delta}(0;n).$$

Clearly $\operatorname{dist}(z, \Omega^c) \ge 0$ if $\Omega \neq \mathbb{C}$, and if $\Omega = \mathbb{C}$, we could perhaps interpret it as ∞. Now K_n is bounded (because $K_n \subseteq \overline{\Delta}(0;n)$ for each n) and K_n is closed because it is an intersection of two closed sets. Thus, K_n is compact for each n. For $z \in \Omega \cap K_n$, we show that $\Delta(z;\delta_n) \subset K_{n+1}$. It follows that

$$\begin{aligned} |z| \le n \text{ and } w \in \Delta(z;\delta_n) &\implies |w| \le |w-z| + |z| < \delta_n + n < 1 + n \\ &\implies w \in \overline{\Delta}(0;n+1). \end{aligned}$$

Also, for each $a \in \Omega^c$, $z \in K_n$ and $w \in \Delta(z;\delta_n)$, we have

$$|z-a| \ge 1/n \quad \text{and} \quad |w-z| < \delta_n$$

so that

$$|w-a| \ge |z-a| - |w-z| > \frac{1}{n} - \delta_n = \frac{1}{n} - \frac{1}{n(n+1)} = \frac{1}{n+1}.$$

Thus, $w \in K_{n+1}$ and so, $\Delta(z;\delta_n) \subset K_{n+1}$. Note that each $z \in K_n$ implies that $\Delta(z;\delta_n) \subset K_{n+1}$. This means that K_n is contained in the interior of K_{n+1}. Hence, (i) and (ii) follow.

To prove (iii), we let $z \in \Omega$. Since Ω is open, there exists some disk $\Delta(z;r) \subset \Omega$. In particular,

$$\Delta(z;1/n) \subset \Delta(z;r) \subset \Omega$$

where n is large enough so that $n \geq |z| > 1/r$. This gives $z \in K_n$. Therefore, $\Omega \subseteq \bigcup_{n=0}^{\infty} K_n$. But $K_n \subset \Omega$ for all n and so we also have

$$\cup_{n=1}^{\infty} K_n \subseteq \Omega.$$

and (iii) follows.

For the proof of (iv), we consider a compact subset $K \subset \Omega$. Then

$$K \subseteq \cup_{n=1}^{\infty} K_n \subset \cup_{n=1}^{\infty} \text{int}\,(K_{n+1})$$

and so we have an open cover of K by the sets $\text{int}\,(K_{n+1})$. According to the Heine-Borel theorem, there exists a finite subcover of K. Thus, there is an N such that

$$K \subset \cup_{n=1}^{N} \text{int}\,(K_{n+1}) = \text{int}\,(K_{N+1}) \subset K_{N+1}$$

and (iv) holds. ■

We prove the following general result which we shall need later on.

12.23. Lemma. *Let Ω be a domain in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ such that $\mathcal{F}$ is locally uniformly bounded in Ω. Then, for each $k \geq 1$, the family*

$$\mathcal{F}^{(k)} = \{f^{(k)} : f \in \mathcal{F}\}$$

is locally uniformly bounded in Ω.

Proof. Let $a \in \Omega$ be an arbitrary point of Ω and $0 < \delta < \text{dist}\,(a, \partial\Omega)$. Then $\overline{\Delta}(a;\delta) \subseteq \Omega$. As $\mathcal{F}$ is locally uniformly bounded in Ω, there exists a constant $M > 0$ such that

$$|f(z)| \leq M \quad \text{for all } z \in C \text{ and for every } f \in \mathcal{F},$$

where C is the boundary of the disk $\Delta(a;\delta)$ oriented positively. By the Cauchy integral formula for derivatives

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta \quad \text{for } z \in \Delta(a;\delta)$$

and for all $f \in \mathcal{F}$. For $z \in \Delta(a;\delta/2)$, as $|\zeta - z| \geq |\zeta - a| - |a - z| > \delta - \delta/2$, we obtain

$$|f^{(k)}(z)| \leq \frac{k!}{2\pi}\frac{M}{(\delta/2)^{k+1}}2\pi\delta = \frac{k!M}{\delta^k}2^{k+1}$$

so that $\mathcal{F}^{(k)}$ is locally uniformly bounded in Ω. ∎

Next we begin with the definition of an equicontinuous family. Let Ω be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of complex-valued functions (not necessarily analytic) in Ω. We say that $\mathcal{F}$ is *equicontinuous* on Ω if for every $\epsilon > 0$ there is a $\delta > 0$ (depending only on ϵ and not on $f \in \mathcal{F}$) such that

$$|f(z) - f(z')| < \epsilon$$

for all $f \in \mathcal{F}$ whenever $z, z' \in \Omega$ and $|z - z'| < \delta$.

For example, consider $\mathcal{F} = \{f : f(z) = 3z + \text{constant}\}$. Then for all $f \in \mathcal{F}$,

$$|f(z) - f(z')| = 3|z - z'| < \epsilon$$

whenever $|z - z'| < \delta = \epsilon/3$ and $z, z' \in \mathbb{C}$. Thus, $\mathcal{F}$ is an equicontinuous family on $\mathbb{C}$.

12.24. Lemma. (Bounded derivatives imply equicontinuity) *Let Ω be a domain in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ such that the family of their derivatives, $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$, is locally uniformly bounded in Ω. Then $\mathcal{F}$ is (locally) equicontinuous.*

Proof. It suffices to prove equicontinuity of $\mathcal{F}$ on an arbitrary bounded closed disk K contained in Ω. As $\mathcal{F}'$ is locally uniformly bounded in Ω, for a closed disk $K \subset \Omega$ there exists a constant $M = M(K) > 0$ such that

$$|f(z)| \leq M \quad \text{for all } z \in K \text{ and for every } f \in \mathcal{F}.$$

Then for any two points $z, z' \in K$, integrating over a straight line path from z' to z gives

$$f(z) - f(z') = \int_{z'}^{z} f'(\zeta)\,d\zeta = \int_{[z',z]} f'(\zeta)\,d\zeta$$

so that

$$|f(z) - f(z')| \leq M|z - z'|.$$

For any $\epsilon > 0$, let $\delta = \epsilon/M$. Then $|f(z) - f(z')| < \epsilon$ whenever $z, z' \in K$ and $|z - z'| < \delta$. Thus, $\mathcal{F}$ is equicontinuous on compact subset of Ω. ∎

12.25. Lemma. *Suppose that $\{f_n(z)\}$ is a sequence of analytic functions that is locally uniformly bounded in a domain Ω. Suppose that $\{f_n(\zeta)\}$ converges at every ζ on a dense subset E of Ω. Then $\{f_n(z)\}$ converges locally uniformly on Ω.*

Proof. Fix a compact set $K \subset \Omega$. By Lemmas 12.23 and 12.24, $\{f_n(z)\}$ is equicontinuous on K. Therefore, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(z) - f_n(z')| < \epsilon/3 \quad \text{for } |z - z'| < \delta \text{ and } z, z' \in K, \tag{12.26}$$

where n is arbitrary. Since K is compact, we can cover K by a finite subcover, say p neighborhoods $\Delta(z_k; \delta/2)$ with $z_k \in K$ and $1 \le k \le p$. In each of these p neighborhoods, choose a point $\zeta_k = z_{m(k)}$ $(1 \le k \le p)$ from the dense subset of K. By hypothesis, $\{f_n(\zeta_k)\}$ converges for each $1 \le k \le p$. As a result, this sequence is Cauchy. Consequently, there is an integer N such that $n > m \ge N$

$$|f_n(\zeta_k) - f_m(\zeta_k)| < \epsilon/3 \quad \text{for } 1 \le k \le p. \tag{12.27}$$

Now, let $z \in K$ be an arbitrary point. Then, $z \in \Delta(z_k; \delta/2)$ for some k. It follows from (12.26) and (12.27) that for the above $n > m \ge N$

$$\begin{aligned} |f_n(z) - f_m(z)| &\le |f_n(z) - f_n(\zeta_k)| + |f_n(\zeta_k) - f_m(\zeta_k)| + |f_m(\zeta_k) - f_m(z)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus, the sequence $\{f_n(z)\}$ is uniformly Cauchy on K, and therefore converge uniformly on K, since K is compact. ∎

Using the notion of equicontinuity and uniform boundedness, we are now prepared to state the following simple version of Montel's theorem which is one of the key ingredients in the proof of the Riemann mapping theorem.

12.28. Theorem. **(Montel's theorem)** *Let Ω be a domain in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ such that $\mathcal{F}$ is locally uniformly bounded in Ω. Then $\mathcal{F}$ is a normal family.*

Proof. Let $\{f_n\}$ be a sequence from $\mathcal{F}$. To prove that $\mathcal{F}$ is normal, we have to construct a subsequence that converges locally uniformly in Ω. Let K be a compact subset of Ω. Then, by Lemma 12.22, K is contained in K_n for some n, where K_n is defined as in Lemma 12.22. Therefore, it suffices to prove that there exists a subsequence that converges uniformly on each K_n. Let $E \subset \Omega$ be a countable dense subset. For instance,

$$E = \{z = x + iy \in \Omega : x \text{ and } y \text{ are rational}\}.$$

As E is countable, we may enumerate the points of E by $z_1, z_2, \dots,$

$$E = \{z_1, z_2, z_3, \dots\}.$$

We are given that $\mathcal{F}$ is uniformly bounded on each compact subset of Ω and hence it is pointwise bounded (because $K_n = \{z_n\}$ consisting of singleton subsets are compact). Thus, $\{f_n(z_1)\}$ is a bounded sequence of complex numbers. Thus (by Bolzano-Weierstrass theorem), there is a convergent subsequence of $\{f_n\}$, which we shall denote by $\{f_{n,1}\}$, that is convergent at z_1. Now consider the sequence $\{f_{n,1}(z_2)\}$ which is bounded, and so we can find a further subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ such that $\{f_{n,2}(z_2)\}$ converges. Since $\{f_{n,2}\}$ is a subsequence of $\{f_{n,1}\}$, $\{f_{n,2}(z_1)\}$ is convergent too. Hence $\{f_{n,2}\}$ actually converges at two points z_1, z_2. Repeating the process, for each $k \geq 1$, we obtain a subsequence $\{f_{n,k}\}$ that converges at $z_1, z_2, \ldots, z_k$ and $\{f_{n,k}\} \subseteq \{f_{n,k-1}\}$. We get a list of lists:

$$\begin{array}{lllllll}
\mathbf{f_{1,1}(z)}, & f_{2,1}(z), & f_{3,1}(z), & \ldots & f_{k,1}(z), & \ldots & \text{conveges at } z_1 \\
f_{1,2}(z), & \mathbf{f_{2,2}(z)}, & f_{3,2}(z), & \ldots & f_{k,2}(z), & \ldots & \text{converges at } z_1, z_2 \\
f_{1,3}(z), & f_{2,3}(z), & \mathbf{f_{3,3}(z)}, & \ldots & f_{k,3}(z), & \ldots & \text{converges at } z_1, z_2, z_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
f_{1,k}(z), & f_{2,k}(z), & f_{3,k}(z), & \ldots & \mathbf{f_{k,k}(z)}, & \ldots & \text{converges at } z_1, \ldots, z_k. \\
\vdots & \vdots & \vdots & \vdots & \vdots & &
\end{array}$$

Now we consider the functions $g_n(z) = f_{n,n}(z)$, $n \in \mathbb{N}$. Then for each fixed k, $\{g_n(z_k)\}$ is a (diagonal) subsequence of the convergent sequence $\{f_{n,k}(z_k)\}$, $n \geq k$, and hence converges at each $z_k \in E$. By Lemma 12.25, $\{g_n(z_k)\}$ converges locally uniformly on Ω and so, the limit function $g(z)$ is analytic on Ω. ∎

Let use consider some simple examples.

1. Let $f_n(z) = z^n$ and $\mathcal{F}$ be the collection of all $f_n(z)$, $z \in \Delta$. Then $f_n \to 0$ locally uniformly to 0 on Δ, but does not converge uniformly on Δ because $\lim_{n\to\infty} \sup_{z\in\Delta} |f_n(z)| \neq 0$. It follows that $\mathcal{F}$ is a normal family of analytic functions in Δ. Also, we remark that $\{z^n\}$ does not converge locally uniformly on the closed disk $|z| \leq 1$, because $\lim_{n\to\infty} |z^n| = 1$ at $z = 1$.
2. Let $f_n(z) = z/n$, and $\mathcal{F}$ be the collection of all $f_n(z)$, $\Omega = \mathbb{C}$. Clearly, there is no $M > 0$ such that
$$|z/n| \leq M$$
for all n and all $z \in \mathbb{C}$. On the other hand, on any fixed compact set K, there exists an $M > 0$ (e.g. $M = \sup_{z\in K} |z|$) such that $|z/n| \leq M$ for all n and all $z \in \mathbb{C}$. Thus, $\mathcal{F}$ is a normal family (as $\mathcal{F}$ is a locally uniformly bounded family of analytic functions in $\mathbb{C}$). Note that $f_n \to 0$ locally uniformly on $\mathbb{C}$.
3. Consider $\mathcal{F} = \{nz^2 : n \in \mathbb{N}\}$. If $f_n(z) = nz^2$, then $f_n(0) = 0 \to 0$ as $n \to \infty$ whereas $f_n(z) \to \infty$ as $n \to \infty$ for $z \neq 0$. This observation

shows that $\mathcal{F}$ cannot be normal in any domain which contains the origin. Also, $\mathcal{F}$ is not normal in any domain that does not contain the origin.

4. Define $\mathcal{F} = \{f \in \mathcal{H}(\Delta) : f(0) = 1 \text{ and } \operatorname{Re} f(z) > 0 \text{ for } z \in \Delta\}$. Then we have (see Theorem 6.49)
$$|f(z)| \leq \frac{1+|z|}{1-|z|} \quad \text{for } z \in \Delta.$$
Thus, $\mathcal{F}$ is locally uniformly bounded and therefore, $\mathcal{F}$ is a normal family.

5. Let $\mathcal{F}$ denote the family of all functions from $\mathcal{H}(\Delta)$ such that
$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq c \quad \text{for each } r \in (0,1)$$
and for some $c > 0$. Then for each compact set $K \subset \Delta$, there exists a ρ such that $K \subset \Delta_\rho$. Now, we fix r with $\rho < r < 1$ and let $m = \max_{|z|=r} |f(z)|$. By the Cauchy integral formula, for each $f \in \mathcal{F}$ and $a \in K$, we have
$$|f(a)| = \left| \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - a}\, d\zeta \right| \leq \frac{1}{2\pi} \frac{m}{r-|a|} 2\pi r = M(K) \quad \text{(say)}.$$
Thus, $\mathcal{F}$ is locally uniformly bounded in Δ and therefore, by Montel's theorem, $\mathcal{F}$ is normal.

12.4 The Riemann Mapping Theorem

The Riemann mapping theorem is concerned with the mapping of a domain in the z-plane onto a domain on the w-plane. In this section, we are concerned with this important problem which has a lot of applications.

12.29. Problem. Given two domains Ω, Ω', is it always possible to find a conformal map of Ω onto Ω'? That is, given two domains Ω and Ω', under what conditions is there a function $f \in \mathcal{H}(\Omega)$ such that $\Omega' = f(\Omega)$?

Does the problem always have a solution? For instance, does it have a solution if Ω is a multiply connected domain and Ω' is a simply connected domain? In general, the answer is no as there exists no map (why!) which takes the multiply connected domain Ω onto the simply connected domain Ω'.

So unless otherwise stated explicitly, from now onwards, we restrict our attention to when Ω and Ω' are simply connected domains. *Two simply connected domains Ω and Ω' in $\mathbb{C}$ are said to be conformally equivalent if* *there exists an analytic function ϕ from Ω to Ω' such that ϕ is one-to-one and onto.* Clearly conformality is an equivalence relation. If the domains Ω and Ω' are conformally equivalent, then $\phi^{-1} : \Omega' \to \Omega$ with $\phi^{-1} \in \mathcal{H}(\Omega')$ so that ϕ and ϕ^{-1} are conformal. However, it is natural to ask whether it is always possible to say that two given simply connected domains are conformally equivalent. Is the complex plane $\mathbb{C}$ conformally equivalent to the unit disk Δ, for example? Note that both $\mathbb{C}$ and Δ are simply connected domains. According to Liouville's theorem, $\mathbb{C}$ and Δ are not conformally equivalent (why!).

12.30. Theorem. (Riemann Mapping Theorem) *Let Ω be a simply connected domain and $\Omega \neq \mathbb{C}$. Then there exists a univalent function $f \in \mathcal{H}(\Omega)$ such that $f(\Omega) = \Delta$; i.e. every simply connected domain which is a proper subset of $\mathbb{C}$ is conformally equivalent to the unit disk.*

The map f in Theorem 12.30 is essentially unique in the following sense: if a is an element of Ω and $\theta \in (-\pi, \pi]$ is an arbitrary angle, then there exists precisely one f as above with the additional properties $f(a) = 0$ and $\arg f'(a) = \theta$. Without loss of generality we may simply assume that $\theta = 0$. Also, we remark that the condition $f(a) = 0$ is simply a convenient normalization. Moreover, the Riemann mapping theorem[15] implies that

- among the simply connected domains, there are exactly two equivalence classes: one consisting of $\mathbb{C}$ alone and the other containing the unit disk (and much more).
- among the doubly connected regions, there are uncountably many equivalence classes, each containing a circular annulus for some unique real (and much more). In particular, the two annuli $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent iff $R_1/r_1 = R_2/r_2$, that is the ratio of outer radius and inner radius is a conformal invariant. We do not discuss these in detail here.
- the function provided by the Riemann mapping theorem is a homeomorphism. Indeed, as a corollary to the Riemann mapping theorem, we say that if Ω and Ω' are conformally equivalent, then they are homeomorphic. The converse is false; for example, $\mathbb{C}$ is homeomorphic to Δ, but not conformally equivalent to Δ.

A number of proofs are available today, see Burckel's survey [5]. Before we present the proof of the Riemann mapping theorem, we recall the following facts:

- every non-constant analytic function is an open map.

[15] Riemann's dissertation, completed under Gauss's supervision in 1851, was on the foundations of complex analysis. It introduced several ideas of fundamental importance, such as the definitions of conformal mapping and simple connectivity.

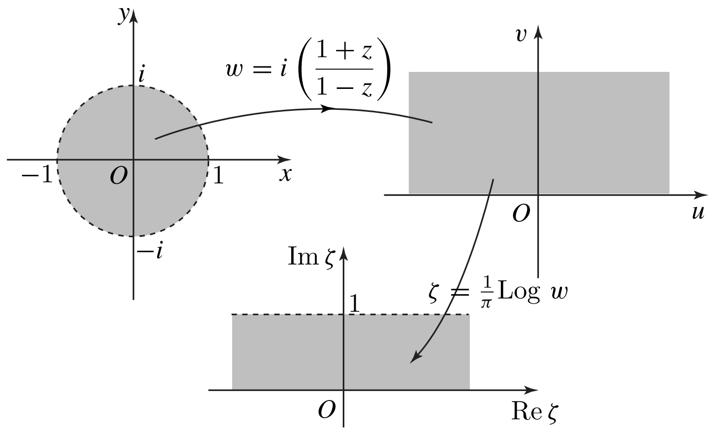


Figure 12.1: A conformal map of Δ onto $\Omega' = \{w : 0 < \operatorname{Im} w < 1\}$.

- every one-to-one analytic function has non-vanishing derivative and, so has an analytic inverse.
- every analytic function with non-vanishing derivative preserves angles and their orientations and so, the function provided by the Riemann mapping theorem is, in particular, conformal. In view of this reasoning the Riemann mapping theorem is frequently called the "conformal mapping theorem".
- frequently the composition of conformal maps makes the problem easier. For instance, it follows that if Ω_1 and Ω_2 are two simply connected domains with $\mathbb{C} \setminus \Omega_k \neq \emptyset$ for $k = 1, 2$ then, by Theorem 12.30, there exist two conformal maps $f_k : \Omega_k \to \Delta$ $(k = 1, 2)$ and therefore, $f = f_2^{-1} \circ f_1 : \Omega_1 \to \Omega_2$ is the desired conformal map from Ω_1 onto Ω_2. Thus, the use of the unit disk in the proof is a convenient intermediate step in the statement of the Riemann mapping theorem.

Before we undertake the proof of the Riemann mapping theorem it is important to examine several examples of such mappings.

12.31. Example. A conformal map which carries the unit disk Δ onto the horizontal strip $\Omega' = \{w : 0 < \operatorname{Im} w < 1\}$ is given by

$$\frac{1}{\pi} \operatorname{Log}\left(\frac{i(1+z)}{1-z}\right).$$

A proof for this fact follows easily from Figure 12.1. ●

12.32. Example. Consider the crescent shaped domain D_1 defined by

$$D_1 = \{z \in \mathbb{C} : |z - i\alpha| > \alpha\} \cap \{z \in \mathbb{C} : |z - 2i\alpha| < 2\alpha\} \quad (\alpha > 0)$$

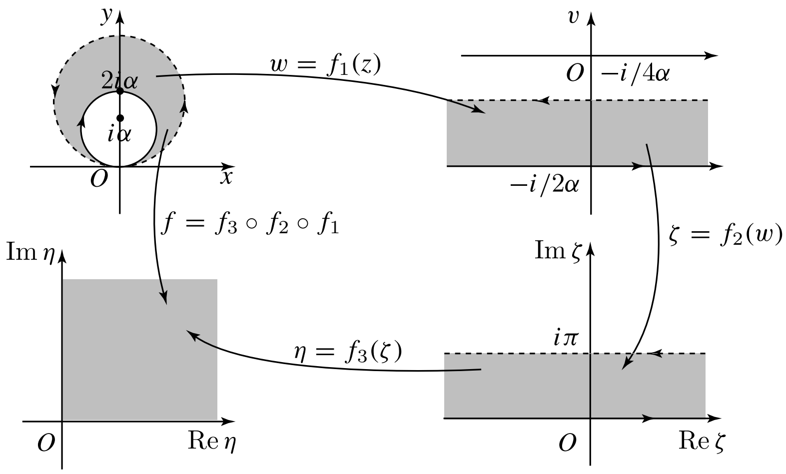


Figure 12.2: A conformal map of D_1 onto D_2.

and D_2, the open first quadrant. The Riemann mapping theorem assures the existence of a univalent analytic function f from D_1 onto D_2 (although it does not give any method for constructing such a mapping). In this special case, we can actually describe the method of finding a conformal mapping which carries D_1 onto D_2. To do this we first see that if $a > 0$ then, under the inversion $w = 1/z$, one has

- $|z - ia| < a$ is mapped onto the half-plane $\mathrm{Im}\,(w) < -1/(2a)$
- $|z - ia| = a$ is mapped onto the straight line $\mathrm{Im}\,(w) = -1/(2a)$
- $|z - ia| > a$ is mapped onto the half-plane $\mathrm{Im}\,(w) > -1/(2a)$.

If we let (see Figure 12.2)

$$w = f_1(z) = \frac{1}{z},\ \zeta = f_2(w) = 4\pi\alpha\left(w + \frac{i}{2\alpha}\right),\ \eta = f_3(\zeta) = \exp(\zeta/2),$$

then the above information help us to conclude that $f = f_3 \circ f_2 \circ f_1$ is a map with the desired property. This gives $f(z) = -\exp(2\pi\alpha/z)$. Also we note that $f(2i\alpha) = 1$ and $f(4i\alpha) = i$. In particular if D_3 is the open upper half-plane $\mathbb{H}^+$, then a conformal mapping which carries D_1 onto D_3 is given by $f(z) = \exp(4\pi\alpha/z)$. ●

12.33. Example. For $k = 1, 2$, let $D_k = \{z : 0 < r_k < |z| < R_k\}$ such that $(r_2/r_1) = (R_2/R_1)$. Then we see that f given by

$$f(z) = (r_2/r_1)z$$

maps the annulus D_1 onto D_2.

Next, we let $A(r) = \{z : r < |z| < 1\}$, $r > 0$, and $B = \Delta\setminus\overline{\Delta}(1/3; 1/3)$. Let us now describe a method of finding a conformal map that takes $A(r)$ onto B. Define

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}, \quad C_1 = \partial\Delta(1/3; 1/3) \text{ and } C_2 = \partial\Delta.$$

First we need to find α such that $\phi_\alpha(C_1) = \partial\Delta_r$. For the moment, let α be real. Then $\phi_\alpha(x)$ is real and ϕ_α maps the real line into itself. Now,

$$\phi_\alpha(0) = \alpha, \quad \text{and} \quad \phi_\alpha(2/3) = \frac{\alpha - (2/3)}{1 - (2/3)\alpha}$$

meets the real axis. Note that C_1 meets the real axis in a perpendicular fashion, real goes to real and that ϕ_α is conformal. As we need $\phi_\alpha(C_1)$ to be the circle $\partial\Delta_r$ centered at the origin, $\phi_\alpha(0) = -\phi_\alpha(2/3)$ which gives

$$\alpha = -\left(\frac{\alpha - (2/3)}{1 - (2/3)\alpha}\right) \quad \text{or} \quad \alpha^2 - 3\alpha + 1 = 0.$$

That is $\alpha = (3 \pm \sqrt{5})/2$. As $|\alpha| < 1$, we choose $\alpha = (3 - \sqrt{5})/2$ which is positive. Thus,

$$\phi_\alpha(z) = \frac{((3 - \sqrt{5})/2) - z}{1 - ((3 - \sqrt{5})/2)z}$$

maps $\partial\Delta$ onto itself, and maps C_1 onto the circle with center at the origin and radius $r = (3 - \sqrt{5})/2$.

Suppose that $\gamma_1 = \partial\Delta(1/4; 1/4)$ and $\gamma_2 = \partial\Delta$. Then, $\phi_\alpha(0) = -\phi_\alpha(1/2)$ gives

$$\alpha = -\left(\frac{\alpha - (1/2)}{1 - (1/2)\alpha}\right) \quad \text{or} \quad \alpha^2 - 4\alpha + 1 = 0$$

so that $\alpha = 2 \pm \sqrt{2}$. Thus, $\phi_{2-\sqrt{2}}(z)$ maps $\partial\Delta$ onto itself, and maps γ_1 onto the circle with center at the origin and radius $r = 2 - \sqrt{2}$. Moreover, as the inverse of ϕ_α is itself,

$$z = \phi_{2-\sqrt{2}}(w) = \frac{(2 - \sqrt{2}) - w}{1 - (2 - \sqrt{2})w}$$

is a conformal map which carries the annulus $A(r) = \{z : r < |z| < 1\}$ $(r = 2 - \sqrt{2})$ onto $B = \Delta\setminus\overline{\Delta}(1/4; 1/4)$. •

12.34. Proof of the Riemann mapping theorem. The proof is long and it involves a considerable number of tricky ideas. So, we divide the proof into several steps.

Uniqueness: We know that the every member of the (analytic) automorphisms of the unit disk Δ which fixes the origin is given by the mapping $f(w) = e^{i\theta}w$ for some real θ. This result makes the uniqueness part easier. Indeed, if Ω is a simply connected domain with $\Omega \neq \mathbb{C}$, $a \in \Omega$ and such that (see Figure 12.3)

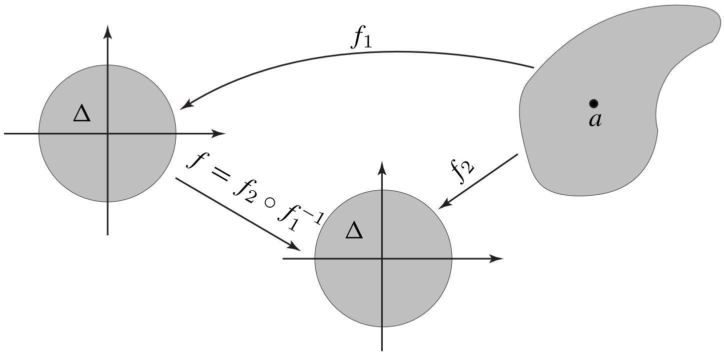


Figure 12.3: Uniqueness of the Riemann map.

- $f_i : \Omega \to \Delta$ is analytic, one-to-one and onto for each $i = 1, 2$,
- $f_i(a) = 0$ and $f_i'(a) > 0$ for each $i = 1, 2$,

then the function $f = f_2 \circ f_1^{-1}$ belongs to $\mathcal{H}(\Delta)$ and maps Δ onto itself conformally. Further,

- $f(0) = f_2(f_1^{-1}(0)) = f_2(a) = 0$
- $f'(0) = f_2'\left(f_1^{-1}(0)\right)\left(f_1^{-1}\right)'(0) = \dfrac{f_2'(a)}{f_1'(a)} > 0.$

Observe that if $z = \Phi^{-1}(w)$, then $w = \Phi(z)$ so that

$$\frac{d\Phi^{-1}(w)}{dw} \cdot \frac{dw}{dz} = 1, \quad \text{i.e. } (\Phi^{-1})'(w) = \frac{1}{\Phi'(\Phi^{-1}(w))}$$

and, for $\Phi(a) = 0$, we have $\left(\Phi^{-1}\right)'(\Phi(a)) = \left(\Phi^{-1}\right)'(0) = 1/[\Phi'(a)]$. Now, by Schwarz' lemma (see also Corollary 6.48), $f(w) = e^{i\theta}w$ for some $\theta \in \mathbb{R}$. Since $f'(0) = e^{i\theta} > 0$, it follows that $f = f_2 \circ f_1^{-1}$ is the identity mapping $f(w) = w$ and so, $f_2(z) = f_1(z)$ for all $z \in \Omega$, as asserted. Having proved the uniqueness part, we now turn to the problem of the existence part.

Existence: Let $a \in \Omega$. Consider the family $\mathcal{F}$ defined by

$$\mathcal{F} = \{f \in \mathcal{H}(\Omega) : f \text{ univalent on } \Omega,\ f(a) = 0,\ f'(a) > 0 \text{ and } |f(z)| < 1\}.$$

We show that the family $\mathcal{F}$ is non-empty.

Case (i): If Ω is bounded, then the assertion is easy. To do this we simply need to find an f meeting the stated specifications. If $f(z) = \alpha(z-a)$ and if $\alpha > 0$ is taken sufficiently small, then $|f(z)|$ can be made less than 1 for all $z \in \Omega$. Indeed,

$$|f(z)| = |\alpha|\,|z-a| \le \alpha(|z| + |a|) \le 2\alpha \sup_{z\in\Omega} |z| = 1$$

provided $\alpha = 1/(2\sup_{z\in\Omega}|z|)$.

Case (ii): Let Ω be unbounded. Since $\Omega \neq \mathbb{C}$, there exists a point $b \in \mathbb{C}\setminus\Omega$. Further, as $z-b \neq 0$ on Ω (by Theorem 4.39), there exists a function $h:\Omega\to\mathbb{C}$ analytic on Ω with $h^2(z) = z-b$ for $z\in\Omega$. We have

- h is univalent on Ω, since, for $z_1, z_2 \in \Omega$,

$$h(z_1) = h(z_2) \;\Rightarrow\; z_1 - b = h^2(z_1) = h^2(z_2) = z_2 - b \;\Rightarrow\; z_1 = z_2.$$

- there exist no two points $z_1, z_2 \in \Omega$ such that $h(z_1) = -h(z_2)$. Indeed if $z_1, z_2 \in \Omega$, then

$$\begin{aligned} h(z_1) = -h(z_2) \;&\Rightarrow\; z_1 - b = h^2(z_1) = h^2(z_2) = z_2 - b\\ &\Rightarrow\; z_1 = z_2\\ &\Rightarrow\; h(z_1) = h(z_2) = -h(z_1)\\ &\Rightarrow\; 0 = h(z_1)\\ &\Rightarrow\; 0 = h^2(z_1) = z_1 - b\\ &\Rightarrow\; z_1 = b \in \mathbb{C}\setminus\Omega \end{aligned}$$

  which is a contradiction to the fact that $z_1 \in \Omega$.

- h, being a (one-to-one) non-constant analytic function, is an open mapping. Now, by the open mapping theorem, $h(\Omega)$ is open and hence, $h(\Omega)$ contains a disk, say $\Delta(w_0;\delta)$ (about a point $w_0 \in h(\Omega)$ with some radius δ), (see Figure 12.4).

- $\Delta(-w_0;\delta)$ fails to meet $h(\Omega)$, i.e. $\Delta(-w_0;\delta)\cap h(\Omega) = \emptyset$. Indeed, if there exists a point $w_1 \in \Delta(-w_0;\delta)\cap h(\Omega)$ then $w_1 = h(z_1)$ for some $z_1 \in \Omega$ and also, $|w_1 + w_0| = |-w_1 - w_0| < \delta$ showing that $-w_1 \in \Delta(w_0;\delta) \subset h(\Omega)$. The last fact infers that

$$-w_1 = h(z_2) \quad \text{for some } z_2 \in \Omega.$$

  Thus, $h(z_1) = -h(z_2)$ which is not possible (as seen above) so that

$$\Delta(-w_0;\delta)\cap h(\Omega) = \emptyset.$$

- for every $z\in\Omega$, $|h(z) - (-w_0)| = |h(z)+w_0| \geq \delta > 0$ so that g defined by

$$g(z) = \frac{\delta k}{h(z)+w_0} \quad (k \in (0,1))$$

  is a one-to-one analytic function (since h is one-to-one on Ω with $h(z)+w_0 \neq 0$ on Ω) defined on Ω satisfying $|g(z)| < 1$ for all $z \in \Omega$.

That is, g is a univalent map of Ω into a subdomain $g(\Omega)$ of Δ, but $g(a) \neq 0$. To construct a function f with $f(a) = 0$ and $f'(a) > 0$, we simply consider

$$f(z) = e^{-i\theta}\left(\frac{g(z)-g(a)}{2}\right), \quad \theta = \mathrm{Arg}\,(g'(a)).$$

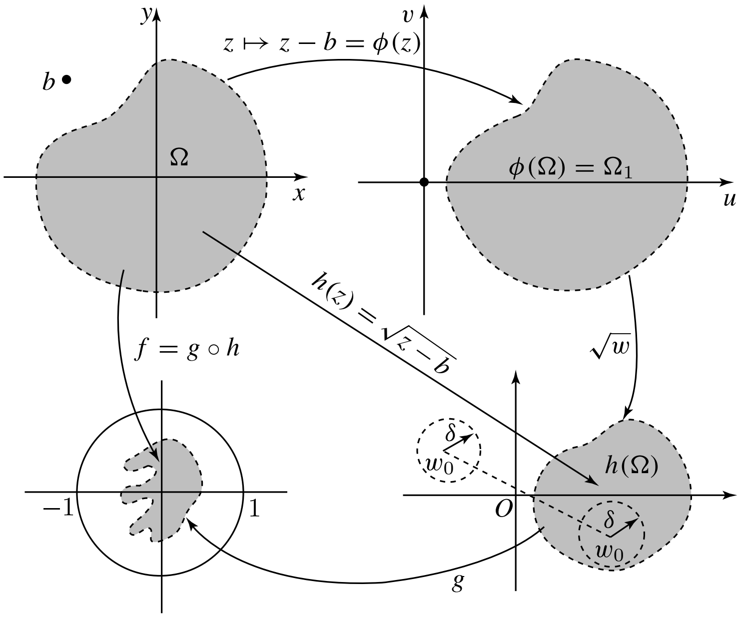


Figure 12.4:

Therefore, $\mathcal{F} \neq \emptyset$. Thus, we have established the existence of a conformal mapping f from Ω into a subdomain of Δ satisfying the desired condition; that is, $f \in \mathcal{F}$.

Onto: Since $\mathcal{F}$ is uniformly bounded, by Montel's Theorem, $\mathcal{F}$ is a normal family. We have already noted in the proof of Lemma 12.23, there is a finite upper bound for $f'(a)$ for all $f \in \mathcal{F}$. Let

$$\lambda = \sup\{f'(a) : f \in \mathcal{F}\}.$$

Clearly, $\lambda > 0$ because $f'(a) > 0$ for the function defined by $\mathcal{F}$. Also, λ is finite. By the definition of supremum, there is a sequence $\{g_n(z)\}$ in $\mathcal{F}$ such that $g_n'(a) \to \lambda$. By Montel's Theorem, there exists a subsequence (which we may as well assume is the whole sequence $\{g_n(z)\}$) of $\{g_n(z)\}$ that converges locally uniformly on Ω (to a limit function $g(z)$, say). But then Weierstrass's theorem (see Theorem 4.84) tells us that $g_n' \to g'$ locally uniformly on Ω, which gives that

$$\lim_{n\to\infty} g_n'(a) = g'(a).$$

Being the local uniform limit of a sequence of univalent analytic functions, $g(z)$ is either analytic and univalent on Ω or $g(z)$ is constant there. As $\{g_n'\}$ is locally uniformly convergent on Ω with limit g', we have $g(a) = 0$ and $g'(a) = \lambda > 0$. Thus, g cannot be constant in Ω, and is therefore univalent

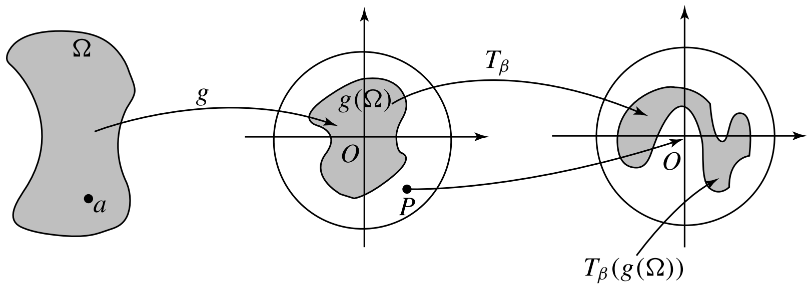


Figure 12.5:

in Ω. Since $g(\Omega)$ is open (by the open mapping theorem), $g(\Omega) \subset \Delta$ and so $|g(z)| < 1$. The above discussion shows that the family $\mathcal{F}$ is compact; i.e. $g \in \mathcal{F}$.

Observe that the map $g \mapsto g'(a) : \mathcal{F} \to \mathbb{R}$ is a continuous function on the compact family $\mathcal{F}$, and so it attains it maximum value. Let $g \in \mathcal{F}$ be a function such that $g'(a)$ is as large as possible. It remains to show that g is an onto map. To prove g is onto, we use Koebe's trick.

Suppose that g is not onto. Then there exists a point $\beta \in \Delta \setminus g(\Omega)$ (see Figure 12.5). We consider the familiar Möbius mapping

$$T_\beta(w) = \frac{\beta - w}{1 - \overline{\beta} w}.$$

We know that T_β is a one-to-one mapping of Δ onto itself with inverse T_β itself. Also $T_\beta(w) = 0$ iff $w = \beta$. Define

$$\Phi(z) = (T_\beta \circ g)(z) = \frac{\beta - g(z)}{1 - \overline{\beta} g(z)}, \quad z \in \Omega.$$

Now, $\Phi \in \mathcal{H}(\Omega)$, $\Phi(z) \neq 0$ on the simply connected domain Ω (since $\beta \notin g(\Omega)$) and so, Ω admits a square root mapping $H \in \mathcal{H}(\Omega)$ such that

$$H^2(z) = \Phi(z), \quad z \in \Omega.$$

Since T_β and g are one-to-one, so is H. Since $H^2(\Omega) \subseteq \Delta$, it follows that $H(\Omega) \subseteq \Delta$. But $H(a) \neq 0$ and hence, H cannot be in $\mathcal{F}$. Now, set $H(a) = \alpha$ and note that $0 < |\alpha| < 1$. Define

$$F(z) = -e^{i\theta}(T_\alpha \circ H)(z) = e^{i\theta} \frac{H(z) - \alpha}{1 - \overline{\alpha} H(z)},$$

where θ is chosen to ensure that $F'(a) > 0$. Indeed, $F \in \mathcal{H}(\Omega)$, $F(a) = 0$ and

$$F'(z) = e^{i\theta} \left(\frac{(1 - |\alpha|^2) H'(z)}{(1 - \overline{\alpha} H(z))^2} \right)$$

so that (note that $H'(a) \neq 0$)

$$F'(a) = e^{i\theta} \frac{H'(a)}{1 - |\alpha|^2}.$$

If we choose θ such that $e^{i\theta} = |H'(a)|/H'(a)$, then we obtain that

$$F'(a) = \frac{|H'(a)|}{1 - |\alpha|^2} = \frac{|H'(a)|}{1 - |H(a)|^2} > 0.$$

Now, F is in $\mathcal{F}$. As $H^2(z) = \Phi(z)$ and $\Phi(z) = (T_\beta \circ g)(z)$, we have

$$2H(z)H'(z) = \Phi'(z) = T'_\beta(g(z))g'(z) \text{ and } T'_\beta(w) = -\frac{1 - |\beta|^2}{(1 - \overline{\beta}w)^2},$$

which, for $z = a$, gives (note that $g(a) = 0$)

$$H'(a) = \frac{T'_\beta(0)}{2H(a)} g'(a) = -\left(\frac{1 - |\beta|^2}{2H(a)}\right) g'(a).$$

Further, $\Phi(a) = T_\beta(g(a)) = T_\beta(0) = \beta$ gives $H^2(a) = \beta$. Finally, using the above observations, we have

$$\begin{aligned} F'(a) &= \frac{|H'(a)|}{1 - |H(a)|^2} \\ &= \frac{1 - |\beta|^2}{2|H(a)|} g'(a) \frac{1}{1 - |H(a)|^2} \\ &= \frac{1 + |\beta|}{2\sqrt{|\beta|}} g'(a) \quad (\text{since } |H(a)|^2 = |\beta|) \\ &> g'(a). \end{aligned}$$

Thus F is in $\mathcal{F}$, but $F'(a)$ is greater than $g'(a)$ contradicts the choice of $g \in \mathcal{F}$ as maximizing $g'(a)$. This contradiction proves that $g(\Omega) = \Delta$. ∎

12.5 Bieberbach Conjecture

We recall that $f \in \mathcal{H}(\Delta)$ admits the Maclaurin series expansion about 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

We note that univalency is preserved under translation. Moreover, if f is univalent on Δ then $f'(z) \neq 0$ on Δ (see Corollary 12.8). In particular, $f'(0) \neq 0$. Since univalency is unaffected by magnification and translation, f is univalent in Δ iff g defined by

$$g(z) = \frac{f(z) - f(0)}{f'(0)} = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n = a_n/a_1),$$

is univalent in Δ. The function g in this form is referred to as a normalized (in the sense that $g(0) = 0 = g'(0) - 1$) analytic function in Δ. It is convenient to introduce the following standard notation:

$$\mathcal{S} = \{f \in \mathcal{H}(\Delta) : f(0) = 0 = f'(0) - 1 \text{ and } f \text{ is univalent on } \Delta\}.$$

Clearly, S is not closed under addition or multiplication of functions although it is closed under a number of elementary transformations (see also Exercise 12.61). Interesting members of $\mathcal{S}$ that can be checked geometrically are

$$z,\ z - \frac{z^2}{2},\ \frac{e^{z\pi} - 1}{\pi},\ \frac{z}{1-z},\ \frac{z}{1-z^2},\ \frac{z}{(1-z)^2}.$$

There are a number of basic questions that arise naturally in the theory of univalent functions. For instance, the problems of finding necessary and sufficient conditions for f to be a member of $\mathcal{S}$.

In 1916, Bieberbach proved that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_2| \leq 2$ and asked whether we always have $|a_n| \leq n$ for all $n = 2, 3, \ldots$. This became his famous conjecture. Although, the cases $n = 3, 4, 5, 6$ have been proved using powerful analysis, the conjecture was proven only in 1984 by de Branges. For a proof of the result that $|a_2| \leq 2$ (due to Bieberbach himself), it will be easier if we define a new class of analytic functions, namely Σ, the set of all analytic and univalent functions on $\{z : 1 < |z| < \infty\}$ with simple pole at ∞ with residue 1. Thus, a typical element $g \in \Sigma$ will be of the form

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \quad 1 < |z| < \infty.$$

12.35. Theorem. **(Area Theorem)** *Suppose that* $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ *is in* Σ. *Then we have* $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. *In particular,* $|b_1| \leq 1$.

Proof. For an arbitrary $r > 1$, let Γ_r be the image of the circle $\gamma_r = \{z \in \mathbb{C} : |z| = r\}$ under $g \in \Sigma$. Since g is univalent, Γ_r is a simple closed curve. Then Green's theorem (see Exercise 12.56) implies that the area A_r of the bounded domain surrounded by Γ_r is given by

$$A_r = \frac{1}{2i}\int_{\Gamma_r} \overline{w}\, dw = \frac{1}{2i}\int_{|z|=r} \overline{g(z)} g'(z)\, dz = \frac{1}{2}\int_0^{2\pi} \overline{g(re^{i\theta})} g'(re^{i\theta}) re^{i\theta}\, d\theta$$

which leads to

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right).$$

Since $A_r > 0$ for $r > 1$, we have $\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n-2} < 1$ which is true for every $r > 1$. The required result follows if we allow $r \to 1^+$. ∎

12.36. Theorem. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, *then we have*

(i) $|a_2| \leq 2$ [Bieberbach Theorem]

(ii) $\Delta_{1/4} \subseteq f(\Delta)$. [Koebe 1/4-Theorem]

The assertions cannot be improved.

Proof. For the proof of (i), we define

$$F(z) = \begin{cases} f(z)/z & \text{if } 0 < |z| < 1 \\ 1 & \text{if } z = 0. \end{cases}$$

Then, $F \in \mathcal{H}(\Delta)$. Since $f \in \mathcal{S}$ and $f(0) = 0$, $f(z) \neq 0$ for $0 < |z| < 1$. Consequently, $F(z) \neq 0$ in Δ and hence, F admits a square root function: $h(z) = (F(z))^{1/2}$ with $h(0) = 1$. Define $g(z) = zh(z^2)$. This in turn shows the existence of $g \in \mathcal{H}(\Delta)$ with

$$g(0) = 0 = g'(0) - 1 \text{ and } g^2(z) = f(z^2) \text{ for all } z \in \Delta.$$

We claim that g is univalent in Δ. Observe that

$$\begin{aligned} g(z_1) = g(z_2) &\implies g^2(z_1) = f(z_1^2) = f(z_2^2) = g^2(z_2) \\ &\implies z_1^2 = z_2^2 \text{ (since } f \text{ is univalent)} \\ &\implies z_1 = \pm z_2 \\ &\implies z_1 = z_2. \end{aligned}$$

Note that if $z_1 = -z_2$, then $g(z_1) = z_1 h(z_1^2) = -z_2 h(z_2^2) = -g(z_2)$ which (together with $g(z_1) = g(z_2)$) gives $g(z_1) = 0 = g(z_2)$. But $0 \notin h(\Delta)$ and so, $z_1 = z_2 = 0$. It is a simple exercise to see that

$$g(z) = z + \frac{a_2}{2}z^3 + \cdots \quad \text{and} \quad \phi(z) = \frac{1}{g(1/z)} = z - \frac{a_2}{2}z^{-1} + \cdots,$$

where $\phi(z) \in \Sigma$. By the area theorem, $|a_2/2| \leq 1$; that is $|a_2| \leq 2$.

(ii) To prove the inclusion $\Delta_{1/4} \subseteq f(\Delta)$ it suffices to show that $w \in \mathbb{C}\backslash f(\Delta) \implies |w| \geq 1/4$. If $w \in \mathbb{C}\backslash f(\Delta)$, then G defined by

$$G(z) = \frac{wf(z)}{w - f(z)}$$

is in $\mathcal{S}$. Indeed, $G \in \mathcal{H}(\Delta)$ with $G(0) = G'(0) - 1 = 0$ and $w \neq 0$ so that

$$\begin{aligned} G(z_1) = G(z_2) &\implies \frac{f(z_1)}{1 - w^{-1}f(z_1)} = \frac{f(z_2)}{1 - w^{-1}f(z_2)} \\ &\implies f(z_1)(1 - w^{-1}f(z_2)) = f(z_2)(1 - w^{-1}f(z_1)) \\ &\implies f(z_1) = f(z_2) \\ &\implies z_1 = z_2 \text{ (since } f \in \mathcal{S}). \end{aligned}$$

Thus, $G \in \mathcal{S}$. Moreover, as $G''(0)/2! = a_2 + w^{-1}$, it follows from Case (i) that $|a_2 + w^{-1}| \leq 2$. This inequality implies that

$$|w^{-1}| \leq |a_2| + 2 \leq 4, \quad \text{i.e. } |w| \geq 1/4,$$

which proves the inclusion $\Delta_{1/4} \subseteq f(\Delta)$.

Finally, consider the Koebe function

$$k_\alpha(z) = \frac{z}{(1 - e^{i\alpha}z)^2} = z + \sum_{n=2}^{\infty} n e^{i(n-1)\alpha} z^n = \frac{e^{-i\alpha}}{4}\left[\left(\frac{1 + e^{i\alpha}z}{1 - e^{i\alpha}z}\right)^2 - 1\right].$$

It follows that $k_\alpha \in \mathcal{S}$ and $k_\alpha(\Delta)$ is the complement of the slit from $-(1/4)e^{-i\alpha}$ to ∞. Note that $|a_2| = 2$ and $\Delta_{1/4}$ is the largest disk centered at 0 and contained in $k_\alpha(\Delta)$. Thus, the conclusion is sharp. ∎

12.6 The Bloch-Landau Theorems

Basically Bloch's theorem leads to two important topics:

- Bloch constants and Landau constants
- Bloch space (more generally, many function spaces associated with it).

For $f \in \mathcal{H}(\Delta)$, we raise the following

12.37. Problem. How big is the image domain $f(\Delta)$ in size?

We can talk about size of image domains in terms of disks, in particular open disks in the following sense. That is, what size open disks (need not be centered at the origin) can be placed inside $f(\Delta)$? Thus, this problem can be answered by means of giving a good estimate rather than finding the exact size of these disks.

Clearly, the answer does not seem to be easy. Intuitively, it is clear that it is not easy to find the precise size of such disks for each function in the space of all analytic functions defined on any arbitrary domain. Nevertheless, we shall try to answer these questions step by step. Before we get into the subject, we shall dispose some of the simple cases. We have already encountered many results concerning the size of the image domains, for example Liouville's theorem, exponential functions, and certain trigonometric functions. Recall that the set of all entire functions is divided into two subsets, namely the set of all polynomials (including constant functions) and the set of all (entire) transcendental functions. Let us consider the following cases:

- *Entire bounded functions*: In this case, by Liouville's theorem, the image domain $f(\mathbb{C})$ is a singleton set. So, no open disk can be placed inside $f(\mathbb{C})$.

- *Entire unbounded functions*: In this case, we see that $f(\mathbb{C})$ is either the whole complex plane $\mathbb{C}$ or the plane $\mathbb{C}$ minus a single point. For example, we know that if $p(z)$ is a non-constant polynomial (see fundamental theorem of algebra), then $p(\mathbb{C}) = \mathbb{C}$. Also, $\sin(\mathbb{C}) = \mathbb{C}$, $\cos(\mathbb{C}) = \mathbb{C}$ and $\exp(\mathbb{C}) = \mathbb{C}\setminus\{0\}$ (see Casorati-Weierstrass Theorem when ∞ is an isolated essential singularity of $\sin z$, $\cos z$ and e^z). So, an open disk of arbitrarily large radius can be placed inside $f(\mathbb{C})$.
- *Analytic functions on the unit disk* Δ: As $f(z)$ is entire iff g defined by $g(z) = f(Rz)$ is analytic in Δ for each $R > 0$, we note that this case leads to the previous case.

Thus, we deal only with the analytic functions on Δ. The reason for this is due to the celebrated Riemann Mapping Theorem. So, it suffices to pay attention to the last case because, by means of compositions, we retrieve the properties of the image domain of the analytic functions defined on any arbitrary simply connected domain. Avoiding constant functions in our discussion, it suffices to consider the family of functions $\mathcal{F} = \{f \in \mathcal{H}(\Delta) : f'(0) = 1\}$. The condition $f'(0) = 1$ is to ensure that f is not a constant function. First we notice that the existence of such a disk in $f(\Delta)$, $f \in \mathcal{F}$, is ensured by the open mapping theorem.

To make the size of such disks more meaningful, we define two constants. Consider $f \in \mathcal{F}$. Choose a point $w \in f(\Delta)$ and find the radii of all possible disks centered at w such that they lie completely in $f(\Delta)$. Repeat this with every point of $f(\Delta)$. Let these radii be named as $r_{\alpha,\beta}$, where α and β belong to some indexed sets Λ and Λ', respectively. If $r_{\alpha,\beta}$ is the radius of the β-th disk centered at $w_\alpha \in f(\Delta)$, then we define

$$L_f = \sup\{r_{\alpha,\beta} : \alpha \in \Lambda, \beta \in \Lambda'\}.$$

Then the Landau[16] constant is defined as

$$L = \inf_{f \in \mathcal{F}} L_f.$$

Clearly, the definition of L is related to the size of the image domains of functions in $\mathcal{F}$.

For L, Ahlfors's [1] ultra-hyperbolic method produced the non-sharp lower bound 1/2. Later Yanahigara [12] improved the lower bound for L to $\frac{1}{2} + 10^{-335}$. Again it is important to mention that, we do not know the exact value of L. The following bounds were determined by Robinson (1938) and independently by Rademacher [10],

$$\frac{1}{2} < L \leq \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)},$$

[16] The name "Landau" is in honor of the great mathematician Landau who contributed a lot to this field of research.

who further conjectured that L is actually the above mentioned upper bound, namely,

$$\frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} = 0.5432588 \cdots .$$

We let $\mathcal{H}_0$ denote the space of all functions in $\mathcal{H}(\Delta)$ such that $f'(0) \neq 0$.

12.38. Theorem. **(Bloch)** *For each $f \in \mathcal{H}_0$, there exists a positive constant b such that f maps some subdomain of Δ bianalytically onto a disk of radius $b|f'(0)|$.*

The disk referred to in this theorem is called a *univalent disk*. Thus, a disk about some point in $f(\Delta)$ is said to be a univalent disk iff it is the one-to-one and onto image of some subdomain of Δ. For $f \in \mathcal{H}_0$, define

$$B_f = \sup\{r_{\alpha,\beta} : \alpha \in \Lambda, \beta \in \Lambda'\},$$

where $r_{\alpha,\beta}$ is the radius of the β-th univalent disk about $w_\alpha \in f(\mathbb{C})$. Then the Bloch constant B is defined as

$$B = \inf_{f \in \mathcal{H}_0} B_f.$$

The following upper and lower estimates for B were found by Ahlfors and Grunsky [1], and Ahlfors [2]:

$$0.43 \cdots = \frac{\sqrt{3}}{4} \leq B \leq \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)(1+\sqrt{3})^{1/2}} \approx 0.4719.$$

It is conjectured that the correct value of B is precisely this upper bound although they did not prove this conjecture. Recently, on the basis of Bonk's work [4] and the Schwarz-Pick lemma, Chen Huaihui and P. M. Gauthier [6] improved the lower bound for Bloch's constant further as follows:

$$\frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} \leq B.$$

Theorem 12.38 yields the following

12.39. Corollary. *The range $f(\mathbb{C})$ of every non-constant entire function contains open disks of arbitrary radius.*

12.40. Example. Consider the following functions:

$$f_1(z) = z, \quad f_2(z) = az, \quad f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \quad f_4(z) = \frac{z}{1-z},$$

where a is some non-zero complex number. Then

$$L_{f_1} = 1 = B_{f_1}, \quad L_{f_2} = |a| = B_{f_2}, \quad L_{f_3} = \frac{\pi}{4} = B_{f_3}, \quad L_{f_4} = \infty = B_{f_4}.$$

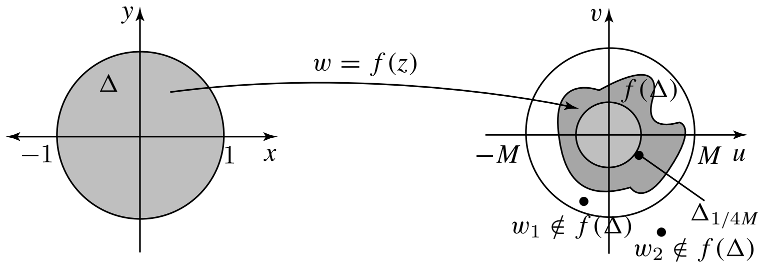


Figure 12.6: Description for a Bloch theorem

Finally, for the familiar Koebe function $k(z) = z/(1-z)^{-2}$ (also for its rotation), we have $L_k = B_k = \infty$. ●

12.41. Theorem. *Let* $f \in \mathcal{H}(\Delta)$ *with* $f(0) = 0 = f'(0) - 1$. *If furthermore* $|f(z)| \leq M$ *for some* $M > 0$, *then* $\Delta(0; 1/4M) \subseteq f(\Delta)$.

Proof. We suppose that $w \notin f(\Delta)$, i.e. $w \neq f(z)$ for any $z \in \Delta$. We wish to show that $|w| \geq 1/(4M)$ (see Figure 12.6). To do this, we define g by

$$g(z) = 1 - \frac{f(z)}{w}, \quad z \in \Delta.$$

Thus $g \in \mathcal{H}(\Delta)$, $g(0) = 1$, $g(z) \neq 0$ on Δ and therefore, by the square root property (see Theorem 4.39), g has an analytic square root function $h \in \mathcal{H}(\Delta)$ with $h(0) = 1$ such that

$$h^2(z) = 1 - \frac{f(z)}{w}.$$

From this, it follows that

$$2h(0)h'(0) = -\frac{f'(0)}{w}, \quad \text{i.e.} \quad h'(0) = -\frac{1}{2w}.$$

By the triangle inequality

$$|h(z)|^2 \leq 1 + \left|\frac{f(z)}{w}\right| \leq 1 + \frac{M}{|w|}, \quad z \in \Delta.$$

The Maclaurin series expansion of h is then given by

$$h(z) = \sum_{k=0}^{\infty} h_k z^k,$$

where $h_0 = h(0) = 1$, $h_1 = h'(0) = -1/(2w)$. Also, for $z = re^{i\theta}$, $r \in (0,1)$ (see Remark 12.43),

$$1 + |h_1|^2 r^2 \leq \sum_{k=0}^{\infty} |h_k|^2 r^{2k} = \frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2 \, d\theta \leq 1 + \frac{M}{|w|}$$

which gives

$$1+\frac{r^2}{4|w|^2} \le 1+\frac{M}{|w|}, \quad \text{i.e.} \;\; |w| \ge \frac{r^2}{4M}.$$

Letting $r \to 1$, we get the desired result. ■

12.42. Corollary. *If $g \in \mathcal{H}(\Delta_R)$, $g(0)=0$, $g'(0) \neq 0$ and $|g(z)| \le M$ for $|z|<R$, then*

$$\Delta\left(0; \frac{R^2|g'(0)|^2}{4M}\right) \subseteq g(\Delta_R).$$

Proof. Define f by

$$f(z) = \frac{g(Rz)}{Rg'(0)} = z + \cdots, \quad z \in \Delta.$$

Then f is a normalized analytic function and $|f(z)| \le M/(R|g'(0)|)$ for $z \in \Delta$. Using Theorem 12.41, we have

$$\Delta\left(0; \frac{1}{4(M/R|g'(0)|)}\right) \subseteq f(\Delta) = \frac{1}{R|g'(0)|} g(\Delta_R)$$

which is equivalent to

$$R|g'(0)|\Delta\left(0; \frac{R|g'(0)|}{4M}\right) \subseteq g(\Delta_R)$$

and the desired conclusion follows. ■

12.43. Remark. For those who are familiar with Hilbert spaces and complex Fourier series, it is easy to show that if $f \in \mathcal{H}(\Delta)$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt.$$

Indeed, this equality is a consequence of Parseval's equality (see for example the book by Ponnusamy [8]) applied to the function

$$t \mapsto f(re^{it}) = \sum_{k=0}^{\infty} a_k r^k e^{ikt}.$$

However, one can provide a direct proof of Bessel's inequality

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt. \tag{12.44}$$

To do this we consider the space X of all complex-valued functions defined on $[0, 2\pi]$ with an inner product given by

$$\langle \phi, \psi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi(t)\overline{\psi(t)}\, dt.$$

If $e_k(t) = e^{ikt},\ k \in \mathbb{Z}$, then

$$\langle e_k, e_l \rangle = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

so that $\{e_k(t) : k \in \mathbb{Z}\}$ forms an orthonormal basis for the inner product space X. We consider the correspondence $f_r : t \mapsto f(re^{it})$ and the orthonormal projection of f_r onto the space $Y_n = \text{span}\,\{e_0, e_1, \dots, e_{n-1}\}$. This projection is given by

$$s_n(t) = \langle f_r, e_0 \rangle e_0 + \langle f_r, e_1 \rangle e_1 + \cdots + \langle f_r, e_{n-1} \rangle e_{n-1}.$$

It follows that

$$\langle f_r, e_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_r(t) e^{-ikt}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{-ikt}\, dt = a_k r^k.$$

Therefore, for each $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} |a_k|^2 r^{2k} = \|s_n\|^2 \leq \|f_r\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta.$$

Letting $n \to \infty$, we get the inequality (12.44). ●

12.45. Theorem. (Landau's Theorem) *Let $f \in \mathcal{H}(\Delta)$ and $f'(0) = 1$. Then $f(\Delta)$ contains a disk of radius $\alpha > 0$, where α is an absolute constant.*

Proof. We may assume that f is analytic on $\overline{\Delta}$, since otherwise we can consider $\rho^{-1} f(\rho z)$ for ρ near 1. For $t \in [0, 1]$, we set

$$M(t) = \max_{|z| \leq t} |f'(z)|.$$

Then $M(t)$ is non-decreasing and, because $f'(0) = 1$, we have $M(t) \geq 1$. In fact, $M(t) = \max_{|z|=t} |f'(z)|$ (by the maximum principle). Now, we put

$$w(t) = tM(1-t).$$

Then $w : [0, 1] \to \mathbb{R}$ is continuous, $w(0) = 0$ and $w(1) = 1$ (see Figure 12.7).

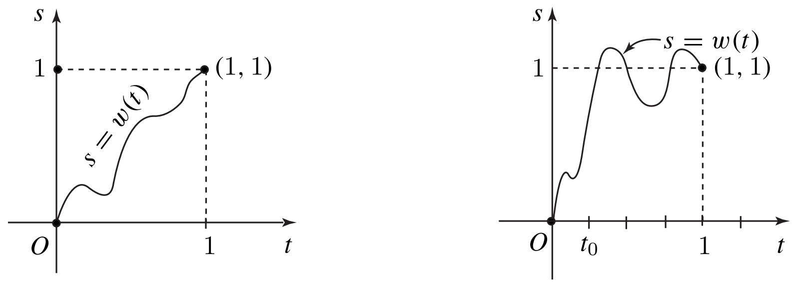


Figure 12.7: Choosing suitable branch.

Therefore, there exists at least one t, say $t_0 > 0$ such that $w(t_0) = 1$ and $w(t) < 1$ if $0 < t < t_0$ (Note that we can choose $t_0 = \inf\{t : w(t) = 1\}$ so that $w(t_0) = 1$ and $w(t) < 1$ for $0 \le t < t_0$). Let a be a point with

$$|a| = 1 - t_0 \quad \text{and} \quad |f'(a)| = M(1 - t_0) = \frac{w(t_0)}{t_0} = \frac{1}{t_0}.$$

Define $\phi(z) = f(z+a) - f(a)$ and note that $|z+a| \le |z| + |a| = |z| + 1 - t_0 < 1$ if $|z| < t_0$. Moreover,

- ϕ is analytic for $|z| < t_0$, $\phi(0) = 0$, and $|\phi'(0)| = |f'(a)| = 1/t_0 > 0$
- $|z + a| \le |z| + 1 - t_0 < 1 - t_0/2$ if $|z| < t_0/2$
- for $|z| < t_0/2$,

$$|\phi'(z)| = |f'(z + a)| \le M(1 - t_0/2) = \frac{w(t_0/2)}{t_0/2} < \frac{2}{t_0}$$

(because $w(t) < 1$ for $0 < t < t_0$). Hence, for $|z| < t_0/2$,

$$|\phi(z)| = |\phi(z) - \phi(0)| = \left| \int_0^z \phi'(\zeta)\, d\zeta \right| \le \frac{2}{t_0}|z| < 1.$$

Finally, as $\phi \in \mathcal{H}(\Delta_{t_0/2})$ with $\phi(0) = 0$, $|\phi'(0)| = 1/t_0$ and $|\phi(z)| < 1$ on $\Delta_{t_0/2}$, we can apply Corollary 12.42 to get

$$\Delta\left(0; \frac{R^2|\phi'(0)|^2}{4M}\right) \subseteq \phi(\Delta_R)$$

where $R = t_0/2$, $M = 1$, $|\phi'(0)| = 1/t_0$. Substituting these values we get

$$\Delta_{1/16} \subseteq \phi(\Delta_{t_0/2}).$$

This expresses the fact that the image of $\Delta(0; t_0/2)$ under $\phi(z)$ covers the disk $\Delta(0; 1/16)$. Equivalently, the image of $|z - a| < t_0/2$ under $\phi(z) = f(z + a) - f(a)$ covers the disk $\Delta(0; 1/16)$. That is,

$$\Delta(f(a); 1/16) \subseteq f(\Delta(a; t_0/2))$$

so that the image of Δ under f contains the disk $\Delta(f(a);1/16)$. ∎

Theorem 12.45 does not say anything about the center about which the disk of radius α can be found in $f(\Delta)$. In general, $f(0)$ is not the center of such a disk as the function

$$f_\epsilon(z) = \epsilon(e^{z/\epsilon} - 1) = z + \frac{z^2}{2!\epsilon} + \cdots$$

shows. Note that $f_\epsilon(0) = 0 = f'_\epsilon(0) - 1$, and $f_\epsilon(z)$ omits the value $-\epsilon$ as $-\epsilon \notin f_\epsilon(\Delta)$. This function shows that for small values of $\epsilon > 0$, $f_\epsilon(\Delta)$ never contains any disk $\Delta(f(0);\alpha)$. Thus, although $f_\epsilon(\Delta)$ contains some disk $\Delta(b;1/16)$, b is not necessarily $f(0)$.

12.46. Corollary. *If $f \in \mathcal{H}(\Delta(c;R))$ and $f'(c) \neq 0$, then the image $f(\Delta(c;R))$ contains a disk of radius $\alpha R|f'(c)|$ with $\alpha = 1/16$.*

Proof. Define

$$g(z) = \frac{f(c+Rz)}{Rf'(c)} = \frac{f(c)}{Rf'(c)} + z + \cdots .$$

Then $g \in \mathcal{H}(\Delta)$ and $g'(0) = 1$. By Theorem 12.45, $g(\Delta)$ contains a disk of radius α with $\alpha = 1/16$. The result follows. ∎

Let us now prove the following stronger version of Theorem 12.45.

12.47. Theorem. **(Bloch's Theorem)** *Let $f \in \mathcal{H}(\Delta)$ and $f'(0) = 1$. Then there exists a subdomain Ω (in fact a disk) in Δ on which f is univalent such that $f(\Omega)$ contains a disk of radius $1/24$.*

Proof. Following the proof of Theorem 12.45, the function ϕ defined by $\phi(z) = f(z+a) - f(a)$ satisfies the following conditions

(i) $\phi \in \mathcal{H}(\Delta_{t_0})$, $\phi(0) = 0$, $|\phi'(0)| = 1/t_0 > 0$

(ii) $|\phi'(z)| < \frac{2}{t_0}$ for $|z| < t_0/2$.

Now, we note that for $|z| < t_0/2$,

$$|\phi'(z) - \phi'(0)| \leq |\phi'(z)| + |\phi'(0)| < \frac{2}{t_0} + \frac{1}{t_0} = \frac{3}{t_0} = 3|\phi'(0)|.$$

Schwarz' lemma (see Corollary 6.28) applied to $F(z) = \phi'(z) - \phi'(0)$ for $|z| < t_0/2$ implies that

$$|\phi'(z) - \phi'(0)| \leq 3|\phi'(0)| \left(\frac{|z|}{t_0/2}\right) = |\phi'(0)| \left(\frac{6|z|}{t_0}\right) < |\phi'(0)|$$

whenever $|z| < r = t_0/6$. By Corollary 12.19, $\phi(z)$ is univalent for $|z| < r$. Thus, the function h defined by

$$h(z) = \frac{f(rz+a) - f(a)}{rf'(a)}$$

is a normalized univalent function in Δ. By the Koebe 1/4-theorem, $h(\Delta)$ covers the disk $\Delta(0; 1/4)$. Consequently, $f(rz+a) - f(a)$ covers the disk of radius $r|f'(a)|/4$ which is $1/24$, as $|f'(a)| = 1/t_0 = 1/6r$. Equivalently, $f(rz+a)$ contains a (univalent) disk of radius $1/24$ about $f(a)$. The desired result follows. ∎

Although the exact value of B is unknown, by our argument, for all functions in $\mathcal{F} = \{f \in \mathcal{H}(\Delta) : f'(0) = 1\}$ it follows that $B \geq 1/24$.

12.7 Picard's Theorem

We begin by deriving some easy consequences of simple connectivity for the representation of functions that omit two values. To do this, we recall that every analytic function on a simply connected domain admits a primitive. Moreover, if $\Omega \subseteq \mathbb{C}$ is a simply connected domain then, each non-vanishing function $f \in \mathcal{H}(\Omega)$ admits an analytic branch of the logarithm and an analytic branch of the square root function in Ω (see Theorems 4.38 and 4.39). That is, there exist $h, g \in \mathcal{H}(\Omega)$ with $g(z) \neq 0$ on Ω such that

- $e^{h(z)} = f(z)$ or $h(z) = \log f(z), \quad z \in \Omega$
- $f(z) = (g(z))^2$ or $g(z) = (f(z))^{1/2}, \quad z \in \Omega.$

We start by proving two lemmas which lead to the classical proof of Picard's little theorem with the help of Bloch's theorem.

12.48. Lemma. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, and let $f \in \mathcal{H}(\Omega)$ be such that $1 \notin f(\Omega)$ and $-1 \notin f(\Omega)$. Then there exists an $F \in \mathcal{H}(\Omega)$ such that $f(z) = \cos F(z)$ for $z \in \Omega$.*

Proof. Consider the function $1 - f^2(z) = (1 + f(z))(1 - f(z))$ which is non-vanishing in Ω and belongs to $\mathcal{H}(\Omega)$. Since $1 - f^2(z)$ has no zeros in Ω, by the square-root property there exists a function $g \in \mathcal{H}(\Omega)$ such that

$$g^2(z) = 1 - f^2(z), \quad \text{i.e.} \quad (f+ig)(f-ig) = f^2 + g^2 = 1, \quad z \in \Omega,$$

which shows that $f + ig$ belongs to $\mathcal{H}(\Omega)$ and has no zeros in Ω. As a consequence, $f + ig = \exp(iF)$ for some $F \in \mathcal{H}(\Omega)$ and it follows that

$$f - ig = \frac{1}{f + ig} = \exp(-iF)$$

so that

$$f = \frac{\exp(iF) + \exp(-iF)}{2} = \cos(F)$$

which completes the proof. ■

12.49. Lemma. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, and let $f \in \mathcal{H}(\Omega)$ be such that $0 \notin f(\Omega)$ and $1 \notin f(\Omega)$. Then there exists a $g \in \mathcal{H}(\Omega)$ such that*

$$f(z) = \frac{1 + \cos(\pi \cos \pi g(z))}{2} \tag{12.50}$$

and with the property that $g(\Omega)$ contains no open disk of radius 1.

Proof. Clearly the function $2f(z) - 1$ omits the values 1 and -1 in Ω. Thus, by Lemma 12.48, there exists a function $F \in \mathcal{H}(\Omega)$ such that

$$2f(z) - 1 = \cos \pi F(z), \quad z \in \Omega.$$

Moreover, as $\cos F(z)$ omits the values 1 and -1, the function $F \in \mathcal{H}(\Omega)$ must omit all integer values which implies the existence of $g \in \mathcal{H}(\Omega)$ with $F = \cos(\pi g)$. Therefore, the desired representation in (12.50) follows. It remains to show that $g(\Omega)$ contains no disk of radius 1 in the w-plane. We define

$$A = \left\{ a_{m,n} = m + i\pi^{-1} \log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z},\ n \in \mathbb{N} \right\}.$$

We claim that the points of A do not belong to $g(\Omega)$, yet every open disk of radius 1 contains one point of A. To do this, we consider a general point in A:

$$a := a_{m,n} \quad \text{for some } m \in \mathbb{Z}, n \in \mathbb{N}.$$

Then, as $i\pi a = i\pi m - \log(n + \sqrt{n^2 - 1})$, we have

$$\begin{aligned} \cos \pi a &= \frac{e^{i\pi a} + e^{-i\pi a}}{2} \\ &= \frac{(-1)^m}{2} \left(\frac{1}{n + \sqrt{n^2 - 1}} + n + \sqrt{n^2 - 1} \right) \\ &= (-1)^m n \end{aligned}$$

so that $\cos \pi(\cos \pi a) = \cos \pi(n(-1)^m) = \cos \pi n = (-1)^n$. Thus, if $a \in g(\Omega)$, there exists a point $z_a \in \Omega$ with $a = g(z_a)$ and therefore,

$$f(z_a) = \frac{1 + \cos \pi(\cos \pi g(z_a))}{2} = \frac{1 + \cos \pi(\cos \pi a)}{2} = \frac{1 + (-1)^n}{2}$$

showing that $f(z)$ assumes the values 0, or 1 or both, which contradicts that $f(z) \neq 0, 1$ for $z \in \Omega$. This observation shows that the first claim $g(\Omega) \cap A = \emptyset$ is verified.

Next we must show that no disk of radius 1 can be disjoint from the set A. For this, we estimate the difference between the consecutive real and imaginary parts of points from A. Clearly, the points of A are simply the vertices of a rectangular grid in $\mathbb{C}$ and the length of every rectangle (i.e. the difference between the real parts of the two consecutive horizontal points) is seen to be 1. Also, each rectangle has a height (i.e. the difference between the imaginary points of the two consecutive vertical points) less than 1. Indeed, by the monotonicity of $\log(x)$, we have

$$\begin{aligned}\log\left(\frac{n+1+\sqrt{n^2+n}}{n+\sqrt{n^2-1}}\right) &= \log\left(\frac{1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}}\right)\\ &\leq \log\left(1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}\right)\\ &\leq \log(2+\sqrt{3}) < \pi\end{aligned}$$

so that

$$\frac{1}{\pi}\log(n+1+\sqrt{(n+1)^2-1}) - \frac{1}{\pi}\log(n+\sqrt{n^2-1}) < 1.$$

These two observations imply that corresponding to every given complex number $w \in \mathbb{C}$ there exists an $a = a_{m,n}$ (the closest one in A) with

$$|\mathrm{Re}\, a - \mathrm{Re}\, w| \leq \frac{1}{2} \quad\text{and}\quad |\mathrm{Im}\, a - \mathrm{Im}\, w| < \frac{1}{2},$$

so that

$$|a-w| \leq |\mathrm{Re}\, a - \mathrm{Re}\, w| + |\mathrm{Im}\, a - \mathrm{Im}\, w| < 1.$$

Thus, every disk of radius 1 intersects A. But we have already shown that $g(\Omega)\cap A = \emptyset$ and therefore, $g(\Omega)$ contains no disk of radius 1. ∎

We can now prove that the range of every non-constant entire function is the complex plane with at most one exception (see Theorem 7.37).

12.51. Proof of Picard's little theorem. We shall show that every $f \in \mathcal{H}(\mathbb{C})$ which omits 0 and 1 is necessarily a constant. For this, we apply Lemma 12.49 with $\Omega = \mathbb{C}$. According to Lemma 12.49, we have

$$f(z) = \frac{1+\cos\left(\pi\left(\cos\, \pi g(z)\right)\right)}{2}$$

where $g \in \mathcal{H}(\mathbb{C})$ and $g(\mathbb{C})$ contains no disk of radius 1. Since f is not constant, g cannot be constant either. Therefore, we can choose c such that $g'(c) \neq 0$. We then define

$$G(z) = \frac{1}{\alpha}g\left(\frac{\alpha}{g'(c)}z + c\right) = \frac{g(c)}{\alpha} + z + \cdots$$

which is entire and $G'(0) = 1$. Since $g(z)$ does not contain any disk of radius 1, $G(z)$ does not contain a disk of radius α.

But Bloch's theorem (see Theorem 12.47 with $\alpha = 1/24$, Landau's Theorem 12.45 with $\alpha = 1/16$, and also Corollary 12.39) implies that the range of every non-constant entire function contains disks of every radius. This is a contradiction, so g is a constant and hence, f is constant. ■

Another application of Lemma 12.49 is Schottky's theorem.

12.52. Theorem. (Schottky's Theorem) *Let* $f(z) = a_0 + a_1 z + \cdots$ *be analytic for* $|z| < R$ *and omit the values* 0 *and* 1*. Then, for any* $0 < \beta < 1$*, there exists a constant* $S(a_0, \beta)$ *depending only on* a_0 *and* β*, such that* $|f(z)| \le S(a_0, \beta)$ *for* $|z| \le R\beta$*.*

Proof. By Lemma 12.49, there exists a $g \in \mathcal{H}(\Delta_R)$ such that

$$f(z) = \frac{1 + \cos(\pi \cos \pi g(z))}{2} = \cos^2\left(\frac{\pi \cos \pi g(z)}{2}\right) \tag{12.53}$$

and with the property that $g(\Delta_R)$ contains no open disk of radius 1.

Suppose $a \in \Delta_{R\beta}$. It is clear that the function

$$G(z) = \frac{g(a + (1-\beta)Rz)}{(1-\beta)Rg'(a)} = \frac{g(a)}{(1-\beta)Rg'(a)} + z + \cdots$$

is analytic for $|z| < 1$ and satisfies the conditions of the Bloch-Landau theorem (see Corollary 12.46), provided that $g'(a) \neq 0$. Since $g(\Delta_R)$ contains no disk of radius 1, $G(\Delta)$ does not cover a disk of radius

$$\frac{1}{(1-\beta)R|g'(a)|}.$$

On the other hand, by Corollary 12.46, it follows that

$$\frac{1}{(1-\beta)R|g'(a)|} > \frac{1}{16}, \quad \text{i.e. } |g'(a)| < \frac{16}{(1-\beta)R}.$$

This is also true if $g'(a) = 0$. If $|z| \le R\beta$, we have

$$|g(z) - g(0)| = \left|\int_0^z g'(\zeta)\, d\zeta\right| \le \frac{16|z|}{(1-\beta)R} \le \frac{16\beta}{1-\beta}$$

so that

$$|g(z)| \le |g(0)| + \frac{16\beta}{1-\beta}.$$

Note that $g(0)$ depends only on a_0. If we let $w = \frac{\pi}{2}\cos(\pi g(z))$, then (12.53) gives

$$\begin{aligned} |f(z)|^{1/2} &= \left|\frac{e^{iw} + e^{-iw}}{2}\right| \le e^{|w|} \le \exp\left((\pi/2)\exp(\pi|g(z)|)\right) \\ &\le \exp\left((\pi/2)\exp(\pi[|g(0)| + 16\beta/(1-\beta)]\right). \end{aligned}$$

■

12.8 Exercises

12.54. Determine whether the following statements are true or false. Justify your answer.

(a) Let D be a domain in $\mathbb{C}$ and $f \in \mathcal{H}(D)$. Suppose that B is a disk of positive radius contained in D. If $\operatorname{Re} f$ or $\operatorname{Im} f$ or $\operatorname{Arg} f$ is constant on B, then f is constant on D.

(b) If f is univalent and analytic in an open set D except for isolated singularities, then f can have at most one singularity and that as a simple pole.

(c) An even function in a domain D is not univalent in D.

(d) The function $f(z) = z/(1-z)^3$ is univalent for $|z| < 1/2$ but not in any larger disk centered at the origin.

(e) If $f \in \mathcal{H}(\Delta)$ and $|f'(z) - 1 - i| < \sqrt{2}$ for $z \in \Delta$, then f is univalent in Δ.

(f) If f is analytic in a convex domain D such that $\operatorname{Re} f'(z) \neq 0$ for all $z \in D$, then f is univalent in D.

(g) If the polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ $(a_n \neq 0)$ is univalent in the unit disk Δ, then $n|a_n| \leq |a_1|$.

(h) The function $f(z) = [\exp(6z) - 1]/6$ omits the value $-1/6$ and this function does not contradict Koebe's one-quarter theorem.

(i) Both $\cos z$ and $\sin z$ are univalent in

- (i) the strip $\Omega = \{z : \alpha < \operatorname{Re} z < \alpha + \pi\}$
- (ii) the semistrip $\Omega = \{z : \alpha < \operatorname{Re} z < \alpha + 2\pi,\ \operatorname{Im} z > 0\}$
- (iii) the semistrip $\Omega = \{z : \alpha < \operatorname{Re} z < \alpha + 2\pi,\ \operatorname{Im} z < 0\}$,

where α is a fixed real number.

(j) If $\Omega = \{z : 0 < \operatorname{Re} z < 2\pi, \operatorname{Im} z > 0\}$, then the function $\cos z = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$ is univalent on the domain Ω and $\cos(\Omega) = \mathbb{C} \setminus [-1, +\infty)$.

(k) If $\Omega = \{z : |\operatorname{Re} z| < \pi/2, \operatorname{Im} z > 0\}$, then the function $\sin z = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$ is univalent on the domain Ω and $\sin(\Omega)$ is the upper half-plane.

(l) The function $\coth(z/2) = \frac{e^z+1}{e^z-1}$ maps the semistrip $\Omega = \{z : \operatorname{Re} z > 0, |\operatorname{Im} z| < \pi\}$ onto $\Omega' = \{w : \operatorname{Re} w > 0\}$.

(m) The function $f(z) = z/(1+|z|)$ is a homeomorphism of $\mathbb{C}$ onto Δ.

(n) The punctured unit disk $\Delta \setminus \{0\}$ is homeomorphic to the punctured plane $\mathbb{C} \setminus \{0\}$ but not conformally equivalent.

(o) If $D = \{z : |z| \leq 1 \text{ or } |z+2| \leq 1\}$ and $D^c = (\mathbb{C} \setminus D) \cup \{\infty\}$, then D^c is conformally equivalent to the open upper half-plane $\mathbb{H}^+$.

(p) There exists an analytic univalent function f that maps the infinite strip $\{z : 0 < \text{Im}\, z < 1\}$ onto the unit disk Δ.

(q) If Ω is a simply connected domain, $f \in \mathcal{H}(\Omega)$ and $0 \notin f(\Omega)$, then f has an analytic n-th root; that is there exists a $g \in \mathcal{H}(\Omega)$ with $g^n(z) = f(z)$ for $z \in \Omega$.

(r) If $\Omega \subset \mathbb{C}$ $(\Omega \neq \mathbb{C})$ is a simply connected domain with the property that every non-vanishing analytic function on Ω has a square root property, then there exists a univalent analytic function g on Ω such that $g(\Omega) \subset \Delta$.

(s) Let Ω be a simply connected domain, a and b be any two points of Ω. Then there exists an analytic automorphism f of Ω such that $f(a) = b$.

(t) If f is an univalent analytic mapping from Δ onto a simply connected domain D such that $f(0) = 0$ and $\alpha = \text{dist}\,(0, \mathbb{C}\backslash D)$, then $|f'(0)| \geq \alpha$.

(u) There exists a unique conformal map $w = f(z)$ which carries Δ onto itself such that $f(1/2) = i/3$ and $f'(i/3) > 0$.

(v) If $w = f(z)$ is a conformal mapping of a domain D onto D', then the area A of the domain D' is given by $A = \int\int_D |f'(z)|^2\, dx\, dy$.

(w) The area of the image of the unit disk Δ under the univalent function $f(z) = z + (z^2/2)$ is $3\pi/2$.

(x) There exists a conformal mapping of the crescent shaped domain $D_1 = \Delta(2; 2)\backslash\overline{\Delta}(1; 1)$ onto the unit disk.

(y) If $A(r) = \{z : 0 < r < |z| < 1\}$ and $\Omega = \Delta\backslash\overline{\Delta}(9/28; 5/28)$, then there exists an r such that $A(r)$ is conformally equivalent to Ω.

(z) If $A(R) = \{z : 1 < |z| < R\}$ and $\Omega = \Delta(1; 5/2)\backslash\overline{\Delta}$, then there exists an R such that $A(R)$ is conformally equivalent to Ω.

12.55. Suppose f is analytic in a neighborhood N of the origin and that $f'(0) \neq 0$. Show that there exists a disk $\Omega \subset N$, a positive integer n, and an analytic function h such that $f(z) = f(0) + (h(z))^n$ in Ω.

12.56. If γ is a positively oriented simple closed curve and D is the region bounded by γ, then $\text{Area}\,(D) = \dfrac{1}{2i}\displaystyle\int_\gamma \overline{z}\, dz$.

12.57. If γ is a closed curve and f is analytic in a domain containing γ, then $\displaystyle\int_\gamma \overline{f(z)} f'(z)\, dz$ is a purely imaginary number.

12.58. If $f : \Omega \to \mathbb{C}$ is one-to-one on Ω, then we know that $f'(z) \neq 0$ on Ω. Does this result hold for a function of a real variable? Explain with an example.

12.59. Find a conformal, one-to-one map f from the unit disk Δ onto the domain $\Omega = \{w : |\mathrm{Im}\, w| < \pi/2\} \setminus \{u : u \leq -1\}$ such that $f(0) = 1$.

12.60. Let $\Omega = \{z \in \Delta : \mathrm{Im}\, z > 1/2\}$. Find a conformal map f which maps Ω one-to-one and onto Δ such that $f(3i/4) = 0$.

12.61. If $f \in \mathcal{S}$, then show that each of the functions G defined below is in $\mathcal{S}$:

(i) $G(z) = \alpha^{-1} f(\alpha z) \quad (|\alpha| \leq 1)$

(ii) $G(z) = \overline{f(\overline{z})}$

(iii) $G(z) = \{f(z^n)\}^{1/n} \quad (n \in \mathbb{N})$

(iv) $G(z) = \dfrac{wf(z)}{w - f(z)} \quad (w \notin f(\Delta))$

(v) $G(z) = \dfrac{f\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)} \quad (|\alpha| \leq 1).$

12.62. Give an example of a normal family of functions in $\mathcal{H}$ which contains unbounded functions.

12.63. Supply the proof of Corollary 12.17.

12.64. Suppose that f is conformal map of a simply connected domain $\Omega\,(\neq \mathbb{C})$ onto Δ, and $a \in \Omega$. Find a conformal mapping g of Ω onto Δ such that $g(a) = 0$ and $g'(a) > 0$.

12.65. Let $\Omega\,(\neq \mathbb{C})$ be a simply connected domain and $b \notin \Omega$. Suppose that h is an analytic branch of $\sqrt{z-b}$. Show that h is univalent in Ω and $h(\Omega)$ is disjoint from $-h(\Omega)$.

12.66. Let $f \in \mathcal{H}(\Delta)$, $f(0) = f'(0) - 1 = 0$ and $|f(z)| \leq M$ for all $z \in \Delta$. Using the Cauchy estimate show that $f(\Delta) \supset \Delta_{1/6M}$.

12.67. For $0 < \lambda < 1$, define

$$f_\lambda(z) = \frac{z}{(1-\lambda z)^2}, \quad g_\lambda(z) = \frac{z}{1-\lambda z}, \quad \text{and} \quad h_\lambda(z) = \frac{z}{1-\lambda z^2}.$$

If f denotes one of these functions, then $f(0) = f'(0) - 1 = 0$, and $f \in \mathcal{H}(\overline{\Delta})$. Find an explicit radius $r > 1/16$ such that $\Delta_r \subseteq f(\Delta)$. Also, how large can you make r?

12.68. Suppose that $f(z)$ is entire and $\mathrm{Im}\, f(z) \neq 0$ whenever $|z| \neq 1$. Prove that $f(z)$ is constant.

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Index of Special Notation

Symbol	Meaning
$\emptyset$	empty set
$a \in S$	a is an element of the set S
$a \notin S$	a is not an element of S
$\{x : \ldots\}$	set of all elements with the property $\ldots$
$X \cup Y$	set of all elements in X or Y; i.e. union of the sets X and Y
$X \cap Y$	set of all elements in X as well as in Y; i.e. intersection of the sets X and Y
$X \subseteq Y$	set X is contained in the set Y; i.e. X is a subset of Y
$X \subset Y$ or $X \subsetneq Y$	$X \subseteq Y$ and $X \neq Y$; i.e. set X is a proper subset of Y
$X \times Y$	Cartesian product of sets X and Y; i.e. $\{(x, y) : x \in X,\ y \in Y\}$
$X \setminus Y$ or $X - Y$	set of all elements that live in X but not in Y
X^c	complement of X
$\Longrightarrow$	implies (gives)
$\not\Longrightarrow$	does not imply
$\Longleftrightarrow$	if and only if, or briefly 'iff'
$\longrightarrow$ or $\to$	converges (approaches) to; into
$\not\longrightarrow$ or $\not\to$	does not converge
$\mathbb{N}$	set of all natural numbers, $\{1, 2, \ldots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$
$\mathbb{Z}$	set of all integers (positive, negative and zero)
$\mathbb{Q}$	set of all rational numbers, $\{p/q : p,\ q \in \mathbb{Z},\ q \neq 0\}$
$\mathbb{R}$	set of all real numbers, real line
$\mathbb{R}_\infty$	$\mathbb{R} \cup \{-\infty, \infty\}$, extended real line
$\mathbb{C}$	set of all complex numbers, complex plane
$\mathbb{C}_\infty$	extended complex plane, $\mathbb{C} \cup \{\infty\}$

$\mathbb{R}^n$	n-dimensional real Euclidean space, the set of all n-tuples $x = (x_1, \ldots, x_n)$, $x_k \in \mathbb{R}$, $k = 1, \ldots, n$
$i\mathbb{R}$	set of all purely imaginary numbers, imaginary axis
domain	open and connected
$\mathbb{H}^+$	upper half-plane
$\mathbb{H}^-$	lower half-plane
$\overline{z}$	$\overline{z} := x - iy$, complex conjugate of $z = x + iy$
$\lvert z\rvert$	$\sqrt{x^2 + y^2}$, modulus of $z = x + iy$, $x, y \in \mathbb{R}$
$\mathrm{Re}\, z$ $(\mathrm{Im}\, z)$	real part x (imaginary part y) of $z = x + iy$
$\arg z$	set of real values of θ such that $z = \lvert z\rvert e^{i\theta}$
$\mathrm{Arg}\, z$	argument $\theta \in \arg z$ such that $-\pi < \theta \leq \pi$; the principal value of $\arg z$
$\limsup \lvert z_n\rvert$	upper limit of the real sequence $\{\lvert z_n\rvert\}$
$\liminf \lvert z_n\rvert$	lower limit of the real sequence $\{\lvert z_n\rvert\}$
$\lim \lvert z_n\rvert$	limit of the real sequence $\{\lvert z_n\rvert\}$
$\sup S$	least upper bound, or the supremum, of the set $S \subset \mathbb{R}_\infty$
$\inf S$	greatest lower bound, or the infimum, of the set $S \subset \mathbb{R}$
$\inf_{x \in D} f(x)$	infimum of f in D
$\max S$	the maximum of the set $S \subset \mathbb{R}$; the largest element in S
$\min S$	the minimum of the set $S \subset \mathbb{R}$; the smallest element in S
$f : D \longrightarrow D_1$	f is a function from D into D_1
$f(z)$	the value of the function at z
$f(D)$	set of all values $f(z)$ with $z \in D$; i.e. $w \in f(D) \Longleftrightarrow \exists\, z \in D$ such that $f(z) = w$
$f^{-1}(D)$	$\{z :\, f(z) \in D\}$, the preimage of D under f
$f^{-1}(w)$	the preimage of one element $\{w\}$
$f \circ g$	composition mapping of f and g
$\mathrm{dist}\,(z, A)$	distance from the point z to the set A i.e. $\inf\{\lvert z - a\rvert :\, a \in A\}$
$\mathrm{dist}\,(A, B)$	distance between two sets A and B i.e. $\inf\{\lvert a - b\rvert :\, a \in A, b \in B\}$
$[z_1, z_2]$	closed line segment connecting z_1 and z_2; $\{z = (1-t)z_1 + tz_2 :\, 0 \leq t \leq 1\}$
(z_1, z_2)	open line segment connecting z_1 and z_2; $\{z = (1-t)z_1 + tz_2 :\, 0 < t < 1\}$

$\Delta(a;r)$	open disk $\{z \in \mathbb{C} : \|z-a\| < r\}$ $(a \in \mathbb{C},\ r > 0)$
$\overline{\Delta(a;r)}$	closed disk $\{z \in \mathbb{C} : \|z-a\| \leq r\}$ $(a \in \mathbb{C},\ r > 0)$
$\partial\Delta(a;r)$	the circle $\{z \in \mathbb{C} : \|z-a\| = r\}$
Δ_r	$\Delta(0;r)$
Δ	$\Delta(0;1)$, unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$
$\partial\Delta$	unit circle $\{z \in \mathbb{C} : \|z\| = 1\}$
e^z	$\exp(z) = \sum_{n\geq 0} \frac{z^n}{n!}$, an exponential function
$\operatorname{Log} z$	$\ln\|z\| + i\operatorname{Arg} z,\ \ -\pi < \operatorname{Arg} z \leq \pi$
$\log z$	$\ln\|z\| + i\arg z := \operatorname{Log} z + 2k\pi i,\ k \in \mathbb{Z}$
$\frac{\partial}{\partial z}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, Cauchy-Riemann operator
$\frac{\partial}{\partial \overline{z}}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
f_z	$\frac{\partial f}{\partial z}$, partial derivative of f w.r.t z
$f_{\overline{z}}$	$\frac{\partial f}{\partial \overline{z}}$, partial derivative of f w.r.t z
$\operatorname{Int}(\gamma)$ $(\operatorname{Ext}(\gamma))$	interior (exterior) of γ
$\gamma_1 + \gamma_2$	sum of two curves γ_1, γ_2
$L(\gamma)$	length of the curve γ
$f^{(n)}(a)$	n-th derivative of f evaluated at a
$f(z) = O(g(z))$ as $z \to a$	there exists a constant K such that $\|f(z)\| \leq K\|g(z)\|$ for all values of z near a
$f(z) = o(g(z))$ as $z \to a$	$\lim_{z\to a} \frac{f(z)}{g(z)} = 0$
$\lim_{n\to\infty} z_n = z$, or $z_n \to z$	sequence $\{z_n\}$ converges to z
$d(z_n, z) \to 0$	sequence $\{z_n\}$ converges to z with a metric d
$\operatorname{Res}[f(z);a]$	residue of f at a
$\sum \operatorname{Res}[f(z);D]$	sum of the residues of f at each singularity of f inside D
$\mathcal{H}(D)$	family of all analytic (holomorphic) functions on D

Hints and Solutions for Selected Exercises

Chapter 1, Exercises 1.7:

1.53:

(k) If $\operatorname{Im} z \neq 0$, then

$$\begin{aligned}\left|\frac{\operatorname{Im} z^n}{\operatorname{Im} z}\right| &= \left|\frac{z^n - (\overline{z})^n}{2i} \cdot \frac{2i}{z - \overline{z}}\right| \\ &= |z^{n-1} + z^{n-2}\overline{z} + \cdots + (\overline{z})^{n-1}| \leq n|z|^{n-1}.\end{aligned}$$

(r) No. For example take $z_1 = i$ and $z_2 = -i$. Note that the two roots of $z^2 + z + 1 = 0$ are $w_1 = (-1 + i\sqrt{3})/2$, $w_2 = (-1 - i\sqrt{3})/2$ and they satisfy $w_1^2 = w_2$ and $w_2^2 = w_1$.

(s) If $\operatorname{Re} z = a > 0$, then $|z - 1| < |z|$ so that

$$\left|\frac{z^{n+1} - 1}{z - 1}\right| > \left|\frac{z^{n+1} - 1}{z}\right| \geq |z|^n - \frac{1}{|z|}$$

from which we obtain $|1 + z + z^2 + \cdots + z^n| > |z|^n - 1/|z|$. Since $1/|z| < 1$ and the left hand side of the above inequality is an integer we can neglect the term $1/|z|$ if we replace the symbol $>$ by $\geq$. Proof of (ii) is similar.

(w) Let $\lim_{n\to\infty} z_n = \ell$. Then, for a given $\epsilon > 0$ there exists an N such that $|z_n - \ell| < \epsilon/2$ for $n > N$. Now the triangle inequality gives

$$|Z_n - \ell| = \frac{1}{n}\left|\sum_{k=1}^{n}(z_k - \ell)\right| \leq \frac{1}{n}\sum_{k=1}^{n}|z_k - \ell|.$$

Therefore, for $n > N$, we have

$$\begin{aligned}|Z_n - \ell| &\leq \frac{1}{n}\sum_{k=1}^{N}|z_k - \ell| + \frac{1}{n}\sum_{k=N+1}^{n}|z_k - \ell| \\ &\leq \frac{1}{n}\sum_{k=1}^{N}|z_k - \ell| + \frac{1}{n}(n - N)\frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{1}{n}\sum_{k=1}^{N}|z_k - \ell|.\end{aligned}$$

Choose n sufficiently large so that $\frac{1}{n}\sum_{k=1}^{N}|z_k-\ell| < \frac{\epsilon}{2}$. Thus, we have $|Z_n-\ell|<\epsilon$ for sufficiently large n and so, $Z_n\to\ell$ as $n\to\infty$. A simple consequence of this is that

$$\frac{1}{n}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}\right]\to 0, \text{ since } \frac{1}{n}\to 0.$$

1.54. If we let $z_j=|z_j|e^{i\operatorname{Arg} z_j}$ $(j=1,2)$, then

$$z_1z_2=|z_1z_2|e^{i(\operatorname{Arg} z_1+\operatorname{Arg} z_2)} \quad \text{with} \quad -2\pi<\operatorname{Arg} z_1+\operatorname{Arg} z_2\leq 2\pi$$

and $z_1z_2=|z_1z_2|e^{i\operatorname{Arg}(z_1z_2)}$ with $-\pi<\operatorname{Arg}(z_1z_2)\leq\pi$. Thus, we must have

$$\operatorname{Arg}(z_1z_2)=\operatorname{Arg} z_1+\operatorname{Arg} z_2+2k_1\pi,$$

where k_1 is some integer in the set $\{-1,0,1\}$ such that

$$-\pi<\operatorname{Arg} z_1+\operatorname{Arg} z_2+2k_1\pi\leq\pi.$$

Now, (a) is clear. Similarly we can establish (b).

1.55. Assume that the result is true for $m=n>1$. For $m=n+1$, we have

$$\begin{aligned}
&\left(\sum_{k=1}^{n+1}|z_k|^2\right)\left(\sum_{k=1}^{n+1}|w_k|^2\right)-\sum_{1\leq k<j\leq n+1}|z_k\overline{w}_j-z_j\overline{w}_k|^2\\
&= \left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|^2\right)+|z_{n+1}|^2|w_{n+1}|^2\\
&\qquad+\sum_{k=1}^{n}|z_kw_{n+1}|^2+\sum_{k=1}^{n}|z_{n+1}w_k|^2\\
&\qquad-\sum_{1\leq k<j\leq n}|z_k\overline{w}_j-z_j\overline{w}_k|^2-\sum_{k=1}^{n}|z_k\overline{w}_{n+1}-z_{n+1}\overline{w}_k|^2\\
&= \left|\sum_{k=1}^{n}z_kw_k\right|^2+|z_{n+1}w_{n+1}|^2\\
&\qquad+\sum_{k=1}^{n}[|z_kw_{n+1}|^2+|z_{n+1}w_k|^2-|z_k\overline{w}_{n+1}-z_{n+1}\overline{w}_k|^2]\\
&= \left|\sum_{k=1}^{n}z_kw_k\right|^2+|z_{n+1}w_{n+1}|^2+2\operatorname{Re}\left\{\sum_{k=1}^{n}z_kw_k\overline{z}_{n+1}\overline{w}_{n+1}\right\}\\
&= \left|\sum_{k=1}^{n+1}z_kw_k\right|^2.
\end{aligned}$$

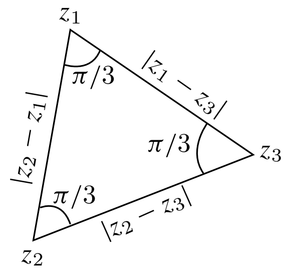

Figure 12.8: Description for equilateral triangle.

1.56. We have $z_2 - z_1 = e^{i\pi/3}(z_3 - z_1)$ and $z_1 - z_3 = e^{i\pi/3}(z_2 - z_3)$ so that

$$(z_2 - z_1)(z_2 - z_3) = (z_1 - z_3)(z_3 - z_1)$$

from which we get the required conclusion (see Figure 12.8). Let $z_1 = 1+i$ and $z_2 = 1-i$. Then $z_1^2 = 2i$, $z_2^2 = -2i$, $z_1 + z_2 = z_1 z_2 = 2$. Since z_1, z_2, z_3 should form an equilateral triangle, we should have

$$2i - 2i + z_3^2 = 2 + 2z_3, \quad \text{i.e.} \quad z_3^2 - 2z_3 - 2 = 0.$$

Solving the last equation gives $z_3 = 1 + \sqrt{3}$ or $1 - \sqrt{3}$.

1.61. $|z - (-7 + ib)| = (48 + b^2)^{1/2}$, $b \in \mathbb{R}$.

Chapter 2, Exercises 2.5:

2.63:

(d) For z_1, z_2 in Δ, we have $f(z_1) = f(z_2) \Longrightarrow (z_1 - z_2)(n + \zeta) = 0$ where $\zeta = z_1^{n-1} + z_1^{n-2} z_2 + \cdots + z_1 z_2^{n-2} + z_2^{n-1}$ lies in Δ_n so that $n + \zeta \neq 0$.

(f) For z_1, z_2 in Δ,

$$\begin{aligned} k(z_1) = k(z_2) &\Longrightarrow z_1(1 - z_2)^2 = z_2(1 - z_1)^2 \\ &\Longrightarrow (z_1 - z_2)(1 - z_1 z_2) = 0 \\ &\Longrightarrow z_1 - z_2 = 0, \text{ since } 1 - z_1 z_2 \neq 0. \end{aligned}$$

(g) Note that $k(z) = (1/4)[((1 + z)/(1 - z))^2 - 1]$.

(j) For $z = re^{i\theta}$, we note that

$$\frac{\text{Re}\,(z^2)}{|z|^2} = \cos 2\theta \quad \text{and} \quad \frac{\text{Im}\,(z^2)}{|z|^2} = \sin 2\theta$$

and allow $r \to 0$ along $\theta = 0, \pi/2, \pi/4$.

(m) Clearly, $f(z)$ is continuous on the cut plane $\mathbb{C}\backslash\{x+iy : x \le 0,\ y = 0\}$. Because $f(-2) = 0 = \lim_{z\to -2}(2+z)$, one may use the $\epsilon - \delta$ notation to see that $z = -2$ is the only removable discontinuity of $f(z)$.

(o) Recall the binomial theorem:

$$(z+h)^n - z^n = \binom{n}{1} z^{n-1}h + \binom{n}{2} z^{n-2}h^2 + \cdots + \binom{n}{n-1} zh^{n-1} + h^n$$

so that $F(z,h) = \binom{n}{2} z^{n-2}h + \cdots + \binom{n}{n-1} zh^{n-2} + h^{n-1}$. The conclusion follows if we use the triangle inequality.

(p) Use the fact that $|f_n(z)-0| \le 1/(n|z|-1) < \epsilon$ whenever $n > N(\epsilon, z) = (1+1/\epsilon)/|z|$.

(t) Fixing $z \ne 0$, one could apply the root/ratio test with $a_n(z) = 3^{-n}z^n$ to obtain that the series converges absolutely for $|z| < 3$.

(v) Let $z = re^{i\theta}$ and $w = u + iv$. Then we find that $u = a\cos\theta$ and $v = b\sin\theta$, where $a = 2^{-1}(r + \alpha^2 r^{-1})$ and $b = 2^{-1}(r - \alpha^2 r^{-1})$. From this we obtain

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

which is an ellipse.

2.66. **Note.** For instance to check the non-uniform continuity of $f(z) = 1/z$ in $U = \{z : \operatorname{Re} z > 0\}$, it is enough to consider

$$\zeta_n = \frac{1+i}{n} \quad \text{and} \quad \eta_n = \frac{1+i}{n+1}.$$

Note that

$$\zeta_n - \eta_n = \frac{1+i}{n(n+1)} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{and} \quad |f(\zeta_n) - f(\eta_n)| = \frac{1}{\sqrt{2}}.$$

2.67. Set $z = x + iy$ and $w = u + iv$. Then

$$z + w = x + u + i(y+v), \quad zw = xu - yv + i(xv + yu).$$

Similarly, if

$$A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

then matrix addition and multiplication show that

$$A+B = \begin{pmatrix} x+u & y+v \\ -(y+v) & x+u \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} xu - yv & yu + xv \\ -(yu+xv) & xu - yv \end{pmatrix}.$$

Thus, if we let $f(z) = A$ and $f(w) = B$, then

$$f(z+w) = A + B, \quad f(zw) = AB, \quad f(\alpha z) = \alpha A$$

showing that there is a one-to-one correspondence between $\mathbb{C}$ and the set of matrices of the given form.

Chapter 3, Exercises 3.7:

3.116:

(g) Define $f : \mathbb{C} \to \Delta$ by $f(z) = z/(1 + |z|)$. Then f is bijective and bicontinuous, and the inverse mapping is given by $f^{-1} : \Delta \to \mathbb{C}$, where $f^{-1}(w) = w/(1 - |w|)$.

(h) Let $f(z) = e^{2\pi iz} - 1$. Then f is entire, $f(0) = 0 = f(1)$, yet there exist no values of z (real or complex) such that

$$f(1) - f(0) = f'(z)(1 - 0), \quad \text{i.e.} \quad 0 = 2\pi i e^{2\pi iz}.$$

(n) As $\lim_{r\to 0+} f(re^{i\pi/4}) = \lim_{r\to 0+} e^{1/r^4} = \infty \neq f(0) = 0$, f is not continuous at the origin.

(o) The implication '$\Longrightarrow$' is trivial. For the converse part, let $f(z) = u + iv$. Then

$$zf(z) = (xu - yv) + i(xv + yv) =: U + iV$$

and u, v, U, V are all harmonic in D. It is now a simple exercise to see that $\nabla^2 U = 0$ and $\nabla^2 V = 0$ imply $u_x = v_y$ and $u_y = -v_x$, respectively. By Theorem 3.26, f is analytic in D.

(p) Consider $f_1(z) = z$ and $f_2(z) = 1$ in $\mathbb{C}$.

(r) If $f(z) = u(x) + iv(y)$ is entire, then $\nabla^2 u = u''(x) = 0$ and $\nabla^2 v = v''(y) = 0$ on $\mathbb{R}$. Thus, $u(x) = ax + b$ and $v(y) = cy + d$, where a, b, c, d are real. Using the C-R equations, we get $a = c$ so that

$$f(z) = (ax + b) + i(cy + d) = az + b + id.$$

What will be the form of f if $f(x + iy) = u(y) + iv(x)$ is entire?

(s) Define $F = f - g = i(v - V)$. Then, $F \in \mathcal{H}(\Omega)$ and the C-R equations give $(v - V)_x = 0 = (v - V)_y$. Thus, $v - V$ (being a real-valued function) is constant on every vertical line segment as well as on every horizontal line segment in Ω. Since every pair of points in the domain Ω can be connected by vertical and horizontal line segments, $v - V$ is necessarily constant in Ω.

(t) From the previous exercise ($f(z) = u + iv$ and $\overline{f(z)} = u - iv$) we get that $v - (-v) = 2v$ is constant.

(u) If $u = y^2 - 2x$ then $u_{xx} + u_{yy} = 2 \neq 0$.

(v) If $v = x^3 - y^3$, then $v_{xx} + v_{yy} = 6(x - y) = 0$ for $z \in \Omega = \{\alpha(1 + i) : \alpha \in \mathbb{R}\}$. Note that Ω is not in an open set in $\mathbb{C}$ and therefore, v cannot be harmonic in any open subset of $\mathbb{C}$.

(w) We have

$$\begin{cases} v_x = 2uu_x \\ v_y = 2uu_y \end{cases} \quad \text{and} \quad \begin{cases} v_{xx} = 2uu_{xx} + 2u_x^2 \\ v_{yy} = 2uu_{yy} + 2u_y^2. \end{cases}$$

Since u and v are harmonic in Ω, the addition of the last two equations show that $u_x^2 + u_y^2 = 0$, i.e. $u_x = 0 = u_y$. Similarly, $v_x = 0 = v_y$. Thus, f is constant.

(z) Consider $f_n(z) = z^n/n^2$.

3.117:

(a) Note that $(n!)^{1/n} = \exp[(1/n)\,\mathrm{Log}\, n!]$ and

$$(1/n)\,\mathrm{Log}\, n! = (1/n)[\ln 1 + \ln 2 + \cdots + \ln n] \to \infty \quad \text{as} \quad n \to \infty.$$

(d) Suppose $f(z) = \sum_{n\geq 0} a_n z^n$ converges for $|z| < R$. Then by the comparison test we see that $F(z) = \sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1}$ converges for $|z| < R$ and $F'(z) = f(z)$ for $|z| < R$.

(f) It suffices to note that $|(\mathrm{Re}\, a_n) z^n| \leq |a_n|\, |z|^n$.

(g) Let $S_n = \sum_{p=1}^{n} (z^k)^p$. Then for $z^k \neq 1$

$$S_n = z^k \frac{(1 - z^{kn})}{1 - z^k} \to \frac{z^k}{1 - z^k} \quad \text{as } n \to \infty.$$

(j) More generally, for $|a| = |c| > 0$ the series $\sum_{n=0}^{\infty} \left(\frac{az+b}{cz+d}\right)^n$ converges whenever $\mathrm{Re}\,(z(c\overline{d} - a\overline{b})) > (|b|^2 - |a|^2)/2$.

(k) For $|z| = 1$, the triangle inequality shows that $\frac{z^n}{n!} + \frac{n^4}{z^n} \to \infty$ and therefore, the series diverges for $|z| = 1$. Note that $\sum_{n=0}^{\infty} \frac{n^4}{z^n}$ converges for $|z| > 1$ and diverges for $|z| \leq 1$.

(l) Let $f_n(z) = nz^n/(1 - z^n)$. Then, for $|z| \leq r$ with $r \in (0, 1)$

$$|f_n(z)| \leq \frac{n|z|^n}{1 - |z|^n} \leq \frac{nr^n}{1 - r^n}.$$

As $\lim_{n\to\infty} r^n = 0$, it follows that $1 - r^n \geq 1/2$ for large n so that $|f_n(z)| \leq 2nr^n$. Thus, $\sum_{n=1}^{\infty} |f_n(z)| \leq 2\sum_{n=1}^{\infty} nr^n$ showing that the series $\sum_{n\geq 1} f_n(z)$ converges absolutely and uniformly for $|z| \leq r$. A similar conclusion holds for the second series. Now,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} &= \sum_{n=1}^{\infty} nz^n \sum_{m=0}^{\infty} (z^n)^m \\ &= \sum_{n=1}^{\infty} n \sum_{m=0}^{\infty} (z^n)^{m+1} \end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n(z^n)^k$$
$$= \sum_{k=1}^{n}\sum_{n=1}^{\infty} n(z^k)^n$$

and

$$\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2} = \sum_{n=1}^{\infty} z^n \left(\sum_{m=0}^{\infty}(m+1)(z^n)^m\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k(z^n)^k.$$

(p) As e^y is strictly increasing in $\mathbb{R}$, we have $e^y > e^0 = 1 > e^{-y}$ for $y > 0$. Also note that the above inequalities yield

$$\lim_{|y|\to\infty} |\sin z| = \lim_{|y|\to\infty} |\cos z| = \infty.$$

(r) Consider $\cos z = c$. Then, by the definition of $\cos z$, we see that $e^{iz} = c + w$, where $w^2 = c^2 - 1$. The conclusion for $\cos z$ now follows from the fact that the exponential function takes every value except zero. Apply the same principle for $\sin z$, $\cosh z$ and $\sinh z$.

(s) If $x > 0$ and $\theta = \cos^{-1}(x)$, then $\theta \in (0, \pi/2)$ so that

$$\cos\theta = x = \sin(\pi/2 - \theta)$$

and, since θ is acute, we have $\pi/2 - \theta = \sin^{-1}(x)$.

(u) Let $z \in \partial\Delta(r;r)\setminus\{0\}$. Then, we have $z = r(1+e^{i\theta})$ with $-\pi < \theta < \pi$. Now

$$\frac{1}{z} = \frac{1}{r(1+e^{i\theta})} = \frac{e^{-i\theta/2}}{2r\cos(\theta/2)} = \frac{1 - i\tan(\theta/2)}{2r}$$

so that $|f(z)| = e^{1/2r}$.

(y) Suppose that such a function f exists. Then $f(0) = 0$. This implies that $z = f^2(z)$ so that $f^2(z)$ has a zero of order n, $n \geq 2$, at 0. This is a contradiction, since z has only a simple zero at 0. In other words, it is not possible to define $\operatorname{Arg} z$ in such a way that $f(z) = z^{1/2}$ is analytic in a neighborhood of 0.

3.120. By the C-R equations, we have $1 = u_x v_y - u_y v_x = |f'(z)|^2$ so that $f'(z) = e^{i\theta}$ for some constant a with $|a| = 1$.

3.121. We have $f'(z) = u_x + iv_x$ and $f'(z) = v_y - iu_y$ in $\mathbb{C}$. By hypothesis, we see that

$$2f'(z) = (u_x + v_y) + i(v_x - u_y) = 0 + i(v_x - u_y), \quad \text{i.e. } \operatorname{Re} f'(z) = 0$$

and therefore, $|e^{f'(z)}| = 1$ in $\mathbb{C}$ showing that $e^{f'(z)}$ is a constant α with $|\alpha| = 1$, i.e. $e^{f'(z)} = e^{i\theta}$ for some real θ. which gives $f'(z) = it$ for some real t.

3.123. The given condition on f for $z_1 = z_2 = 0$ implies that $f(0) = 0$. Then, as $f(z+h) - f(z) = (f(z) + f(h)) - f(z) = f(h)$, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

and so, $f(z) = f(0) + f'(0)z = f'(0)z$.

3.124. $f(z) = e^{iaz^2}$

3.125. If $f = u + iv$ and $\nabla^2|f| = 0$, then we find that $u_x + v_x = u_y + v_y = 0$ so that $f'(z) = 0$ in Ω.

3.126. For $u(x,y) = \ln\sqrt{x^2+y^2}$, $z = x + iy \neq 0$, we have $u_{xx} + u_{yy} = 0$, so u is an harmonic in $\mathbb{C}\backslash\{0\}$. For a function v to become a harmonic conjugate in $\mathbb{C}\backslash\{0\}$ we must have $f = u + iv$ analytic in $\mathbb{C}\backslash\{0\}$. In particular, u and v must satisfy the C-R equations:

$$u_x = v_y = \frac{x}{x^2+y^2}, \quad -u_y = v_x = -\frac{y}{x^2+y^2}.$$

Therefore, we have the total derivative

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \frac{-ydx + xdy}{x^2+y^2}.$$

Using the change of variable $x = r\cos\theta$, $y = r\sin\theta$ (where $\theta = \text{Arg}\,(x+iy)$), the above becomes $dv = d\theta$ and so $v = \text{Arg}\,(x+iy)+$ some constant. But, as we know, v is not continuous on the set $D_\pi = \{(x,y) : y = 0, x \leq 0\}$. However, v is harmonic conjugate to u in $\mathbb{C}\backslash D_\pi$ and so $f = u+iv$ is analytic in $\mathbb{C}\backslash D_\pi$. Hence in the cut plane (with the negative real axis removed) the solution is $f(z) = \text{Log}\,z$ so that $f'(z) = 1/z$ for $z \in \mathbb{C}\backslash D_\pi$. Again note that there is no analytic function f for all $z \in \mathbb{C}\backslash\{0\}$ such that $f'(z) = 1/z$ (see also Example 4.89).

3.128. From the geometric series, we note that the region of convergence for the first series is given by $|z/(1+z)| < 1$, i.e. $\text{Re}\,z > -1/2$, whereas the region of uniform convergence is obtained from $|z/(1+z)| \leq r$ for $0 < r < 1$, which is given by

$$\left| z - \frac{r^2}{1-r^2} \right| < \frac{r}{1-r^2}.$$

For the second series, we may use Theorem 1.48.

3.129. Let $f(z) = \sum_{n\geq 1} a_n z^n$ and compute the coefficients using the given relation.

3.130. Indeed, if $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(|z| < \delta)$, then we see that

$$g'(2z) + g(z) = 0 \implies \sum_{k=0}^{\infty}(k+1)a_{k+1}2^k z^k + \sum_{k=0}^{\infty} a_k z^k = 0$$

$$\Longrightarrow \quad a_{k+1} = -\frac{a_k}{(k+1)2^k} \quad \text{for } k = 0, 1, 2, \ldots$$

$$\Longrightarrow \quad a_{k+1} = \frac{(-1)^{k+1} a_0}{(k+1)! 2^{k(k+1)/2}}, \quad k = 0, 1, 2, \ldots .$$

Note that if $a_0 = 0$, then $f(z) \equiv 0$. If $a_0 \neq 0$, then by the ratio test

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{(k+1)2^k} \to 0 \quad \text{as} \quad k \to \infty$$

showing that the radius of convergence of the series $\sum_{k=0}^{\infty} a_k z^k$ is ∞, and so g is entire.

3.136. If the principal values of z^{a_1} and z^{a_2} are used, then, for a_1, a_2 complex constants and $z \in D_\pi = \mathbb{C} \backslash \{z : z = x, x \leq 0\}$, we have

$$z^{a_1} z^{a_2} = \exp\{a_1 \operatorname{Log} z\} \cdot \exp\{a_2 \operatorname{Log} z\} = \exp\{(a_1 + a_2) \operatorname{Log} z\} = z^{a_1 + a_2}.$$

If the principal values of z_1^a and z_2^a are used, then for a, a complex constant, and $z_1, z_2 \in \mathbb{C} \backslash \{0\}$,

$$\begin{aligned} z_1^a z_2^a &= \exp\{a \operatorname{Log} z_1\} \cdot \exp\{a \operatorname{Log} z_2\} \\ &= \exp\{a(\operatorname{Log} z_1 + \operatorname{Log} z_2)\} \\ &= \exp\{a[\operatorname{Log}(z_1 z_2) + 2k_1(z_1, z_2)\pi i]\} \\ &= (z_1 z_2)^a \exp\{2ak_1(z_1, z_2)\pi i\}, \end{aligned}$$

where $k_1(z_1, z_2)$ has one of the values of $\{-1, 0, 1\}$ (see Exercise 1.54(a)).

3.137. Write

$$\begin{aligned} F_k(1) &= e^{(1/4)[\operatorname{Log}(1+i) + 2k\pi i]} \\ &= |1+i|^{1/4} e^{(1/4)i \operatorname{Arg}(1+i)} e^{k\pi i/2} \\ &= 2^{1/8} e^{i\pi/16} e^{k\pi i/2}. \end{aligned}$$

Thus, the desired branch corresponds to the case $k = 2$. This means that $F(z) = -e^{(1/4) \operatorname{Log}(z+i)}$.

3.140. Consider $f(z) = \exp((1/3) \operatorname{Log} z)$.

Chapter 4, Exercises 4.13:

4.144:

(d) For an arbitrary fixed $\zeta \in D$, let γ be defined by

$$\gamma(t) = (1-t) \cdot 0 + t \cdot \zeta, \quad t \in [0, 1],$$

and $f(\zeta) = e^{\zeta}$. Then,

$$\int_{\gamma} f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 e^{t\zeta}\zeta\,dt = e^{\zeta} - 1$$

and for $\zeta \in D$,

$$M = \max_{t\in[0,1]} |f(\gamma(t))| = \max_{t\in[0,1]} |e^{t\zeta}| = \max_{t\in[0,1]} e^{t\mathrm{Re}\,\zeta} < 1.$$

So we have,

$$|e^{\zeta} - 1| = \left|\int_{\gamma} f(z)\,dz\right| \le ML(\gamma) < 1.\,|\zeta|.$$

Also, we note that $|e^z - 1| \le |z|$ for all $z \in \overline{D} = \{w : \mathrm{Re}\,w \le 0\}$.

(e) This is a slightly general version of the previous statement. Clearly (see Theorem 4.16),

$$|e^a - e^b| = \left|\int_b^a e^z\,dz\right| \le \max_{x\le 0} e^x\,|b-a| = |b-a|.$$

(f) On the circle $|z| = r$, we have $z = re^{i\theta}$ ($\theta \in [0, 2\pi]$) and $dz = izd\theta$. So, $\mathrm{Re}\,z = r\cos\theta$, $|dz| = r\,d\theta$ and

$$|z-r|^2 = |z|^2 + r^2 - 2r\mathrm{Re}\,z = 2r^2(1-\cos\theta) = 4r^2\sin^2(\theta/2).$$

Thus, $|z-r| = 2r\sin(\theta/2)$ for $\theta \in [0, 2\pi]$ and hence,

$$I = \int_0^{2\pi} 2r\sin(\theta/2)r\,d\theta = 2r^2\left(-\frac{\cos(\theta/2)}{1/2}\right)\Bigg|_0^{2\pi} = 8r^2.$$

(g) Let $z = e^{i\theta}$. Then, $dz = iz\,d\theta$ so that

$$\begin{aligned} I &= i\int_0^{2\pi} [f(e^{i\theta}) - f(e^{-i\theta})]\,d\theta \\ &= i\left[\int_0^{2\pi} f(e^{i\theta})\,d\theta - \int_0^{2\pi} f(e^{i\phi})\,d\phi\right] \quad (\theta = 2\pi - \phi), \end{aligned}$$

which is zero.

(h) Let $p(z) = \sum_{k=0}^n a_k z^k$ and $z = e^{it}$. Then,

$$I = \sum_{k=0}^n a_k \int_0^{2\pi} e^{-ikt} ie^{it}\,dt = i\sum_{k=0}^n a_k \int_0^{2\pi} e^{-i(k-1)t}\,dt = 2\pi i a_1.$$

Also, we observe that $\int_{|z|=1} p(z)\,dz = i\sum_{k=0}^n a_k \int_0^{2\pi} e^{i(k+1)t}\,dt = 0$.

(i) Note that

$$\frac{e^{2z}+1}{\cos(iz)} = \frac{e^z(e^z+e^{-z})}{\cos(iz)} = 2e^z \neq 0 \text{ for } z \in \mathbb{C}.$$

The existence of an entire function f follows from Theorem 4.38 (see also Theorem 4.40). Clearly, entire functions satisfying the given equation are given by $f(z) = \ln 50 + z + 2k\pi i$, $k \in \mathbb{Z}$.

(l) Let $f(z) = g(z) - z$. Apply the Cauchy integral formula for the first derivative to f and obtain

$$g'(a) - 1 = \frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)-z}{(z-a)^2}\,dz$$

so that (as $|z| = 1$ implies that $|z-a| \geq |z| - |a| = 1 - |a|$)

$$|g'(a) - 1| \leq \frac{1}{2\pi}\frac{2\pi}{(1-|a|)^2} = \frac{1}{(1-|a|)^2}.$$

Thus, we have $|g'(a)| \leq 1 + (1-|a|)^{-2}$. For example, if $a = 1/2$ then we have $|g'(1/2) - 1| \leq 4$. In particular, $|g'(1/2)| \leq 5$.

Note: If we use the Schwarz-Pick lemma for $f(z)$, one can obtain an improved estimate.

(m) Apply the Cauchy integral formula for the second derivative at $a = 1$.

(n) Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $z \in \mathbb{C}$. Then, for any $r > 0$,

$$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^{n+1}} r\,d\theta \leq \frac{1}{2\pi r^{n-\alpha}}.$$

For $n > \alpha$, allow $r \to \infty$ to obtain $a_n = 0$ for $n > \alpha$. For $n < \alpha$, allow $r \to 0$ to obtain $a_n = 0$ for $n < \alpha$. Thus, $f(z) = 0$ in $\mathbb{C}$. What happens when α is a positive integer?

(o) Consider

$$f(z) = \begin{cases} 1 & \text{if } z \in \mathbb{C}\backslash\{1\} \\ 0 & \text{if } z = 1 \end{cases} \quad \text{and} \quad f(z) = i\left(\frac{1+z}{1-z}\right).$$

(p) Clearly, the radius of convergence is 1. We may set the sum of the series as $f(z)$. It follows that

$$f'(z) = \sum_{n=1}^{\infty} (z-3)^n = \frac{1}{1-(z-3)} = \frac{1}{4-z}, \quad |z-3| < 1.$$

Integration gives $f(z) = -\operatorname{Log}(4-z) + c$, where the principal branch cut is chosen. The branch cut is $\mathbb{C}\backslash\{x + i0 : x \geq 4\}$. As $0 = f(3)$, we have $c = 0$ and therefore, the sum is $-\operatorname{Log}(4-z)$.

(q) The convergence of $\sum_{n=0}^{\infty} f^{(n)}(a)$ implies that the n-th term of this series, namely, $f^{(n)}(a) = n!a_n$, approaches zero as $n \to \infty$ showing that the radius of convergence of the power series about a is ∞.

(r) As $|b_n|^{1/n} < n^{2/n}|a_n|^{1/n}$, the Root test implies that the radius of convergence of the series is at least 1.

(s) Note that the disk of convergence of $\sum_{n=0}^{\infty} a_n(z+2)^n$ is $|z+2| < 3$ and therefore, the transformed series $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 3$. What is its sum?

(v) As $f(z) = f(-z)$, we have $f^{(n)}(z) = (-1)^n f^{(n)}(-z)$ which (at $z = 0$) gives $(1-(-1)^n)f^{(n)}(0) = 0$. Thus, $a_n = f^{(n)}(0)/n! = 0$ whenever $n > 0$ is an odd integer. The Taylor series of f about 0 will be of the form $f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}$.

(y) By the Cauchy inequality, we have

$$\frac{\left|f^{(n)}(0)\right|}{n!} \leq \frac{M}{R^n}, \text{ where } M = \max_{|z|=R} |f(z)|.$$

If the inequality $\left|f^{(n)}(0)\right| \geq n!n^n$ were true, then we would have $n^n < MR^{-n}$ for all $n \in \mathbb{N}$. However, this inequality cannot hold for all n since n^n grows faster than R^{-n} for any fixed $R \in (0,1)$.

Alternately, we observe that the given condition implies that $|a_n|^{1/n} \geq n$. As $n \to \infty$, we see that the radius of convergence of $f(z) = \sum a_n z^n$ is zero and hence, $f(z)$ cannot be analytic at $z = 0$ which is a contradiction.

4.145:

(a) By definition

$$\sin z = 0 \iff (e^{iz} - e^{-iz})/2 = 0 \iff e^{i2z} = 1 \iff z = k\pi, \quad k \in \mathbb{Z}.$$

(b) By the definition of the cosine hyperbolic function, $\cosh x \neq 0$. Therefore,

$$\cosh z = 0 \iff \cosh x \cos y = 0 \quad \text{and} \quad \sinh x \sin y = 0.$$

Since $\cosh x \neq 0$, the first equation implies that $\cos y = 0$. Hence $y = (k+1/2)\pi$, $k \in \mathbb{Z}$. However, $\sin(k+1/2)\pi \neq 0$ for each $k \in \mathbb{Z}$. The second then becomes $\sinh x = 0$ and this has only one root $x = 0$. Therefore, $\cosh z = 0 \iff z = i(k+1/2)\pi$, $(k \in \mathbb{Z})$. Note that this can be quickly obtained from the zeros of $\cos z$, since $\cos(iz) = \cosh z$.

(d) Find two distinct harmonic functions in a domain D which coincides in a set $S \subseteq D$ which has a limit point in D. On the other hand, one can prove that if u and v are harmonic functions in a domain D and $u = v$ on in open set S in D, then $u = v$ in D.

(h) Apply the Uniqueness theorem by equating the coefficients of f^2 and f.

(j) Suppose that such an f were to exist in Δ satisfying the desired property. Then the function g defined by

$$g(z) = f(z) - z^{2k-1}$$

belongs to $\mathcal{H}(\Delta)$ and $g(1/n) = 0 = g(-1/n)$ for $n = 2, 3, \ldots$. It follows from the uniqueness theorem that $g(z) \equiv 0$ in Δ; i.e. $f(z) \equiv z^{2k-1}$ in Δ. But then $f(-1/n) = -1/n^{2k-1}$ for $n \geq 2$, which is not possible. Thus there exists no analytic function with the desired property.

(q) By the Uniqueness theorem, we have

$$(f'/f)(z) = (g'/g)(z), \quad \text{i.e.} \quad g(z)f'(z) - g'(z)f(z) = 0 \text{ in } \Delta$$

which gives $(f/g)'(z) = 0$ in Δ. Thus, f/g is constant in Δ.

4.149. See Example 4.68.

4.150. Use the method of proof of Example 4.68 by choosing a suitable auxiliary function $\phi(z)$.

4.151. On the unit circle $|z| = 1$, we have $z = e^{i\theta}$ ($\theta \in [0, 2\pi]$) and $dz = iz d\theta$. So, $\cos\theta = (z^2 + 1)/(2z)$,

$$\cos^2(\theta/2) = \frac{1 + \cos\theta}{2} = \frac{1}{2} + \left(\frac{z^2+1}{4z}\right) \quad \text{and} \quad \sin^2(\theta/2) = \frac{1}{2} - \left(\frac{z^2+1}{4z}\right).$$

Thus, I_c can be rewritten as

$$\begin{aligned} I_c &= \int_{|z|=1} f(z)\left(\frac{1}{2} + \frac{z^2+1}{4z}\right)\frac{dz}{iz} \\ &= \frac{1}{2i}\int_{|z|=1} \frac{f(z)}{z}\,dz + \frac{1}{4i}\int_{|z|=1} f(z)\,dz + \frac{1}{4i}\int_{|z|=1} \frac{f(z)}{z^2}\,dz \\ &= \frac{2\pi i f(0)}{2i} + \frac{0}{4i} + \frac{2\pi i f'(0)}{4i} \\ &= \pi(f(0) + f'(0)/2). \end{aligned}$$

Similarly, one can obtain that $I_s = \pi(f(0) - f'(0)/2)$.

4.152. If $|a| > 4$, then Cauchy's theorem shows that the value of the integral is zero. So, $f(a) = 0$ for each a with $|a| > 4$. On the other hand, if $|a| < 4$ then the Cauchy integral formula for derivatives implies that the value of the integral is $2\pi i F'(a) = f(a)$, where $F(z) = z^2 + 3z - 7$. Thus, for each a with $|a| < 4$, one has $f(a) = 2\pi i(2a + 3)$.

4.154. Fix $a \in \Delta$. Then, by the Cauchy integral formula,

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^2}\, dz$$

where $\gamma : |z-a| = r$, a circle which is completely inside the unit disk Δ. For example, $r = (1-|a|)/2$ will do. By choosing r in this way, we see that (as $z = a + re^{i\theta}$ implies that $|z| < |a| + r$)

$$1 - |z| \geq 1 - (|a| + r) = (1-|a|) - r = \frac{1-|a|}{2}$$

and

$$|dz| = |ire^{i\theta}\, d\theta| = r d\theta, \quad \frac{|dz|}{|z-a|^2} = \frac{d\theta}{r} = \frac{2}{1-|a|}\, d\theta$$

so that

$$\begin{aligned} |f'(a)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1-|z|)^\alpha} \frac{1}{|z-a|^2} |dz| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{2}{1-|a|}\right)^\alpha \frac{2}{1-|a|}\, d\theta \\ &= \frac{2^{\alpha+1}}{(1-|a|)^{\alpha+1}}. \end{aligned}$$

4.155. Follows from $f(z) - f(0) = \int_0^z f'(z)\, dz$.

4.157. (a) If there were a function f satisfying the given condition, then, by the Cauchy estimate ($a = -1$, $R = M = 5$), we would have

$$1 = |f''(-1)| \leq \frac{5(2!)}{25} = \frac{2}{5}$$

which is clearly not true.

4.158. We write $f(z)$ as

$$\frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{6+(z-5)} - \frac{1}{4+(z-5)}\right)$$

and therefore, for $|z-5| < 4$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-5)^n, \quad a_n = \frac{1}{2}\left(\frac{(-1)^n}{6^{n+1}} - \frac{(-1)^n}{4^{n+1}}\right).$$

Using this and the root/ratio test, one can see that the radius of convergence of the desired series is $R = 4$.

4.159. Rewrite $f(z)$ as

$$f(z) = \frac{1-z^3}{1-(z^3)^5}, \text{ with } z^3 \neq 1 \text{ (i.e. } \; z \neq 1, e^{2\pi i/3}, e^{4\pi i/3}).$$

Then f has simple poles at $z_k = e^{2k\pi i/15}$, $k \in \{1, 2, \dots, 14\} \setminus \{0, 5, 10\}$, so that the distance from 1 to the nearest singularity is $|1 - e^{2\pi i/15}| = R$.

4.160. By the partial fraction decomposition, we easily find that

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{5}}\left[\frac{\alpha}{z-\beta} - \frac{\beta}{z-\alpha}\right] \qquad \left(\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}\right). \\ &= \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left(\frac{\beta}{\alpha^{n+1}} - \frac{\alpha}{\beta^{n+1}}\right) z^n, \quad |z| < |\beta|. \end{aligned}$$

Note that $|\beta| < \alpha$ and $\alpha\beta = -1$.

4.161. By hypothesis, we may write

$$f(z) = (z-a)^m \phi(z) \quad \text{and} \quad g(z) = (z-a)^n \psi(z)$$

where $m \geq 0$, ϕ and ψ are analytic at a with $\phi(a) = f^{(m)}(a)/m!$ and $\psi(a) = g^{(n)}(a)/n! \neq 0$. It follows that

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \lim_{z\to a} (z-a)^{m-n} \frac{\phi(z)}{\psi(z)} = \begin{cases} \dfrac{\phi(a)}{\psi(a)} & \text{if } \; m = n \\ 0 & \text{if } \; m > n \\ \infty & \text{if } \; m < n. \end{cases}$$

Note that the limit exists whenever $m \geq n$.

4.162. Note that $\sqrt{z} = 0$ implies that $z = 0$ which is not to be considered as a zero because $\sqrt{z}$ is not analytic at $z = 0$. The point $z = 0$ is a singular point (but it cannot be an isolated singularity as we shall see in Chapter 7).

4.163. By the Uniqueness theorem, $f''(z) + e^z = 0$ for all $z \in \Delta$. From this one can get an explicit form of $f(z)$. How about if $e^{1/n}$ is replaced by $f'(1/n)$ or $f(1/n)$ or $\sin(1/n)$ or $\cos(1/n)$?

4.164. Define $g(z) = f''(z) - 10 + 3z$. Then, $g \in \mathcal{H}(\mathbb{C})$ and $g(1 + 1/n) = 0$. The Uniqueness theorem implies that $f''(z) = 10 - 3z$ and therefore, $f'(z) = 10z - 3z^2/2 + a$. As $f'(0) = 0$, $a = 0$. So, $f(z) = 5z^2 - z^3/2 + b$ (b-constant).

Chapter 5, Exercises 5.8:

5.82:

(f) Choose for example, $T(z) = z/(z+1)$ and $S(z) = 1+z$. Then

$$T(S(z)) = \frac{z+1}{z+2} \text{ and } S(T(z)) = 1 + \frac{z}{z+1} = \frac{2z+1}{z+2}.$$

(h) Note that T is one-to-one (if it is not a constant).

(k) If $(z, z_1, z_2, z_3) = \alpha$, then from the definition of the cross-ratio, z_3 cannot be ∞ and that this equation reduces to

$$\left(\frac{z_2 - z_3}{z - z_3}\right)\beta = \alpha \quad \left(\beta = \frac{z - z_2}{z_2 - z_1}\right).$$

Solving this for z_3 yields $z_3 = (\beta z_2 - \alpha z)/(\beta - \alpha)$.

(m) $\Longleftarrow$: This is Theorem 5.38.

$\Longrightarrow$: Let $(z_4, z_1, z_2, z_3) = (w_4, w_1, w_2, w_3)$, and $g(w) = (w, w_1, w_2, w_3)$. Then g is a unique Möbius transformation with $g(w_1) = 0$, $g(w_2) = 1$, and $g(w_3) = \infty$. By Theorem 5.38, there exists a unique Möbius transformation f such that

$$f(z) = (z, z_1, z_2, z_3), \quad f(z_j) = w_j \text{ for } j = 1, 2, 3.$$

It remains to show that $f(z_4) = w_4$. In view of the invariance property of the cross-ratio for $f(z)$, the definition of g gives,

$$\begin{aligned}(z_4, z_1, z_2, z_3) &= (f(z_4), f(z_1), f(z_2), f(z_3)) \\ &= (f(z_4), w_1, w_2, w_3) \\ &= g(f(z_4)).\end{aligned}$$

By assumption, the L.H.S equals $(w_4, w_1, w_2, w_3) = g(w_4)$ and so, $g(w_4) = g(f(z_4))$ which, because g is univalent, implies that $w_4 = f(z_4)$.

(p) If

$$T(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

is a Möbius transformation which fixes 0 and ∞, then we must have $b = 0$ and $c = 0$ so that $T(z) = (a/d)z =: \alpha z$. For T to have no other fixed points, we must have $\alpha \neq 1$. Again, we want the map T to take the upper half-plane to itself. Hence, α must be a positive real number such that $\alpha \neq 1$.

(q) If T is a Möbius transformation and $\infty \in \operatorname{Fix}(T)$, then T has the form $T(z) = \alpha z + \beta$ ($\alpha \in \mathbb{C}\setminus\{0\}$, $\beta \in \mathbb{C}$). If T fixes ∞ and no other points, then the fixed point equation $T(z) = z$ gives that $z = \beta/(1-\alpha)$, which should be possible only if $z = \infty$ showing that $\alpha = 1$ (otherwise $T(z) = \alpha z + \beta$ will have a fixed point in $\mathbb{C}$). Therefore, $T(z) = z + \beta$. Again, 0 is not a fixed point, so $\beta \neq 0$. We require T to be a self mapping of the upper half-plane, which gives the condition $\operatorname{Im}\beta > 0$.

(r) If T is a Möbius transformation and $\infty \in \text{Fix}\,(T)$, then T has the form $T(z) = \alpha z + \beta$ $(\alpha \in \mathbb{C}\backslash\{0\},\ \beta \in \mathbb{C})$. Since the required Möbius map carries $\mathbb{R}$ onto $\mathbb{R}$, we must have

$$T(0) = \beta \in \mathbb{R} \text{ and } T(1) = \alpha + \beta \in \mathbb{R}$$

so that both $\alpha \neq 0$ and β are real.

(s) For example, $f(z) = e^z + z$ and $g(z) = e^{z^2} + z$. What is the set of fixed points of the composite function $f \circ f$?

(t) Choose $\beta = 2i$ and $\alpha = \pi$ (so that $e^{i\alpha} = -1$) in Theorem 5.47.

(u) Follows from the proof of Theorem 5.69. In fact, if $T(z)$ is the Möbius transformation whose coefficients a, b, c, d are non-zero and $ad - bc = 1$, then we note that

$$T(0) = \frac{b}{d},\ T(\infty) = \frac{a}{c}.$$

Suppose that T takes $\mathbb{R}$ into $\mathbb{R}$. Then b/d and a/c are real. So we conclude that a, b, c, d are real or purely imaginary.

(v) Note that $g(z) = z^n$ carries the infinite sector of angle π/n onto the upper half-plane $\mathbb{H}^+$. Now compose with a Möbius map which carries $\mathbb{H}^+$ onto Δ.

(x) First map the $\Delta(a; r)$ onto $\Delta(0; r)$ by $w(z) = z - a$ and then map the disk $\Delta(0; r)$ onto $\mathbb{C}_\infty \backslash \Delta(0; R)$ by $\zeta(z) = Rr/w$. Finally, map $|w| > R$ onto $|w - b| > R$ by $\eta(z) = \zeta + b$. The desired Möbius mapping T is given by $T(z) = (\eta \circ \zeta \circ w)(z)$.

(z) Clearly, $f_1(z) = z^6$ maps Ω onto the upper half-plane $\mathbb{H}^+$. A map which carries $\mathbb{H}^+$ onto Δ is given by Theorem 5.47. The composition will give us the desired map.

5.89. Rewrite the given transformation in the form $z = i(w + 1)/(1 - w)$. Then, we find that

$$|z| = r \implies |1 + w|^2 = r^2|1 - w|^2 \implies \left| w + \frac{1 + r^2}{1 - r^2} \right| = \frac{2r}{|1 - r^2|}.$$

Chapter 6, Exercises 6.8:

6.75:

(a) Use Example 6.15.

(c) Clearly, $f \in \mathcal{H}(\overline{\Delta})$ and therefore, $|f(z)|$ attains its maximum value on $|z| = 1$. For $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$,

$$|f(z)|^2 \leq \frac{1}{4p^2 + 4p\cos n\theta + 1} \leq \frac{1}{4p^2 - 4p + 1} = \frac{1}{(2p - 1)^2}$$

and the maximum is attained when $\theta = \pm\pi/n$.

(d) Let $0 \leq x, y \leq 1$. Then, it suffices to observe that

$$\begin{aligned} |f(x+i0)| &= |x^2-2x| = 1-(x-1)^2 \leq 1 \\ |f(0+iy)| &= |-y^2-2iy| = \sqrt{y^4+4y^2} \leq \sqrt{5} \\ |f(1+iy)| &= |(1+iy)^2-2(1+iy)| = |-1-y^2| = 1+y^2 \leq 2 \\ |f(x+i)| &= |(x+i)^2-2(x+i)| = \sqrt{(x^2-2x-1)^2+(2x-2)^2} \\ &\qquad \leq \sqrt{1+2^2} = \sqrt{5}. \end{aligned}$$

(e) Note that $|f(0)| = 2$ and so, $|f(z)|$ attains its maximum at the origin which is an interior point.

(f) We can write $f(z) = (z^2+4)g(z)$, where g is also entire. On $|z| = 3$, $M \geq |f(z)| = |(z^2+4)g(z)| \geq (|z|^2-4)|g(z)| = 5|g(z)|$ so that $|g(z)| \leq M/5$. Therefore, on $|z| = 3$, one has

$$|f(z)| = |(z^2+4)|\,|g(z)| \leq (M/5)|z^2+4|$$

and the desired conclusion follows from the Maximum modulus principle.

(g) We write $f(z) = g(z)\prod_{k=0}^{n-1}(z^n - \omega_k)$, where g is also entire. Apply the Maximum modulus principle.

(k) Choose $a = 1/2$ and $b = 0$ in (6.35). Can one prove this without using the formula (6.35)? Note that among all functions which are analytic and bounded by 1 in the unit disk, $\max_{z\in\Delta}|f'(z)|$ is assumed when $f(1/2) = 0$.

(m) Use $a = 1/2$, $b = 0$ in (6.35). According to (6.35), with $f(a) = b$, it follows that

$$|f'(1/2)| = |f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2} = \frac{1-9/16}{1-1/4} = \frac{7}{12}.$$

But if $f'(1/2) = 3/4$, then we would have $3/4 \leq 7/12$ which is not possible.

(n) As $|f'(z)| \leq |z|$, we have $f'(0) = 0$ and $|f'(1)| \leq 1$. By Theorem 6.60, $f'(z)$ is a polynomial of degree at most one. So, $f'(z) = a_0 + a_1 z$. As $f'(0) = 0$, we have $f'(z) = a_1 z$ which gives that $f(z) = a + bz^2$ with $b = a_1/2$. As $|f'(1)| = |a_1| \leq 1$, we have $|b| \leq 1/2$.

(o) For $|z| \leq 1/2$, we note that

$$\begin{aligned} |p(z)| &\geq 1 - |p(z)-1| \\ &\geq 1 - \{|z|^n + |a_{n-1}|\,|z|^{n-1} + \cdots + |a_2|\,|z|^2 + |a_1|\,|z|\} \\ &\geq 1 - \left\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\right\}, \\ &= 1 - \frac{1}{2}\left\{\frac{1-(1/2)^n}{1-1/2}\right\} = \frac{1}{2^n} \end{aligned}$$

and so $p(z)$ cannot have zeros for $|z| \leq 1/2$. For $|z| \geq 2$, that is $|1/z| \leq 1/2$, we note that

$$\frac{p(z)}{z^n} = \left(\frac{1}{z}\right)^n + a_1 \left(\frac{1}{z}\right)^{n-1} + a_2 \left(\frac{1}{z}\right)^{n-2} + \cdots + a_{n-1} \left(\frac{1}{z}\right) + 1$$

and so proceeding as above, we obtain $|z^{-n}p(z)| > 0$ for $|z| \geq 2$. Thus, all the zeros of $p(z)$ must lie in the annulus D.

(q) As f is entire, $\phi(z) = \exp(f(z))$ is entire and $|\phi(z)| = e^{\mathrm{Re}\, f(z)}$. As $\mathrm{Re}\, f(z)$ is bounded as $|z| \to \infty$, ϕ is bounded. Thus, by Liouville's theorem, ϕ and hence, f is constant.
Note: The condition $|f(z)| \leq M$ on $\mathbb{C}$ implies that

$$-M \leq \mathrm{Re}\, f(z) \leq M; \quad -M \leq \mathrm{Im}\, f(z) \leq M.$$

Apply the hint with f, $-f$, if and $-if$ to show that any one of the four inequalities suffices to prove that f is constant, if f is entire.

(r) Let $f(z) = \sum_{n\geq 0} a_n z^n$. Since $f(\mathbb{R}) \subset \mathbb{R}$, each a_n is real. Since $f(i\mathbb{R}) \subset i\mathbb{R}$, $a_n = 0$ for n even.

(s) Suppose that f does not assume values in a disk, say $\Delta(c; M)$ for some complex number c and $M > 0$. Then, $|f(z) - c| > M$ for all $z \in \mathbb{C}$. Thus, f is constant by Liouville's theorem.

(u) For example if $k = 2$, then

$$f(2z) = f(z) = f(z/2) = f(z/2^2) = \cdots = f(z/2^n) = \cdots.$$

Define $g(z) = f(z) - f(z/2^n)$. Then, $g \in \mathcal{H}(\mathbb{C})$ and $g(z) = 0$ for $z \in \mathbb{C}$. Moreover,

$$0 = g(z) = \lim_{n\to\infty} (f(z) - f(z/2^n)) = f(z) - f(0), \quad \text{or} \quad f(z) = f(0).$$

(w) First we observe that every zero of $f(z)$ is also a zero of $f'(z)$. Thus, $f'(z)/f(z)$ is entire and bounded by 2 so that it is a constant, say a, with $|a| \leq 2$. Integration gives the desired form.

(x) Note that $\phi(z) = \exp(f^2(z))$ is entire and $|\phi(z)| = e^{u^2 - v^2}$ on $\mathbb{C}$. By Liouville's theorem, ϕ and hence f is a constant.

(y) Without loss of generality, we may assume that $0 < a < 1$. Since $u(z)$ is harmonic in $\mathbb{C}$, there exists a harmonic conjugate $v(z)$ in $\mathbb{C}$ such that $g(z) = u(z) + iv(z)$, where $g \in \mathcal{H}(\mathbb{C})$. Now, $f(z) = \exp(g(z))$ is an entire function, and for $|z| > 1$,

$$|f(z)| \leq |e^{u+iv}| = e^u \leq e^b e^{a \ln |z|} = \beta |z|^a$$

so that f is a polynomial of degree not greater than a. Thus, f is a constant and so, $u(z)$ is constant. Does the same conclusion hold even if a and b are fixed real numbers (see the previous Exercise)?

6.77. Suppose that $f \in \mathcal{H}(\Delta)$, $|f(z)| < 1$ and that there exist two distinct points a, b in Δ such that $f(a) = a$ and $f(b) = b$. Define

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z} \quad \text{and} \quad h = \phi_a \circ f \circ \phi_a.$$

Then, we easily have

- $h \in \mathcal{H}(\Delta)$, $|h(z)| < 1$ and $h(0) = 0$
- As ϕ_a is onto, there exists an $\alpha \in \Delta$ such that $\phi_a(\alpha) = b$.
- As $\phi_a^{-1} = \phi_a$, $\alpha = \phi_a^{-1}(b)$ and $h(\alpha) = \phi_a(f(b)) = \phi_\alpha(b) = \phi_\alpha^{-1}(b) = \alpha$
- Since a and b are distinct, $\alpha \neq 0$.

By Schwarz' lemma, $h(z) = e^{i\theta}z$ for some θ. But $h(\alpha) = \alpha$ shows that $\theta = 0$ and so,

$$h(z) = \phi_a(f(\phi_a(z))) = z, \quad \text{i.e.} \quad \frac{a - f(\phi_a(z))}{1-\overline{a}f(\phi_a(z))} = z$$

and this gives $f(\phi_a(z)) = \phi_a(z)$. Since $\phi_a(z)$ is an automorphism of the unit disk, it follows that $f(z) = z$ for $z \in \Delta$.

6.78. By Schwarz' lemma (see Example 6.41), $|f'(0)| \leq 3/4$. Thus, the function

$$f(z) = \frac{z - 1/2}{1 - z/2}$$

has the desired properties for the first part of our problem. If one insists that $f'(0) = 4/5$, then we must have $4/5 = |f'(0)| \leq 3/4$ which is not possible. This observation shows that there exists no analytic function $f : \Delta \to \overline{\Delta}$ with $f(0) = \pm 1/2$ and $f'(0) = 4/5$.

6.79. Let $w = f(z)$ and $f(0) = a$. Define $G = g \circ f$ by

$$G(z) = (g \circ f)(z) = \frac{a - f(z)}{\overline{a} + f(z)}, \quad \text{i.e.} \quad g(w) = \frac{a - w}{\overline{a} + w}.$$

Then g maps $\text{Re}\, w < 0$ onto Δ and so, $G(z)$ maps Δ onto Δ with $G(0) = 0$. Apply Schwarz' lemma to G to obtain the inequality $|G'(0)| \leq 1$ which gives $|f'(0)| \leq 2|\text{Re}\, f(0)|$.

6.80. Note that $f_1(z) = i\pi z/2$ maps $\Omega_1 = \{z : |\text{Re}\, z| < 1\}$ onto $\Omega_2 = \{w : |\text{Im}\, w| < \pi/2\}$, and $f_2(w) = e^w$ maps Ω_2 onto $\Omega_3 = \{\zeta : \text{Re}\, \zeta > 0\}$. Finally, $f_3(\zeta) = (\zeta - 1)/(\zeta + 1)$ maps Ω_3 onto Δ. Therefore,

$$g(z) = (f_1 \circ f_2 \circ f_3)(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1}$$

maps Ω_1 onto Δ with $g(0) = 0$. Thus, $F = g \circ f$ maps Δ onto itself with $F(0) = 0$. By Schwarz' lemma,

$$|F'(0)| = |g'(f(0))f'(0)| = |g'(0)f'(0)| \leq 1$$

and a computation gives, $|f'(0)| \leq 1/|g'(0)| = 4/\pi$.

6.81. Set $g(z) = (\alpha - f(z))/\alpha$. Since $g(0) = 1$ and $\operatorname{Re} g(z) > 0$, by Theorem 6.49, it follows that $|g'(0)| = |f'(0)/\alpha| \leq 2$ and

$$-1 + \left|\frac{f(z)}{\alpha}\right| \leq \left|1 - \frac{f(z)}{\alpha}\right| = |g(z)| \leq \frac{1+|z|}{1-|z|}, \quad \text{i.e. } |f(z)| \leq \frac{2\alpha|z|}{1-|z|}.$$

6.82. Define $g(z) = f(Rz)/M$ for $z \in \Delta$. Then, $g \in \mathcal{H}(\Delta)$ and $|g(z)| \leq 1$ for $z \in \Delta$. Let

$$\alpha = g(z_0) = \frac{f(Rz_0)}{M} = \frac{w_0}{M} \text{ with } |z_0| < 1 \text{ and } \phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

By hypothesis, the analytic function h defined by $h = \phi_\alpha \circ g$ is bounded by 1 and has a zero at $z_0 \in \Delta$. By the Schwarz-Pick lemma (see (6.36)), we have

$$\rho(h(z), 0) = \rho(h(z), h(z_0)) \leq \rho(z, z_0)$$

which is equivalent to

$$|\phi_\alpha(g(z))| = |h(z)| \leq \left|\frac{z - z_0}{1 - \overline{z}_0 z}\right| \quad \text{for } z, z_0 \in \Delta.$$

Substituting back the initial substitutions, this inequality is equivalent to

$$\left|\frac{(w_0/M) - (f(Rz)/M)}{1 - (\overline{w}_0/M)(f(Rz)/M)}\right| \leq \left|\frac{z - z_0}{1 - \overline{z}_0 z}\right| \quad \text{for } z, z_0 \in \Delta;$$

that is

$$M\left|\frac{w_0 - f(z)}{M^2 - \overline{w}_0 f(z)}\right| \leq R\frac{|z - z_0|}{|R^2 - \overline{z}_0 z|} \quad \text{for } z, z_0 \in \Delta_R.$$

6.83. Consider the function $F(z) = f(z)\prod_{j=1}^{N}\left(\frac{1 - \overline{z}_j z}{z - z_j}\right)$ and use the extended version of Schwarz' lemma.

6.84. Choose $N = 3$, $z_1 = 0$, $z_2 = 1/3$ and $z_3 = -1/3$ in Exercise 6.83.

6.86. As $|f(z)| \leq e^{\operatorname{Re} z} = |e^z|$, we have $|e^{-z} f(z)| \leq 1$ for $z \in \mathbb{C}$. By Liouville's theorem, $e^{-z} f(z) = a$ where a is a constant with $|a| \leq 1$. Similarly, an entire function g such that $|g(z)| \leq e^{-\operatorname{Im} z}$ for $z \in \mathbb{C}$ must be of the form $g(z) = ae^{iz}$ with $|a| \leq 1$.

6.88. We have already proved the first inequality (see Section 6.6). The second inequality follows similarly. Indeed, by the triangle inequality for $|z| \geq 1$ (so that $|z|^n \geq |z|^{n-1} \geq \cdots \geq |z|$), we have

$$\begin{aligned} |p(z) - z^n| &\leq |z|^n \left\{ \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|} \right\} \\ &\leq |z|^n \left(\frac{1}{|z|}(|a_0| + |a_1| + \cdots + |a_{n-1}|) \right) \\ &\leq \frac{|z|^n}{2} \quad \text{if } |z| > R \geq R_0, \end{aligned}$$

$R_0 = \max\{1, 2(|a_0| + \cdots + |a_{n-1}|)\}$, which shows that $|p(z)| \leq \frac{3}{2}|z|^n$.

6.89. For $|z| = r \geq R$, let $A(r) = \max_{|z|=r} \operatorname{Re} f(z)$. Then, the hypothesis implies that $A(r) \leq \beta r^\alpha$ for $|z| = r \geq R$. Taking $R = 2r$ in Theorem 6.32, we find that

$$|f(z)| \leq 3|f(0)| + 2A(2r) \leq 3|f(0)| + 2\beta(2r)^\alpha,$$

that is, $|f(z)| \leq 3|f(0)| + 2^{\alpha+1}\beta|z|^\alpha$ for $|z|$ large. Now use the method of proof of Theorem 6.60 to complete the proof.

Chapter 7, Exercises 7.8:

7.49:

(a) If $f(z)$ is entire and has a removable singularity at $z = \infty$, then g defined by $g(z) = f(1/z)$ has a removable singularity at the origin and therefore, $g(z)$ is bounded in a deleted neighborhood of 0. Equivalently, $f(z)$ is bounded on $|z| > R$ for some $R > 0$. By Liouville's theorem, $f(z)$ is constant.

(b) Let $f(z) = \sum_{k=0}^\infty a_k z^k$ be entire and have a pole of order n at ∞. If we define $g(z) = f(1/z)$, then $g(z)$ has a pole of order n at the origin. It follows that $z^n g(z)$ is bounded near 0; i.e. $z^{-n} f(z)$ is bounded near ∞. That is, $f(z)$ is entire such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Consequently, by Theorem 6.60, $f(z)$ is a polynomial of degree n. The converse part is trivial.

(d) If $f(z)$ is a nonconstant entire function, then $g(z) = \exp\{f(z)\}$ is also entire. If $g(z)$ has a removable singularity at ∞, then $g(z)$ (and hence, $f(z)$) is constant. If $g(z)$ has a pole at ∞, then $1/g(z)$, being a bounded entire function, is a constant. In either case, $f(z)$ is a constant which is a contradiction. Thus, ∞ is neither a removalble singularity nor a pole for g.

(h) See Remark 7.13 and Theorem 4.102.

(j) False. The denominator of $f(z)$, namely $z^2(\alpha - \overline{z})(\beta - \overline{z})$, is not analytic unless $\alpha = \beta = 0$.

(l) If z_0 is a pole of f of order k then $f(z) = (z - z_0)^{-k}g(z)$, where g is analytic at z_0, $g(z_0) \neq 0$. This gives $f'(z) = (z - z_0)^{-k-1}\phi(z)$, where ϕ defined by $\phi(z) = (z - z_0)g'(z) - kg(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

(n) Define $f_1(z) = \exp(1/z)$ and $f_2(z) = \sin(1/z)$. Consider its reciprocals $F_1(z) = \exp(-1/z)$ and $F_2(z) = 1/\sin(1/z)$. Then $z = 0$ is an isolated essential singularity for both $f_1(z)$ and $F_1(z)$. On the other hand, $z = 0$ is an isolated essential singularity for $f_2(z)$ but not for $F_2(z)$.

(o) If there does not exist such a sequence, f would then be bounded on a deleted neighbourhood of z_0. By the removability theorem, z_0 would be a removable singularity contradicting the hypothesis.

(r) For z near 1, consider the Laurent expansion of $f(z)$ about 1. Indeed,

$$f(z) = -\frac{e^{1/(z-1)}}{1 - ee^{z-1}} = -\left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(z-1)^n}\right)\left(\sum_{m=0}^{\infty} \frac{1}{m!} e^m e^{m(z-1)}\right)$$

and notice that there are infinitely many negative powers of $z - 1$.

(s) The given inequality shows that $|zf(z)| \to 0$ as $z \to 0$ so that $z = 0$ is a removable singularity of $zf(z) \in \mathcal{H}(\mathbb{C}\setminus\{0\})$. Define

$$F(z) = \begin{cases} zf(z) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

Then, F is entire and $|F(z)| \leq a|z|^{3/2} + b|z|^{1/2} \leq M|z|^{3/2}$ showing that $F(z) = cz + d$, by Theorem 6.60. As $F(0) = 0$, we have $d = 0$ so that $F(z) = cz$ or $f(z) = c$.

(u) Suppose that there exists such a function f. Then, for $z \neq 0$, $1/|f(z)| \leq |z|^{\alpha}$ and therefore, $1/|f(z)| \to 0$ as $z \to 0$. Define

$$F(z) = \begin{cases} \dfrac{1}{f(z)} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

Then, F is entire and $|F(z)| \leq |z|^{\alpha}$ for large value of $|z|$. It follows that F (and, hence, f) is constant. But this contradicts the hypothesis that $|f(z)| \geq |z|^{-\alpha}$ for z near 0. So, no such function exists.

(x) Rewrite the given equation as $e^g = 1 - e^f$, where the left hand side function never assumes zero. This means that $f(z) \neq 2k\pi i$ for each $k \in \mathbb{Z}$. That is, $\{2k\pi i : k \in \mathbb{Z}\} \subset \mathbb{C}\setminus f(\mathbb{C})$ so that the entire function f omits $2k\pi i$ for each $k \in \mathbb{Z}$. By Picard's theorem, f is constant. Interchanging the role of f and g shows that g is also constant.

(y) We know that $z = 0$ is a branch point of $\sqrt{z}$. On the other hand if z goes along a little circle $\partial\Delta_r$ once in the counterclockwise direction, then $\sqrt{z}$ is changed to $-\sqrt{z}$, whereas $f(z)$ is changed to

$$\frac{e^{-\sqrt{z}} - e^{\sqrt{z}}}{\sin(-\sqrt{z})} = \frac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{\sin\sqrt{z}} = f(z).$$

Therefore, $z = 0$ is not a branch point but is an isolated singularity of f. It can be seen that $z = 0$ is a removable singularity of f.
Note: It can be seen in the same spirit that $\sin\sqrt{z}$ is not an entire function whereas $\cos\sqrt{z}$ is entire.

7.50. Note that $|z^3 f(z)| \to 0$ as $z \to 0$. Define

$$F(z) = \begin{cases} z^3 f(z) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

Then, F is entire and $|F(z)| \leq a|z|^5 + b|z| \leq M|z|^5$ for large values of $|z|$. It follows that $F(z)$ is a polynomial of degree at most 5 with $F(0) = 0$. If f is odd, then f must be of the form $f(z) = cz + d/z$.

7.52. The point at infinity is not an isolated singularity of f as it is a limit point of poles and so, it cannot be pole.

7.53. $z = 0$ is not a pole but is an essential singularity. We may write $f(z)$ as $\sin 1 \cos(1/z) - \cos 1 \sin(1/z)$ for $|z| > 0$.

7.54. Consider $f(z) = e^z - \frac{1}{z}$. The point at infinity is an essential singularity of $f(z)$. Note that there is no non-constant meromorphic function which is analytic on the Riemann sphere −a consequence of Liouville's theorem.

7.60. Let ϕ satisfy the above conditions. Then the function f defined by

$$f(z) = (z - z_1)[(z - z_2)(z - z_3)(z - z_4)]^{-1}\phi(z)$$

is analytic at all points except at z_j, $j = 1, 2, 3, 4$. Since $\lim_{z\to z_j} f(z)$ exists, f is analytic in $\mathbb{C}$. Further, condition (iv) implies that f is analytic at infinity. As a consequence of Liouville's theorem f reduces to a constant, and by (iv) this constant must be 2 and hence ϕ takes the form:

$$\phi(z) = 2(z - z_1)^{-1}(z - z_2)(z - z_3)(z - z_4).$$

7.61. We find that

$$\lim_{z\to 1}(z - 1)f(z) = \lim_{z\to 1}\left(\frac{1}{nz^{n-1} + \cdots + 2z + 1} + c\right) = \frac{2}{n(n+1)} + c$$

and so $c = -2/n(n + 1)$ will do.

7.62. Observe that both $\lim_{z\to 0} f(z)$ and $\lim_{z\to 2}(z-2)f(z)$ exist and are non-zero. Further, if $g(z) = f(1/z)$ then

$$\lim_{\substack{z\to 0\\ z=x>0}} z^n g(z) = \lim_{\substack{x\to 0\\ x>0}} \frac{x^{n+2}(e^{1/x}-1)}{1-2x} = \infty$$

and

$$\lim_{\substack{z\to 0\\ z=-x>0}} z^n g(z) = (-1)^n \lim_{\substack{x\to 0\\ x>0}} \frac{x^{n+2}(e^{-1/x}-1)}{1+2x} = 0$$

showing that ∞ is neither a removable singularity nor a pole of $g(z)$. It follows that $z = 0$ is an essential singularity for $g(z)$.

7.63. Let $g(z) = z/\sin z$. Then g has singularities only at $z = k\pi$, $k \in \mathbb{Z}$. We observe that

(i) $\lim_{z\to 0} g(z) = 1$ so that $\lim_{z\to 0} f(z) = e$

(ii) $\lim_{\substack{z=x\to k\pi-\\ k-\text{even}}} g(z) = -\infty$ so that $\lim_{\substack{z=x\to k\pi-\\ k-\text{even}}} f(z) = \exp(-\infty) = 0$

(iii) $\lim_{\substack{z=x\to k\pi+\\ k-\text{even}}} g(z) = \infty$ so that $\lim_{\substack{z=x\to k\pi+\\ k-\text{even}}} f(z) = \exp(\infty) = \infty$.

Thus, (ii) and (iii) reveal the fact that $k\pi$ (k-even and $k \neq 0$) is neither removable nor a pole for f. A similar conclusion holds when k-is odd.

7.64. If $k(z) = z/(1-z)^{-2}$, then f defined by

$$f(z) = e^{i\alpha}k(e^{-i\alpha}z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \left(a_n = ne^{-i(n-1)\alpha}\right),$$

satisfies the hypotheses. Note that $|a_n| = n$ so that the sequence $\{a_n\}$ is not bounded. On the other hand, if $g(z) = (1 - ze^{-i\alpha})^{-1}$ then $g \in \mathcal{H}(\mathbb{C} \setminus \{e^{i\alpha}\})$ where $e^{i\alpha}$ is a simple pole for g. Note that

$$b_n = \frac{g^{(n)}(0)}{n!} = e^{-in\alpha} = \cos n\alpha - i \sin n\alpha$$

and $|b_n| = 1$. Clearly, $\{b_n\}$ does not converge although it is a bounded sequence.

Chapter 8, Exercises 8.6:

8.60:

(a) By definition

$$\begin{aligned}
&\operatorname{Res}[af(z)+bg(z);z_0]\\
&= \frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)}[af(\zeta)+bg(\zeta)]d\zeta\\
&= a\left[\frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} f(\zeta)d\zeta\right]+b\left[\frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} g(\zeta)d\zeta\right]\\
&= a\operatorname{Res}[f(z);z_0]+b\operatorname{Res}[g(z);z_0],
\end{aligned}$$

where r is chosen so that z_0 is the only singularity of f and g in $\Delta(z;r)$.

(f) Suppose not. Then, $|f(z)-1/z|<|a_{-1}-1|$ for all z with $|z|=1$. Moreover,

$$|a_{-1}-1| = \left|\frac{1}{2\pi i}\int_{|z|=1}\left(f(z)-\frac{1}{z}\right)dz\right| < |a_{-1}-1|$$

which is a contradiction. Note that if $f(z)=a_{-1}/z$, then, for $|z|=1$, we have the equality $|f(z)-1/z|=|a_{-1}-1|$.

(g) Use (8.23).

(h) Use (8.23).

(j) As in the previous exercise, we conclude that $F(z)=e^z-1-2z$ has only one zero inside the unit circle $|z|=1$. As $F(0)=0$, the only zero in Δ is at $z=0$ and so $1/F(z)$ has a simple pole at the origin with residue

$$\operatorname{Res}\left[\frac{1}{F(z)};0\right] = \lim_{z\to 0}\frac{z}{e^z-1-2z} = \lim_{z\to 0}\frac{1}{e^z-2} = -1.$$

(k) Justify your answer by constructing explicit examples.

(n) Suppose that such a function exists. Then, we must have

$$2\pi i = \int_{|z|=1}\frac{dz}{z} = 0$$

which is absurd; so no such f can exist with the given property.

(o) See Example 8.17.

(p) Since $\operatorname{Res}[\cosh z\cot z;k\pi]=\cosh k\pi,\ k\in\mathbb{Z}$, and $k\pi$ lies inside the circle C, the assertion follows by the residue theorem.

(q) The Laurent expansion of e^z yields

$$f(z) = \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{z^{-m}}{m!}\right),\ |z|>0.$$

Truth of the assertion now follows by obtaining the coefficients of $1/z$, using the Cauchy product of the two convergent series.

(s) Set $f(z) = e^z - 2$ and $g(z) = -3z$. For $|z| = 1$,

$$|e^z - 2| = \left| -1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} = e < 3 = |-3z|.$$

Since $g(z)$ has only one zero in the unit disk, $f(z)+g(z)$ also has only one zero in Δ. In the same way, we see that the number of solutions of $\sin z = z + 3z^2$ in the unit disk $|z| < 1$ is two.

(t) On $|z| = 1$, $|e^{z-a}| = |e^{z-a} - z - (-z)| = e^{\mathrm{Re}\, z-a} \leq e^{1-a} < |-z|$.

(u) On $|z| = 2$, $|p(z)| \leq 1 + 2 + 2^2 + 2^3 + 2^4 = 31 < 2^5 = |z|^5$.

(v) As $|-f(0)| < m < |f(z)|$ on $|z| = 1$, $f(z)$ and $f(z) - f(0)$ have the same number of zeros in Δ.

(w) Since $|f(z)| = |(f(z)-z)-(-z)| < 1 = |-z|$ for $|z| = 1$, $f(z)-z$ and $-z$ the have same number of zeros in Δ. Thus, there exists a point $z_0 \in \Delta$ such that $f(z_0) = z_0$.

(x) Let $f_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ and $C = \partial\Delta_R$. Then, we see that $f_n(z) \to (1-z)^{-2}$ as $n \to \infty$, $|z| < 1$. Applying the method of Example 8.59 we can immediately conclude that for $R < 1$ and n sufficiently large, the polynomial f_n has no zeros in $|z| < R$ (Also we can use Hurwitz' theorem with $f(z) = (1-z)^{-2}$).

(z) $N_f - P_f$ is the difference of the zeros and the poles of $f(z)$ in Δ plus the difference of the zeros and the poles of $f(1/z)$ in Δ. One may need to consider a disk other than Δ for certain functions.

8.62. As $f'(z) \neq 0$, $f(z) - f(a) \neq 0$ in $D \setminus \{a\}$. Now

$$\lim_{z \to a} (z-a) \frac{g(z)}{f(z) - f(a)} = \frac{g(a)}{f'(a)}.$$

Apply the residue theorem.

8.64. (iii) Set $f(z) = \sin \pi z$ and apply the argument principle.
(iv) Inside the circle $|z| = 3$, the function has two singularities at $z = 1$ and $z = 2$. Since $\mathrm{Res}\,[f(z); \infty] = -1$, by Theorem 8.32, the value of the integral is $2\pi i$.

8.66. Let $f \in \mathcal{H}(\overline{\Delta})$ and $f(z) \in \mathbb{R}$ for $|z| = 1$. Suppose that $c = a + ib$ with $b \neq 0$. We claim that $f(z) - c \neq 0$ for $|z| < 1$. To do this, we apply the argument principle and obtain

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - c}\, dz = N$$

where N denotes the number of zeros of $f(z) - c$ in Δ. Observe that

$$\mathrm{Im}\,(f(z) - c) = \mathrm{Im}\, f(z) - b = -b$$

implying that Γ (the image of $\partial\Delta$ under $f(z)-c$), which describes a closed contour, lies either in the lower half-plane or in the upper half-plane depending upon whether $b>0$ or $b<0$. Thus, in either case, $n(\Gamma,0)=0$ so that $N=0$, i.e. $f(z)-c\neq 0$ in $|z|<1$ whenever $\operatorname{Im} c\neq 0$. It follows that $f(z)$ never assumes non-real values and hence, $f(z)\in\mathbb{R}$ in $|z|<1$ which implies that f is constant (see Theorem 3.6).

8.74. For $|z|=R$, we have $|-e^z|=e^{\operatorname{Re}(z)}<e^R<|a|R^n=|az^n|$ (since $|\operatorname{Re} z|\leq R$). The first part follows from Rouché's theorem. If $R=1$ and $|a|>e$, the equation $az^n-e^z=0$ has n roots in $|z|<1$. If z_0 is a root of order k, $k\geq 2$, then

$$az_0^n-e^{z_0}=0 \quad\text{and}\quad naz_0^{n-1}-e^{z_0}=0$$

which means that $z_0^n-nz_0^{n-1}=0$ and this gives either $z_0=0$ or $z_0=n$; i.e. either $0=e^{z_0}=e^0=1$, or $\operatorname{Re} z_0\geq 1$ and this is a contradiction. Therefore the equation $az^n-e^z=0$ has exactly n simple roots with positive real part located in $|z|<1$.

8.77. Let $z=e^{i\theta}$. Then, we have

$$\begin{aligned}\int_{|z|=1}\frac{\overline{f(z)}}{z-a}\,dz &= \int_0^{2\pi}\frac{\overline{f(e^{i\theta})}}{e^{i\theta}-a}ie^{i\theta}\,d\theta\\ &= \int_0^{2\pi}\frac{\overline{f(\overline{e^{-i\theta}})}}{1-ae^{-i\theta}}i\,d\theta\\ &= \int_{|\zeta|=1}\frac{\overline{f(\overline{\zeta})}}{(1-a\zeta)\zeta}\,d\zeta,\quad \zeta=e^{-i\theta}.\end{aligned}$$

Note that $f(z)$ is analytic for $|z|<2$ if and only if $\overline{f(\overline{z})}$ is analytic for $|z|<2$. The result follows if we apply Cauchy's integral formula or Cauchy's residue theorem.

Chapter 9, Exercises 9.7:

9.75. It suffices to consider $b=1$. With this,

$$\operatorname{Re}\left\{\int_0^{2\pi}\frac{e^{in\theta}}{a+\cos\theta}\,d\theta\right\}=\operatorname{Re}\left\{\frac{2}{i}\int_{|z|=1}\frac{z^n}{z^2+2az+1}\,dz\right\}.$$

9.76(1). All the zeros $z=0,-2,i$ of $z(z+2)(z-i)^2$ lie inside the circle $|z|=3$. So $I=-2\pi i\operatorname{Res}[f(z);\infty]=0$.

9.78(1). The case $\alpha=0$ is the error integral while $\alpha\in[-1,0)$ follows from the case $\alpha\in(0,1]$ by conjugating both the sides of

$$\int_0^{\infty}e^{-(1+i\alpha)^2t^2}\,dt=\left(\frac{1}{1+i\alpha}\right)\frac{\sqrt{\pi}}{2}.$$

Thus, it suffices to prove this whenever $\alpha \in (0,1]$. For the proof, use a triangular contour with $0 < \alpha < 1$: $C = [0,R] \cup \{R + it : 0 \leq t \leq \alpha R\} \cup \{(1+i\alpha)t : t \in [0,R]\}$ and $f(z) = e^{-z^2}$.

9.78(2). One may use the contour shown in Figure 9.9. For $\lambda = 0$, one can find the value of the integral by taking the limit on the right as $\lambda \to 0$.

9.78(3). Choose $f(z) = \operatorname{Log} z/(z^2 + a^2) = (\ln|z| + i\operatorname{Arg} z)/(z^2 + a^2)$.

Chapter 10, Exercises 10.5:

10.54:

(a) Clearly, $f(z) = z$ and $f(z) = \sin z$ satisfies this relation. The given relation may be rewritten as

$$f(z) = 2f(z/2)f'(z/2) \quad \text{for } |z| < R.$$

As f is analytic for $|z| < R$, the right hand side is analytic for $|z| < 2R$. Consequently, by iteration, f is analytic for $|z| < 2^n R$ for arbitrary n.

(b) The functions f and F which represent the first and the second series are given by

$$f(z) = \frac{1}{1 - \alpha z} \quad \text{and} \quad F(z) = \frac{1 - z}{1 - z + (1 - \alpha)z},$$

respectively. Note that the first series converges for $|z| < 1/|\alpha|$ while the second for all z with $|(1-\alpha)z| < |1 - z|$.

(d) Clearly, we may write $f(z) = \operatorname{Log}(1 + z)$ which is analytic in the cut plane $\mathbb{C} \setminus (-\infty, -1]$. Similarly,

$$F(z) = \ln 2 + \operatorname{Log}\left(1 - \frac{1-z}{2}\right) = \ln 2 + \operatorname{Log}\left(\frac{1+z}{2}\right)$$

which is also analytic on the same cut plane. If $|z| < 1$, then both $1 + z$ and 2 lie in the right half-plane $\operatorname{Re} z > 0$ so that

$$F(z) = \ln 2 + \operatorname{Log}(1 + z) - \ln 2 = \operatorname{Log}(1 + z)$$

which implies that $f(z) = F(z)$ for $|z| < 1$.

10.56. Set $f(z) = \exp(z \operatorname{Log} z)$ and observe that $f(x)$ is real for $x > 0$. Apply Theorem 10.50.

10.58. Consider $u(x,y) = Axy + B$. By hypothesis $A + B = 3$ and $2A + B = 7$. Solve for A and B.

Chapter 11, Exercises 11.9:

11.112:

(c) Suppose that there exists a meromorphic function with poles at $1/n$ for each $n \in \mathbb{N}$. Then, 0 (which is a limit point of these poles), is a non-isolated singularity. Thus, the limit point of the poles of a meromorphic function must be the point at infinity.

(d) Suppose that there are infinitely many a_n's in some closed disk $|z| \leq R$ $(0 < R < \infty)$. Then there must exist a subsequence $\{a_{n_k}\}$ converging to a point a in this disk. Continuity of f then shows that $f(a) = 0$; but then this zero is not isolated. Thus, $f(z) \equiv 0$ which is a contradiction.

(f) If $f(z) = e^z$, then $f'(z) \neq 0$ in $\mathbb{C}$. If $f(z) = e^{p(z)}$, where $p(z)$ is a fixed polynomial of degree at least two, then $f'(z) = p'(z)e^{p(z)}$ and hence $f'(z)$ has zero(s) at the point(s) where $p'(z)$ has zero(s). For instance, if $f(z) = e^{z^3}$ then $f'(z)$ has a zero of order two at $z = 0$.

(k) The series $\sum_{n=2}^{\infty} |-n^{-\alpha}| = \sum_{n=2}^{\infty} n^{-\mathrm{Re}\,\alpha}$ converges iff $\mathrm{Re}\,\alpha > 1$.

(l) As $\sum_{n=1}^{\infty} |a^n z| = |z|\frac{|a|}{1-|a|} < \infty$, the product defines an entire function.

(n) Suffices to observe that for $|\alpha| < 1$,

$$P_n = \prod_{k=1}^{n} z^{\alpha^k} = z^{\alpha+\alpha^2+\cdots+\alpha^n} = z^{\alpha(1-\alpha^n)/(1-\alpha)}$$

showing that $P_n \to z^{\alpha/(1-\alpha)}$. In particular, for $\alpha = 1/2$ and $\alpha = 1/3$ gives that

$$\prod_{n=1}^{\infty} z^{2^{-n}} = z \quad \text{and} \quad \prod_{n=1}^{\infty} z^{3^{-n}} = z^{1/2} = \exp((1/2)\,\mathrm{Log}\,z).$$

(p) Note that

$$\begin{aligned} |f_k(z)| = \left|\frac{2}{k^z + k^3 - 1}\right| &\leq \frac{2}{|k^z| - |k^3 - 1|} \\ &\leq \frac{2}{k^{\mathrm{Re}\,z} - k^3 + 1} \\ &\leq \frac{2}{k^{\mathrm{Re}\,z} - k^3} = \frac{2}{k^3(k^{\mathrm{Re}\,z-3} - 1)} \end{aligned}$$

so that $\sum_{k=1}^{\infty} |f_k(z)|$ converges uniformly on every compact subset $\mathrm{Re}\,z \geq 3 + \delta$, $\delta > 0$.

(r) For instance, for $z = -1$, $\prod_{n=1}^{\infty}(1 + 1/n) = \infty$.

(s) The desired function is given by

$$\prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n} + \frac{z^2}{2n^2}\right)\right]^n = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)^n \exp\left(z + \frac{z^2}{2n}\right).$$

(v) Similar to the proof for infinite series.

(w) For example, $f(z) = \dfrac{\sin \pi\sqrt{z}}{\pi\sqrt{z}}$. Is it entire?

(z) Choose an entire function g with zeros at $z = n$, $n \in \mathbb{N}$ and consider $f(z) = g(1/(1+z))$.

11.113:

(c) Note that (as, for each $\epsilon > 0$, $|z| \le e^{\epsilon |z|}$ for $|z|$ large enough)

$$|ze^z| = |z|e^x \le |z|e^{|z|} \le e^{(\epsilon+1)|z|} \le \exp\left(|z|^{1+\epsilon}\right).$$

(d) As $\max_{|z|=r} |f(z)+g(z)| \le \max_{|z|=r} |f(z)| + \max_{|z|=r} |g(z)|$, we have

$$\begin{aligned} M(r, f+g) &\le M(r,f) + M(r,g) \\ &\le \exp\left(r^{\lambda(f)+\epsilon}\right) + \exp\left(r^{\lambda(g)+\epsilon}\right) \quad \text{for large } r \\ &\le 2\exp\left(r^{\max\{\lambda(f),\lambda(g)\}+\epsilon}\right) \\ &\le \exp\left(r^{\max\{\lambda(f),\lambda(g)\}+2\epsilon}\right) \quad \text{for large } r \end{aligned}$$

so that $\ln M(r, f+g) \le r^{\max\{\lambda(f),\lambda(g)\}+2\epsilon}$ and

$$\frac{\ln\ln M(r, f+g)}{\ln r} \le \max\{\lambda(f), \lambda(g)\} + 2\epsilon \quad \text{as } r \to \infty.$$

Since ϵ is arbitrary, $\lambda(f+g) \le \max\{\lambda(f), \lambda(g)\}$. Thus, the order of the sum of two entire functions f, g cannot exceed the order of f and g.

(e) Without loss of generality we may assume that $\lambda(g) < \lambda(f)$. Then the previous exercise gives

$$M(r, f+g) \le \exp(r^{\lambda(f)+2\epsilon})$$

so that $\lambda(f+g) \le \lambda(f) + 2\epsilon$. Therefore, it remains to show that $\lambda(f+g) \ge \lambda(f) - \epsilon$. First, by definition of lim sup

$$M(r, f) \ge \exp(r^{\lambda(f)-\epsilon}).$$

Now we assume that $\lambda(f) > \lambda(g)$. We can choose $\epsilon > 0$ so small such that

$$\lambda(f) - \epsilon > \lambda(g) + \epsilon.$$

Then there exists a sequence of numbers r_n such that $r_n \to \infty$ and

$$\begin{aligned} M(r_n, f+g) &\ge M(r_n, f) - M(r_n, g) \\ &\ge \exp(r_n^{\lambda(f)-\epsilon}) - \exp(r_n^{\lambda(g)+\epsilon}) \\ &= \exp(r_n^{\lambda(f)-\epsilon}) \left[1 - \exp(r_n^{\lambda(g)+\epsilon} - r_n^{\lambda(f)-\epsilon})\right] \\ &\ge \frac{1}{2}\exp(r_n^{\lambda(f)-\epsilon}) \end{aligned}$$

because,

$$r_n^{\lambda(g)+\epsilon} - r_n^{\lambda(f)-\epsilon} = r_n^{\lambda(f)-\epsilon}\left[r_n^{\lambda(g)-\lambda(f)+2\epsilon} - 1\right] \to -\infty \quad \text{as } n \to \infty.$$

That is, $\lambda(f+g) \geq \lambda(f)$. Thus, the addition of a function of lower order does not alter the order of the original function.

(f) Use the previous two exercises.

(g) As $|fg| \leq |f|\,|g|$ implies that $M(r, fg) \leq M(r, f)M(r, g)$, for every $\epsilon > 0$ and r sufficiently large, we have

$$\begin{aligned} M(r, fg) &\leq M(r, f)M(r, g)) \\ &\leq \exp\left(r^{\lambda(f)+\epsilon}\right) \exp\left(r^{\lambda(g)+\epsilon}\right) \\ &\leq \exp\left(2r^{\max\{\lambda(f),\lambda)g\}+\epsilon}\right). \end{aligned}$$

Thus, $\lambda(fg) \leq \max\{\lambda(f), \lambda(g)\}$ and so the order of the product of two entire functions f and g cannot exceed the maximum of the order of f and g.

(h) As $\max_{|z|=r}|h(z)| = \max_{|z|=r}|f(az)| = \max_{|\zeta|=|a|r}|f(\zeta)|$,

$$\begin{aligned} \frac{\ln\ln M(r,h)}{\ln r} &= \frac{\ln\ln M(|a|r, f)}{\ln r} \\ &= \frac{\ln\ln M(|a|r, f)}{\ln(|a|r)} \cdot \frac{\ln(|a|r)}{\ln r} \\ &\to \lambda(f)\cdot 1 \quad \text{as } r \to \infty. \end{aligned}$$

(i) As $\max_{|z|=r}|h(z)| = \max_{|z|=r}|f(z^n)| = \max_{|\zeta|=r^n}|f(\zeta)|$,

$$\begin{aligned} \frac{\ln\ln M(r,h)}{\ln r} &= \frac{\ln\ln M(r^n, f)}{\ln(r^n)} \cdot \frac{\ln(r^n)}{\ln r} \\ &\to \lambda(f)\cdot n \quad \text{as } r \to \infty. \end{aligned}$$

For example, as $\lambda(e^z) = 1$, we have $\lambda(\exp z^n) = n$. Similarly, the order of $\sin(z^n)$ is n.

(j) As $\max_{|z|=r}|h(z)| = \max_{|z|=r}|z^n f(z)| = r^n \max_{|z|=r}|f(z)|$,

$$\begin{aligned} \frac{\ln\ln M(r,h)}{\ln r} &= \frac{\ln\ln(r^n M(r,f))}{\ln r} \\ &= \frac{\ln(n\ln r + \ln M(r,f))}{\ln r} \\ &\geq \frac{\ln(\ln M(r,f))}{\ln r} \end{aligned}$$

which shows that $\lambda(h) \geq \lambda(f)$. Also, we know that every polynomial has order 0 and, h being a product of two entire functions, $\lambda(h) \leq \max\{\lambda(z^n), \lambda(f)\} = \lambda(f)$. Combining the last two inequalities gives $\lambda(h) = \lambda(f)$. For instance, the order of $z^n \sin(z^m)$ is m.

(k) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$ converges for every $\epsilon > 0$, so $\lambda(f) = 1$.

(m) We have $\lambda(e^z) = 1 = \lambda(-e^z)$ and $\lambda(e^z - e^z) = \lambda(0) = 0$.

(n) By substituting $s = 1$ in the definition of the ζ function, we may write

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n}\right)^{-1} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{p_n - 1}\right)$$

in the sense that the series and the product both diverges to ∞. Such an observation implies that the series $\sum_{n=1}^{\infty} \frac{1}{p_n - 1}$ diverges. It follows that there are infinitely many primes.

11.116. (i) By the Weierstrass factorization theorem, the desired entire function is given by $\prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right) e^{-z/n}$.

(iv) $\prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{5/4}}\right)$.

(v) $\prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{4/5}}\right) \exp\left(z/n^{4/5}\right)$.

(vi) $\prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{1/2}}\right) \exp\left(z/n^{1/2} + z^2/(2n)\right)$.

11.117. Because $\prod_{k=1}^{\infty}(1 + a_k)$ converges, we must have $a_k \to 0$ as $k \to \infty$ (by Proposition 11.16). As $\prod_{k=1}^{\infty}(1 + |a_k|)$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ does not converge absolutely (by Corollary 11.22) so the series may or may not converge. For example, if $a_k = (-1)^{k-1}/k$ then the product $\prod_{k=1}^{\infty}(1 + a_k)$ converges whereas $\prod_{k=1}^{\infty}(1 + |a_k|)$ diverges, see 11.15(i) and (iv). In this choice, the series $\sum_{k=1}^{\infty} a_k$ converges but not absolutely.

11.118. The proof is apparently in the proof of Theorem 11.30. Alternatively, we let $\sum_{k=1}^{\infty} |f_k(z)| < M$, $z \in \Omega$. Then

$$\tilde{p}_n(z) := \prod_{k=1}^{n}(1 + |f_k(z)|) \le \exp\left(\sum_{k=1}^{n} |f_k(z)|\right) < \exp(M)$$

so that $\tilde{p}_{n+1}(z) - \tilde{p}_n(z) = \tilde{p}_n(z)|f_{n+1}(z)| < e^M |f_{n+1}(z)|$. Since

$$\sum_{n=1}^{\infty}(\tilde{p}_{n+1}(z) - \tilde{p}_n(z)) \le e^M \sum_{n=1}^{\infty} |f_{n+1}(z)|$$

and $\sum_{n=1}^{\infty} |f_n(z)|$ converges uniformly, it follows that $\sum_{n=1}^{\infty}(\tilde{p}_{n+1}(z) - \tilde{p}_n(z))$ converges uniformly, and so does $\{\tilde{p}_n\}$. Hence, $\prod_{k=1}^{\infty}(1 + |f_k(z)|)$ is uniformly convergent.

(i) If $a_k = (k/(k+1))^k$, then $\lim_{k\to} |a_k|^{1/k} = 1$ so that by the root test the series $\sum_{k=1}^{\infty} a_k z^k$ converges absolutely for $|z| < 1$. Thus, the product $\prod_{k=1}^{\infty}(1 + f_k(z))$ converges absolutely for $|z| < 1$. What about for $|z| \ge 1$?

(ii) For $|z| < 1$, $f_k(z) \to 1 - z \neq 0$ so that the product diverges for $|z| < 1$.

11.119. Define $\phi_a(z) = (a - z)/(1 - \overline{a}z)$ and consider

$$F(z) = f(z) \left(\prod_{i=1}^{m} \frac{1}{\phi_{a_i/R}(z/R)} \right) \left(\prod_{j=1}^{n} \phi_{b_j/R}(z/R) \right).$$

Apply Theorem 10.31 to $\operatorname{Log} F(z)$ and then equate the real parts on both sides of the resulting equation. Finally, replace z by a to get the desired formula.

11.120. For instance, consider

$$f(z) = \cosh \sqrt{z} = \frac{1}{2}(e^{\sqrt{z}} + e^{-\sqrt{z}}) = \sum_{n=1}^{\infty} \frac{z^n}{(2n)!}.$$

Then f is an entire function of order $1/2$. Indeed,

$$|f(z)| < \frac{1}{2}(e^{\sqrt{r}} + e^{-\sqrt{r}}) \leq e^{\sqrt{r}} < e^{\sqrt{r}+\epsilon} \quad \text{for large values of } r$$

and for $z = r > 0$,

$$|f(z)| = \frac{1}{2}(e^{\sqrt{r}} + e^{-\sqrt{r}}) > e^{\sqrt{r}-\epsilon}, \quad \text{i.e.} \quad M(r) > e^{\sqrt{r}-\epsilon}$$

for large r. Similarly,

$$\cos \sqrt{z} = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots \quad \text{and} \quad \frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are entire functions of order $1/2$.

11.121. The zeros of $f(z) = \sin \pi(z + \alpha)$ are $a_n = n - \alpha$, $n \in \mathbb{Z}$. Since $\sum_{n\in\mathbb{Z}} |a_n|^{-1}$ diverges while $\sum_{n\in\mathbb{Z}} |a_n|^{-(1+\alpha)}$ converges for each $\alpha > 0$, we set $p = 1$. Since $\sin z$ is of order 1, by Theorem 11.100, the genus of $f(z)$ is 1. Hence, we have the representation

$$\sin \pi(z + \alpha) = z^0 e^{h(z)} \prod_{n\in\mathbb{Z}} E_1(z/(n - \alpha))$$

where $h(z)$ is a polynomial of degree at most 1. Thus, we write

$$\begin{aligned} \sin \pi(z + \alpha) &= e^{az+b} \prod_{n\in\mathbb{Z}} \left(1 + \frac{z}{n - \alpha}\right) e^{z/(n-\alpha)} \\ &= (\sin \pi\alpha) e^{az} \prod_{n\in\mathbb{Z}} \left(1 + \frac{z}{n - \alpha}\right) e^{z/(n-\alpha)} \end{aligned}$$

(because $z = 0$ gives that $e^b = \sin \pi\alpha$). In order to determine a, we take the logarithmic derivative on both sides. We find

$$\pi \cot \pi(z+\alpha) = a + \sum_{n\in\mathbb{Z}} \left[-\frac{1}{n-a-z} + \frac{1}{n-a} \right]$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $n - \alpha$. Allow $z \to 0$ to obtain $a = \pi \cot \pi\alpha$. Substituting this value and then changing n into $-n$ in the product gives the desired formula.

11.122. The order and the genus of f is 1 and 0, respectively.

11.123. Note that for $a = 1$, this function reduces to the (ordinary) zeta function (11.64). Use the method of proof of Theorem 11.67, and compare with (11.68).

11.124. For $n = 2$, the result reduces to Legendre's duplication formula. Note that $\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$ and $\Gamma(nz)$ have the same set of poles, namely at $z = 0, -\frac{1}{n}, -\frac{2}{n}, \cdots$ so that the quotient of these functions is entire without zeros. So, as in the proof of the case for $n = 2$, we have

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = e^{az+b}\Gamma(nz).$$

Replace z by $z + 1/n$ and compare the resulting equation with the former to compute the values of a and b.

11.125. Rewrite the functional equation (11.75) using the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = \frac{\pi}{2\sin(\pi s/2)\cos(\pi s/2)}.$$

Chapter 12, Exercises 12.8:

12.54:

(f) This statement is simply a reformulation of Theorem 12.18.

(g) By Corollary 12.8, $p'(z) \neq 0$ in Δ so that $a_1 \neq 0$ and each zero α_k of $p'(z)$ must lie outside the unit disk Δ. We may write

$$p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1} = na_n \prod_{k=1}^{n-1} (z - \alpha_k).$$

In particular, $a_1 = na_n \prod_{k=1}^{n-1}(-\alpha_k)$ and so $|a_1| \geq |na_n|$, since $|\alpha_k| \geq 1$ for each $k = 1, 2, \ldots, n-1$.

(r) See Step 1 in the proof of the Riemann mapping theorem. Note that the last part of Step 1 proves the existence of such a function f satisfying $f(a) = 0$ and $f'(a) > 0$.

(s) We need to distinguish the cases when $\Omega = \mathbb{C}$ and $\Omega \neq \mathbb{C}$. When $\Omega = \mathbb{C}$, the desired automorphism is given by

$$f(z) = z + b - a.$$

If $\Omega \neq \mathbb{C}$, then according to the Riemann mapping theorem there are two bijective maps $g : \Omega \to \Delta$ and $h : \Delta \to \Omega$ such that $g(a) = 0$ and $h(0) = b$. Then the desired map is given by $f = h \circ g$. Alternatively, by the Riemann mapping theorem, if $\Omega \neq \mathbb{C}$ then there exists a $G \in \mathcal{H}(\Omega)$ such that G is bijective. Choose $\phi \in \mathrm{Aut}\,(\Delta)$ with $\phi(G(a)) = G(b)$. Then, $f = G^{-1} \circ \phi \circ G$ is the desired map.

(t) By the open mapping theorem, the condition on α implies that $\Delta_\alpha \subseteq f(D)$. So, the preimage of Δ_α will be a subregion Ω of Δ. Define $g(z) = \alpha z$. Then, $F = f^{-1} \circ g$ is a one-to-one mapping of Δ onto Ω,

$$F(0) = f^{-1}(g(0)) = f^{-1}(0) = 0 \quad \text{and} \quad F'(0) = \frac{g'(0)}{f'(0)} = \frac{\alpha}{f'(0)}.$$

By Schwarz' lemma, $|F'(0)| \leq 1$ which gives $|f'(0)| \geq \alpha$.

(u) Consider $\phi : \Delta \to \Delta$ given by

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Then, $\phi_a(a) = 0$ and $\phi_a'(a) = -1/(1 - |a|^2)$. Thus, $g(z) = -\phi_{1/2}(z)$ maps Δ onto itself with $g(1/2) = 0$ and $g'(1/2) > 0$. Similarly, $h(z) = -\phi_{i/3}(z)$ maps Δ onto itself with $h(i/3) = 0$ and $h'(i/3) > 0$. The desired mapping is then given by $f = h^{-1} \circ g$. Note that

$$f(1/2) = h^{-1}(g(1/2)) = h^{-1}(0) = i/3 \quad \text{and} \quad f'(i/3) = \frac{g'(i/3)}{h'(g(i/3))} > 0.$$

(v) It suffices to note that

$$A = \int\!\!\int_{D'} du\, dv = \int\!\!\int_D J_f(z)\, dx\, dy = \int\!\!\int_D |f'(z)|^2\, dx\, dy.$$

(w) Now $f(z) = z + z^2/2$ is univalent and $|f'(z)|^2 = |1 + z|^2$. As

$$\begin{aligned} A &= \int\!\!\int_D |1 + z|^2\, dz \\ &= \int_0^{2\pi}\!\!\int_0^1 |1 + re^{i\theta}|^2 r\, dr d\theta \\ &= \int_0^{2\pi}\!\!\int_0^1 (1 + 2r\cos\theta + r^2) r\, dr d\theta = \frac{3\pi}{2}. \end{aligned}$$

(y) As in Example 12.33, just consider $\phi_\alpha(z)$ defined there with $|\alpha| < 1$ and α-real. Solve $\phi_\alpha(1/7) = -\phi_\alpha(1/2)$ which gives $\alpha = 3$ or $\alpha = 1/3$. Thus, $\phi_{1/3}(z) = (1-3z)/(3-z)$ maps $\partial\Delta$ onto itself, and maps the circle $\partial\Delta(9/28; 5/28)$ onto the circle with center at the origin and radius $r = |\phi_{1/3}(1/7)| = 1/5$.

(z) Using the idea of Example 12.33, we get $R = 2$ and

$$\phi(z) = \frac{ze^{i\theta} - (1/4)}{1 - (1/4)e^{i\theta}z}, \quad \theta \in \mathbb{R}.$$

12.56. Note that

$$\begin{aligned}\frac{1}{2i}\int_\gamma \overline{z}\, dz &= \frac{1}{2i}\int_\gamma (x\, dx + y\, dy) + \frac{1}{2}\int_\gamma (x\, dy - y\, dx)\\ &= \frac{1}{2i}\int\int_D 0\, dx\, dy + \frac{1}{2}\int\int_D 2\, dx\, dy\\ &= \int\int_D dx\, dy = \text{Area}\,(D).\end{aligned}$$

12.57. Set $w = f(z)$ and $\Gamma = f(\gamma)$. Then $f'(z)\, dz = dw$ and

$$\frac{1}{2i}\int_\gamma \overline{f(z)} f'(z)\, dz = \frac{1}{2i}\int_\Gamma \overline{w}\, dw,$$

which is a real number(see Exercise 12.56).

12.61. The proof for the cases (iii) for $n = 2$, and (iv) have already been dealt with in the proof of Theorem 12.36. We provide the proof of (iii) for $n \in \mathbb{N}$ as the remaining cases are easy.

Let $f \in \mathcal{S}$. As $F(z) = z^{-n} f(z^n) \neq 0$ on Δ, there exists $h \in H(\Delta)$ such that $(h(z))^n = F(z)$ with $h(0) = 1$. Then g defined by $g(z) = zh(z^n)$ belongs to $\mathcal{H}(\Delta)$ and $g(0) = 0 = g'(0) - 1$. For the univalency of g,

$$\begin{aligned}g(z_1) = g(z_2) &\implies f(z_1^n) = f(z_2^n)\\ &\implies z_1 = \omega z_2 \quad (\omega^n = 1)\\ &\implies g(z_1) = g(\omega z_1) = \omega z_1 h(\omega^n z_1^n) = \omega g(z_1)\\ &\implies (1-\omega) g(z_1) = 0, \quad \text{i.e. } \omega = 1 \text{ or } z_1 = 0.\end{aligned}$$

12.64. See the proof of the Riemann mapping theorem.

12.65. See the proof of the Riemann mapping theorem.

12.68. Clearly $f(\partial\Delta)$ is compact (as f is continuous and $\partial\Delta$ is a compact set). Then $f(\mathbb{C}) = f(\partial\Delta) \cup f(\mathbb{C}\backslash\partial\Delta) =: A \cup B$, where $A = f(\partial\Delta)$ is a compact set and $B = f(\{z : |z| \neq 1\})$ is a subset of $\mathbb{C}\backslash\mathbb{R}$, by hypothesis. Thus, $f(\mathbb{C}) = \mathbb{C}\backslash\{\mathbb{R}\backslash A\}$ and so f misses more than one point in its image. By Picard's little Theorem, f must be constant.

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